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## CHAPTER 5: TORSION

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### 5.1 Introduction

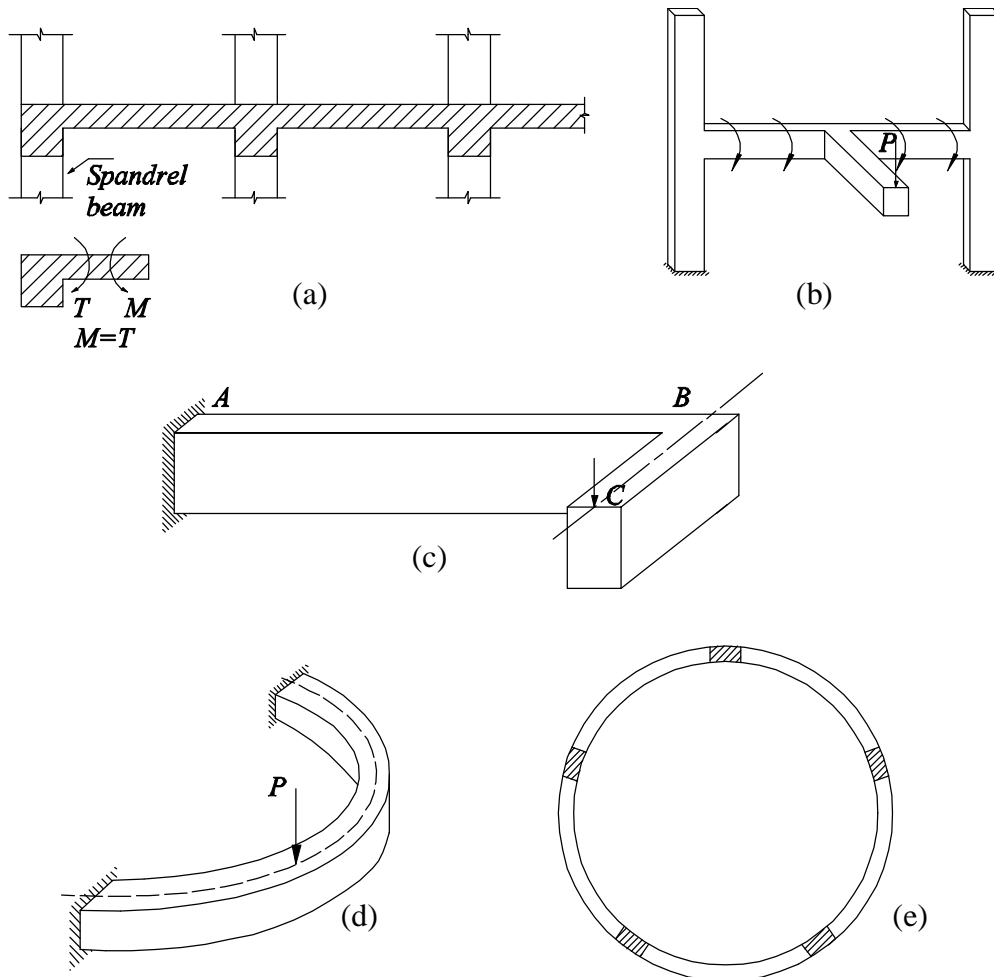
If external loads act far away from the vertical plane of bending, the beam is subjected to twisting about its longitudinal axis, known as torsion, in addition to the shearing force and bending moment.

Torsion on structural elements may be classified into two types; statically determinate, and statically indeterminate.

In Figures 5.1.a through 5.1.e several examples of beams subjected to torsion are shown. In these figures, torsion results from either supporting a slab or a beam on one side only, or supporting loads that act far away transverse to the longitudinal axis of the beam.

Shear stresses due to torsion create diagonal tension stresses that produce diagonal cracking. If the member is not adequately reinforced for torsion, a sudden brittle failure can occur.

Since shear and moment usually develop simultaneously with torsion, a reasonable design should logically account for the interaction of these forces. However, variable cracking, the inelastic behavior of concrete, and the intricate state of stress created by the interaction of shear, moment, and torsion make an exact analysis unfeasible. The current torsion design approach assumes no interaction between flexure, shear and torsion. Reinforcement for each of these forces is designed separately and then combined.



**Figure 5.1: Reinforced concrete members subjected to torsion: (a) spandrel beam; (b)&(c) loads act away from the vertical plane of bending; (d) curved beam; (e) circular beam**

## 5.2 Shear Stresses Due to Torsion

In a rectangular solid section, assuming elastic behavior, the shearing stresses vary in magnitude from zero at the centroid to a maximum at midpoints of the long sides as shown in Figure 5.2. The maximum shear stress  $t_{\max}$  is given as

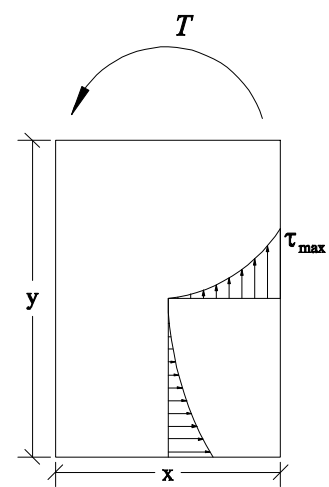
$$t_{\max} = \frac{T}{a x^2 y} \quad (5.1)$$

where  $x$  is the shorter side of the section,  $y$  is the longer side of the section, and  $a$  is a constant in terms of  $\frac{y}{x}$ . A close approximation to  $a$  is

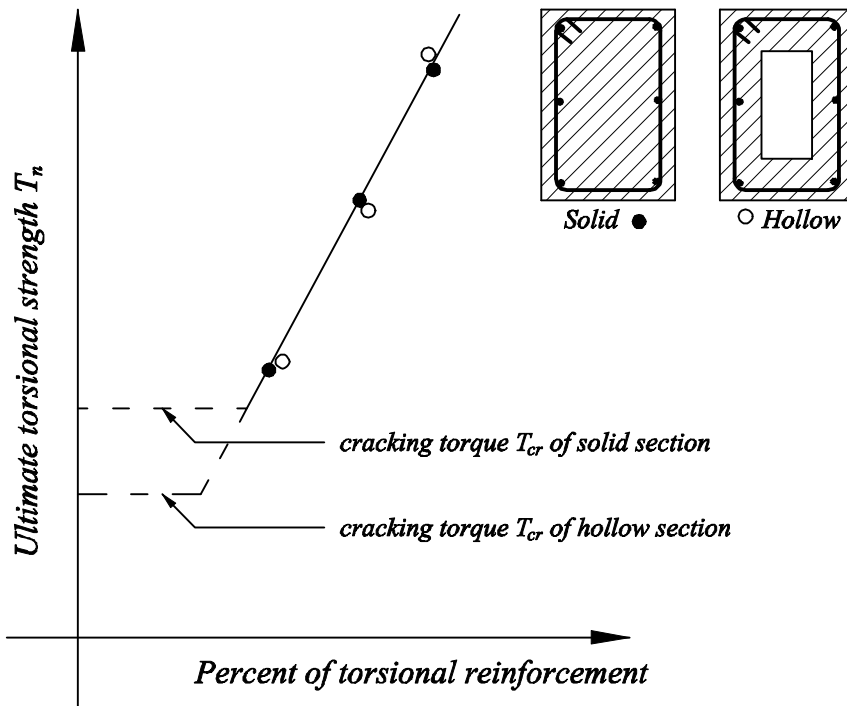
$$a = \frac{1}{3 + \frac{1.8 y}{x}} \quad (5.2)$$

Uncracked concrete members behavior is neither perfectly elastic nor perfectly plastic. However, elastic-based formulas have been satisfactorily used to predict torsional behavior.

Both solid and hollow members are considered as tubes. Experimental test results for solid and hollow beams with the same outside dimensions and identical areas of torsion reinforcement, shown in Figure 5.3, suggest that once torsional cracking has occurred, the concrete in the center of the member has a limited effect on the torsional strength of the cross section and thus can be ignored. In 1995, the *ACI Code* analyzed solid beams as hollow beams for which equations for evaluating shear stresses are easier to develop.



**Figure 5.2: Shear stresses in a rectangular section**



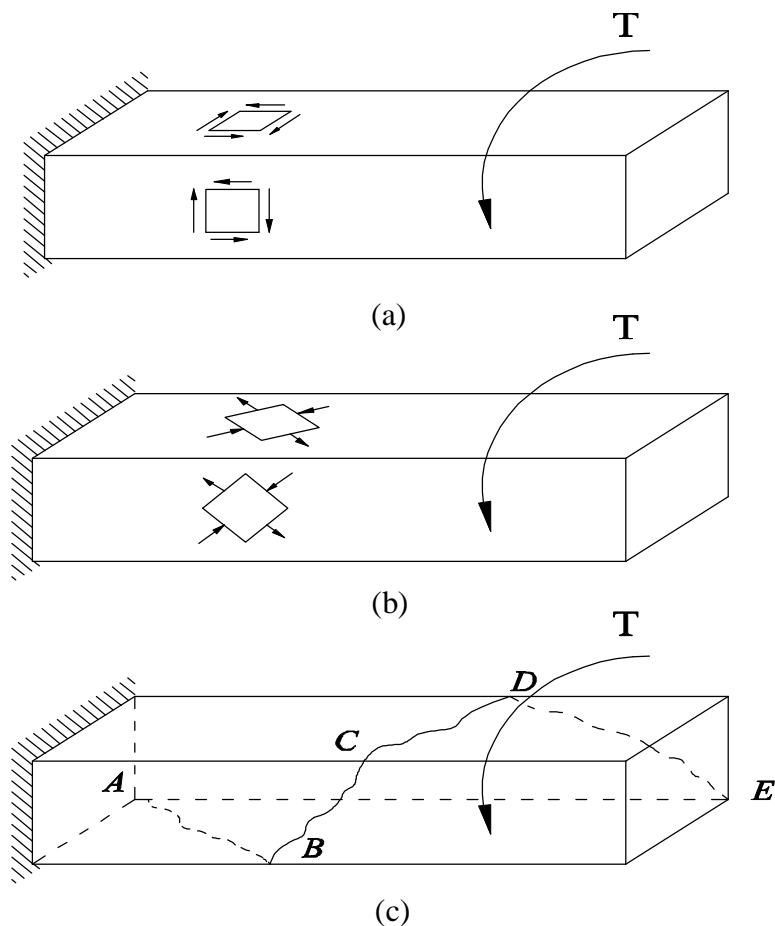
**Figure 5.3: Ultimate torsional strength of solid and hollow sections of the same size**

### 5.2.1 Principal Stresses Due to Pure Torsion

When the beam shown in Figure 5.4.a is subjected to pure torsion, shearing stresses develop in the four faces as shown by the elements. The principal stresses on these elements are shown in Figure 5.4.b.

The principal tensile strength is equal to the principal compressive stress and both are equal to the shearing stress  $t$ . Ultimately, when the principal tensile strength exceeds the maximum tensile strength of the beam, cracking will occur spiraling around the outside surface of the beam as shown in Figure 5.4.c.

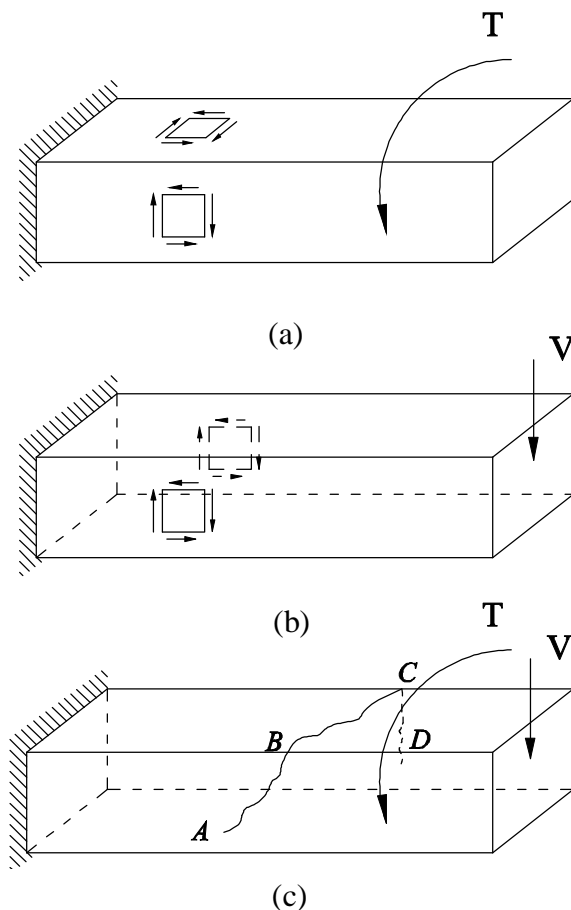
In a reinforced concrete member, such a crack would cause brittle failure unless torsional reinforcement is provided to limit the growth of this crack. Closed stirrups and longitudinal bars in the corners of the section are usually used as torsional reinforcement.



**Figure 5.4: Principal stresses and cracking due to pure torsion: (a) shear stresses; (b) principal stresses; (c) crack**

### 5.2.2 Principal Stresses Due to Torsion, Shear, and Moment

If a beam is subjected to torsion, shear, and bending, the two shearing stresses add on one side face and counteract each other on the opposite face, as shown in Figure 5.5. Therefore, inclined cracks start at the face where the shear stresses add (crack  $AB$ ) and extend across the extreme tension fiber. If the bending moment is large, the crack will extend almost vertically across the back face (crack  $CD$ ). The compressive stresses at the bottom of the cantilever beam prevent the cracks from extending all the way down the full height of the front and back faces.



**Figure 5.5: Combined shear, torsion and moment: (a) shear stresses due to pure torsion; (b) shear stresses due to direct shear; (c) crack**

#### 5.2.4 Torsion in Thin-walled Tubes

Thin-walled tubes of any shape can be quite simply analyzed for the shear stresses caused by a torque applied to the tube. We will consider here an arbitrary cross-sectional shape subjected to pure torsion by torques  $T$  applied at the ends. Furthermore, all cross sections of the tube are assumed to be closed and have similar dimensions and the longitudinal axis is a straight line.

The shear stresses  $t$  acting on the cross section are shown in Figure 5.6, which shows an element of the tube cut out between two cross sections at distance  $dx$ . The intensity of the shear stresses varies across the thickness of the tube. Since the tube is thin, we may assume  $t$  to be constant across the thickness of the tube.

From equilibrium of forces in the  $x$ -direction,

$$F_b = F_c \quad (5.3)$$

Where

$$F_b = t_b t_b dx \quad (5.4)$$

and

$$F_c = t_c t_c dx \quad (5.5)$$

Where  $t_b$  and  $t_c$  is tube thickness at points  $b$  and  $c$ , respectively.

Equating Eq. (5.4) and (5.5) gives

$$t_b t_b dx = t_c t_c dx \text{ or,}$$

$$t_b t_b = t_c t_c \quad (5.6)$$

Therefore, the product of the shear stress  $t$  and the thickness of the tube  $t$  is constant at every point in the cross section. This product is known as the shear flow and denoted by the letter  $q$ , and Eq. (5.6) can be written as

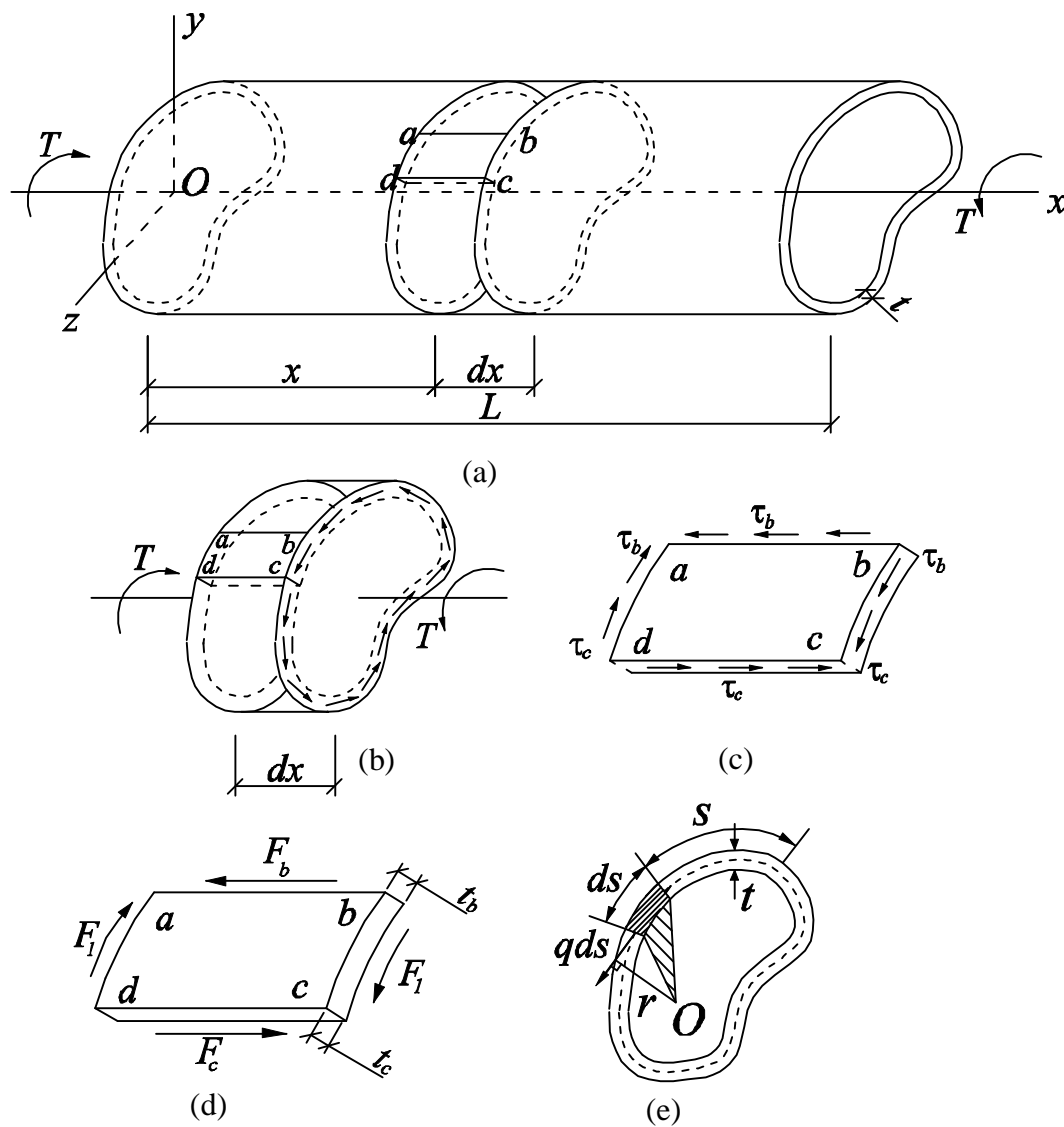
$$q = t t = \text{constant} \quad (5.7)$$

The largest shear stress occurs where the thickness of the tube is smallest, and vice versa. When the thickness is constant, the shear stress is also constant.

To relate the shear flow  $q$  to the torque  $T$ , consider an element of area of length  $ds$ , where  $ds$  is measured along the centerline of the cross section. The total shear force acting on this element of is  $q ds$ , and the moment of this force about any point "O" is  $dT = r q ds$ , where  $r$  is the perpendicular distance from point "O" to the line of action of the force.

The torque produced by shear is obtained by integrating along the entire length of centerline of the cross section, given by

$$T = q \oint r ds \quad (5.8)$$



**Figure 5.6: Shear stresses in a thin-walled tube**

The quantity  $r ds$  represents twice the area of the shaded triangle shown in Figure 5.6.e. Therefore, the integral  $\oint r ds$  represents double the area  $A_0$  enclosed by the centerline of the cross section, or

$$\oint r ds = 2 A_0 \quad (5.9)$$

Substituting Eq. (5.9) into Eq. (5.8) gives

$$T = 2 q A_0 \quad (5.10)$$

Using Eq. (5.7) and (5.10), one gets



$$q = \frac{T}{2 A_o} = t t \quad (5.11)$$

From *Eq. (5.11)* the shear stress  $t$  is given by

$$t = \frac{T}{2 A_o t} \quad (5.12)$$

*Eq. (5.11)* and *(5.12)* apply to any shape in the elastic range. In the inelastic range *Eq. (5.12)* applies only if the thickness  $t$  is constant.

### 5.3 Current ACI Code Design Philosophy

The current design procedure for torsion is based on the following assumptions:

- § Concrete strength in torsion is neglected.
- § Torsion has no effect on shear strength of concrete.
- § Torsion stress determination is based on thin-walled tube, space truss analogy. Both solid and hollow members are considered as tubes before and after cracking, and resistance is assumed to be provided by the outer part of the cross section centered around the stirrups.
- § No interaction exists between moment, shear, and torsion. Reinforcement for each of the three forces is calculated separately and then combined.

The basic design equation for torsion is

$$T_u = \Phi T_n \quad (5.13)$$

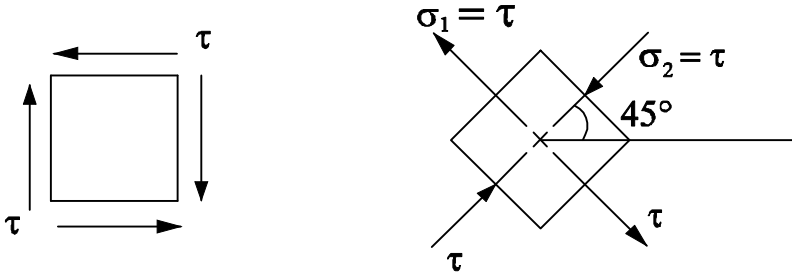
Where  $T_u$  is the factored torque,  $T_n$  is the nominal torsional capacity, and  $\Phi$  is the strength reduction factor for torsion, taken as 0.75.

### 5.4 Limit on Consideration of Torsion

In pure torsion, the principal tensile stress  $s_1$ , shown in Figure 5.7, is equal to the shear stress  $t$  at a given location. From *Eq. (5.12)* for a thin-walled tube,

$$s_1 = t = \frac{T}{2 A_o t} \quad (5.14)$$

Where  $t$  is the wall thickness at a point where the shear stress  $\tau$  is being computed and  $A_o$  is the area enclosed by the centerlines of the wall thicknesses.



**Figure 5.7: Principal stresses due to pure torsion**

It is noteworthy that Eq. (5.14) is derived exclusively for hollow sections. To apply this to solid uncracked section, the actual section is replaced by an equivalent thin-walled tube with a wall thickness  $t$  prior to cracking of  $3 A_{cp} / 4 p_{cp}$ , and an area enclosed by the wall centerline  $A_o$  equals  $2 A_{cp} / 3$ , where  $p_{cp}$  is the perimeter of the concrete section and  $A_{cp}$  is the area enclosed by this perimeter. Substituting these into Eq. (5.14) gives

$$s_1 = t = \frac{T}{2 \left( \frac{2 A_{cp}}{3} \right) \left( \frac{3 A_{cp}}{4 p_{cp}} \right)}$$

and

$$s_1 = t = \frac{T p_{cp}}{(A_{cp})^2} \quad (5.15)$$

Torsional cracking is assumed to occur when the principal tensile stress  $s_1$  reaches the tensile strength of concrete in biaxial tension-compression, taken as  $1.06 \sqrt{f'_c}$ , since the tensile strength under biaxial tension is less than that under uniaxial tension. Substituting this in Eq. (5.15), gives the cracking torque  $T_{cr}$  as

$$T_{cr} = 1.06 \sqrt{f'_c} \frac{(A_{cp})^2}{p_{cp}} \quad (5.16)$$

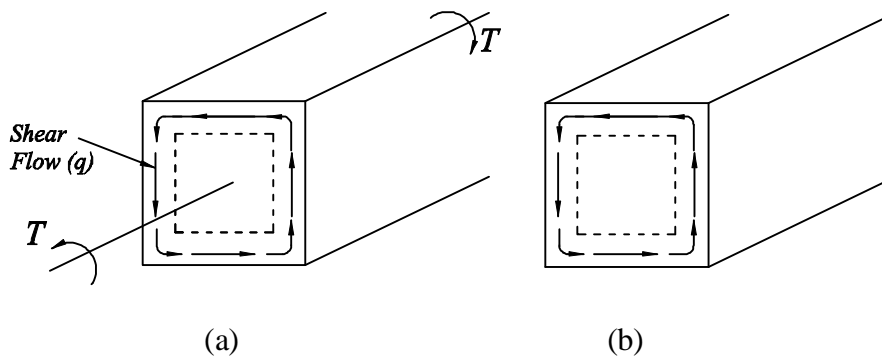
The *ACI Code* requires that torsion be considered in design if  $T_u$  exceeds  $0.25 T_{cr}$  given by

$$T_u \geq f 0.27 I \sqrt{f_c'} \frac{(A_{cp}^2)}{p_{cp}} \quad (5.17)$$

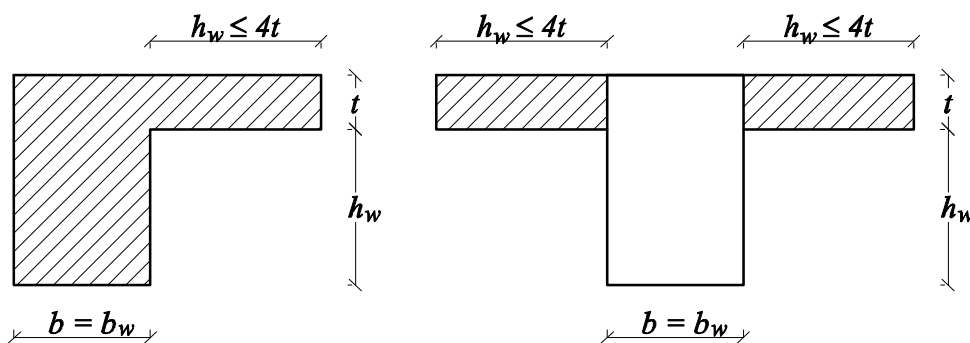
Torques that do not exceed approximately one-quarter of the cracking torque  $T_{cr}$  will not cause a structurally significant reduction in either flexural or shear strength and can be ignored.

For an isolated member with or without flanges,  $A_{cp}$  is the area of the entire cross section including the area of voids in hollow cross sections, and  $p_{cp}$  is the perimeter of the entire cross section as shown in Figure 5.8. For a T-beam cast monolithically with a slab,  $A_{cp}$  and  $p_{cp}$  can include portions of the adjacent slab conforming to the following:

For monolithic construction, a beam includes that portion of slab on each side of the beam extending a distance equal to the projection of the beam above or below the slab, whichever is greater, but not greater than four times the slab thickness. (See Figure 5.9)



**Figure 5.8: Definition of  $A_{cp}$ : (a) thin walled tube; (b) area enclosed by shear flow path**



**Figure 5.9: L and T beams in monolithic construction**

## 5.5 Ensuring Ductile Mode of Failure

The size of the cross section is limited for two reasons, first, to reduce unpleasant cracking and second to prevent crushing of the concrete due to principal compressive stresses resulting out of shear and torsion.

For solid sections, *ACI Code 11.5.3.1* requires that the following equation be satisfied

$$\sqrt{\left(\frac{V_u}{b_w d}\right)^2 + \left(\frac{T_u p_h}{1.7 A_{oh}^2}\right)^2} \leq f \left( \frac{V_c}{b_w d} + 2\sqrt{f_c'} \right) \quad (5.18)$$

For hollow sections, *ACI Code 11.5.3.1* requires that

$$\left( \frac{V_u}{b_w d} \right) + \left( \frac{T_u p_h}{1.7 A_{oh}^2} \right) \leq f \left( \frac{V_c}{b_w d} + 2\sqrt{f_c'} \right) \quad (5.19)$$

where:

$T_u$  = factored torsional moment at section

$V_u$  = factored shear force at section

$b_w$  = web width

$d$  = effective depth

$p_h$  = perimeter of centerline of outermost closed transverse torsional reinforcement

$A_{oh}$  = area enclosed by centerline of outermost closed transverse torsional reinforcement

*ACI 11.5.3.2* requires that if the wall thickness varies around the perimeter of a hollow section, *Eq. (5.19)* be evaluated at the location where the left-hand side of this equation is a maximum.

Furthermore, if the wall thickness is less than  $\frac{A_{oh}}{p_h}$ , *ACI Code 11.5.3.3* requires that the

second term in *Eq. (5.19)* be taken as  $\left( \frac{T_u}{1.70 A_{oh} t} \right)$ , where  $t$  is the thickness of the wall of

the hollow section at the location where the stresses are being checked.

## 5.6 Critical Section for Torsion

According to *ACI Code 11.5.2.4*, sections located less than a distance  $d$  from the face of a support are designed for not less than the factored torque computed at a distance  $d$ . If a concentrated torque occurs within this distance, the critical section for design must be taken at the face of the support. A concentrated torque occurs when a cross beam frames into one side of a girder near the support of the girder.

## 5.7 Torsional Reinforcement

Both longitudinal and transverse reinforcement are required to resist diagonal tension stresses due to torsion. The stirrups must be closed, since torsion cracks can spiral around the beam. *ACI Code 11.5.4.1* requires the use of longitudinal reinforcing bars in addition to closed stirrups, perpendicular to the axis of the member or spiral reinforcement.

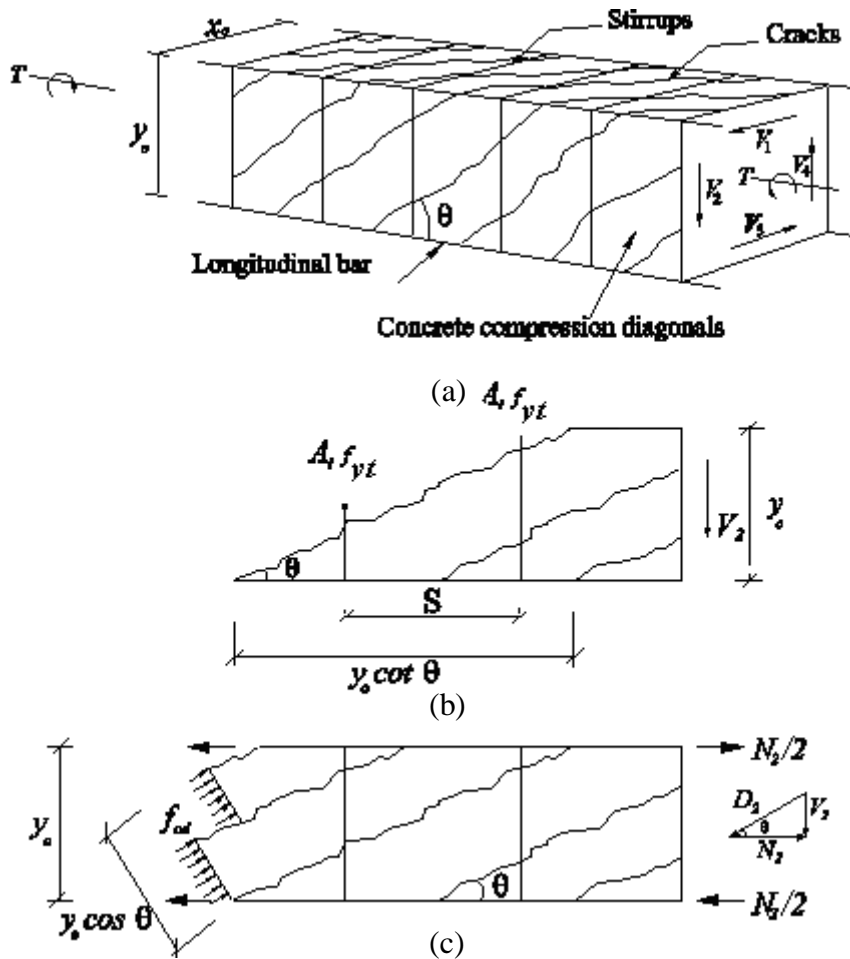
### 5.7.4 Transverse Reinforcement

A beam subjected to pure torsion can be modeled as a hollow-tube space truss consisting of closed stirrups, longitudinal bars in the corners, and diagonal concrete compression members which spiral around the beam between cracks. The height and width of the truss are  $y_0$  and  $x_0$ , measured between the centers of the corner bars. The angle of the crack is  $q$ , generally taken as  $45^\circ$  for reinforced concrete.

The shear flow  $q$ , is given by

$$q = \frac{T}{2 A_0} = t t \quad (5.11)$$

The total shear force due to torsion along each of the two vertical sides of the truss shown in Figure 5.10.a is equal to the product of the shear flow  $q$  and the distance  $y_0$  between the centerlines of these two sides



**Figure 5.10: Torsional reinforcement: (a) Space truss analogy; (b) forces in stirrups; (c) resolution of shear force  $V_2$**

$$V_2 = V_4 = q y_0 \quad (5.20)$$

Substituting Eq. (5.11) into Eq. (5.20) gives

$$V_2 = V_4 = \frac{T y_0}{2 A_0} \quad (5.21)$$

Similarly, the shear force due to torsion along each of the two horizontal sides is given as

$$V_1 = V_3 = \frac{T x_0}{2 A_0} \quad (5.22)$$

The top crack in Figure 10.5.b intersects a number of stirrups  $n$ , where

$$n = \frac{y_o \cot q}{S} \quad (5.23)$$

where  $S$  is the spacing of the stirrups.

The force in each stirrup at ultimate torque, assuming yielding of all stirrups is equal to  $A_t f_{yt}$ , where  $A_t$  is the area of one leg of a closed stirrup resisting torsion within a distance  $S$ , and  $f_{yt}$  is the yield stress of the transverse reinforcement.

From equilibrium of forces in the vertical direction

$$V_2 = n A_t f_{yt} \quad (5.24)$$

Substituting Eq. (5.23) into Eq. (4.24) gives

$$V_2 = \frac{y_o \cot q A_t f_{yt}}{S} \quad (5.25)$$

Equating Eq. (5.25) and (5.21) results in

$$\frac{T_n y_o}{2 A_o} = \frac{y_o \cot q A_t f_{yt}}{S}$$

where  $T_n$  is the nominal torsion capacity, or

$$\frac{A_t}{S} = \frac{T_n}{2 A_o f_{yt} \cot q} \quad (5.26)$$

Where  $q$  may be taken any angle between  $30^\circ$  and  $60^\circ$ . *ACI Code 11.5.3.6* permits  $q$  to be taken as  $45^\circ$  and the area  $A_o$  to be taken as  $0.85 A_{oh}$ , where  $A_{oh}$  is the area enclosed by the centerline of outermost closed stirrups.

### 5.7.5 Longitudinal Reinforcement

The force  $V_2$  in Figure 5.10.c can be resolved into a diagonal compression force,  $D_2$ , parallel to the concrete struts and an axial tension force,  $N_2$ , where  $D_2$  and  $N_2$  are given by

$$N_2 = V_2 \cot q \quad (5.27)$$

Similarly, on the top and bottom faces

$$N_1 = V_1 \cot \mathbf{q} \quad (5.28)$$

The total longitudinal force is

$$N = 2 (N_1 + N_2) \quad (5.29)$$

Substituting Eq. (5.27) and (5.28) into Eq. (5.29) gives

$$N = 2 (V_2 \cot \mathbf{q} + V_1 \cot \mathbf{q}) \quad (5.30)$$

Substituting Eq. (5.21) and (5.22) into Eq. (5.30) gives

$$N = 2 \left( \frac{T y_o}{2 A_o} + \frac{T x_o}{2 A_o} \right) \cot \mathbf{q}$$

or,

$$N = \frac{T_n}{2 A_o} [2(x_o + y_o)] \cot \mathbf{q} \quad (5.31)$$

The force in the corner bars at ultimate strength is given as

$$N = A_l f_y \quad (5.32)$$

where  $A_l$  is total area of longitudinal torsion reinforcement, and  $f_y$  is yield stress of longitudinal torsion reinforcement.

But  $2(x_o + y_o) = p_h \cong$  perimeter of the closed stirrup. Using this and equating Eq. (5.31) and (5.32) gives

$$A_l = \frac{T_n p_h \cot \mathbf{q}}{2 A_o f_y} \quad (5.33)$$

Substituting Eq. (5.26) into Eq. (5.33) gives

$$A_l = \left( \frac{A_t}{S} \right) p_h \left( \frac{f_{yt}}{f_y} \right) \cot^2 \mathbf{q} \quad (5.34)$$



## 5.7.6 Minimum Amount of Torsion Reinforcement

### 5.7.6.1 Longitudinal Reinforcement

To ensure that concrete beams will fail in a ductile manner, *ACI Code 11.5.5.3* requires that the minimum area of longitudinal steel should not be less than

$$A_{l,min} = \frac{1.33\sqrt{f'_c} A_{cp}}{f_y} - \left(\frac{A_t}{S}\right) p_h \frac{f_{yt}}{f_y} \quad (5.35)$$

where  $\frac{A_t}{S} \geq 0.175 \frac{b_w}{f_{yt}}$

### 5.7.6.2 Transverse Reinforcement

*ACI Code 11.5.5.2* specifies that where torsion reinforcement is required, the minimum area of transverse closed stirrups for combined action of shear and torsion is computed by:

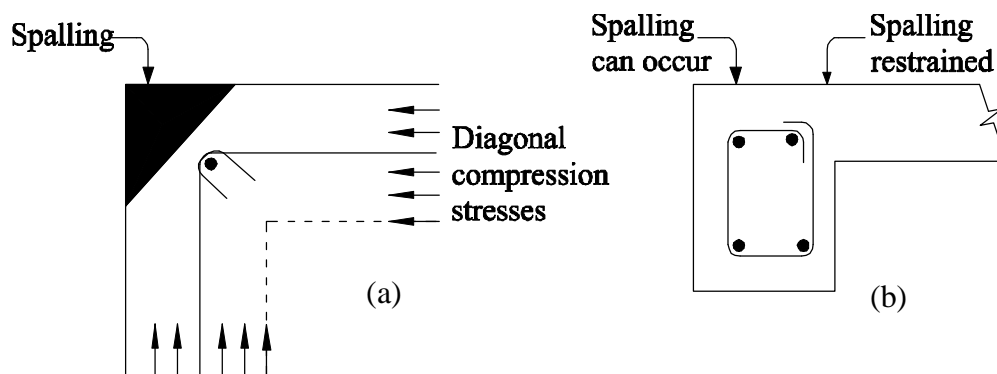
$$(A_v + 2A_t) = \frac{0.2\sqrt{f'_c} b_w S}{f_{yt}} \geq \frac{3.5b_w S}{f_{yt}} \quad (5.36)$$

where  $A_v$  is the area of two legs of a closed stirrup while  $A_t$  is the area of one leg of a closed stirrup.

## 5.7.7 Details of Torsion Reinforcement

### 5.7.7.1 Transverse Reinforcement

When a rectangular beam fails in torsion, the corners of the beam tend to spall off due to the compressive stresses in the concrete diagonals of the space truss. In tests, closed stirrups anchored by 90-degree hooks failed when this occurred. For this reason, 135-degree hooks are preferable for torsional stirrups in all cases. In regions where this spalling is prevented by an adjacent slab or flange, *ACI Code 11.5.4.2* relaxes this and allows 90 degree hooks, as shown in Figure 5.11.



**Figure 5.11: Spalling of corners of beams loaded in torsion**

*ACI Code 11.5.4.4* requires that for hollow sections in torsion, the distance between the centerline of the closed stirrups to the inside face of the wall of the hollow section should not be less than  $\frac{A_{oh}}{2p_h}$ .

### 5.7.7.2 Longitudinal Reinforcement

*ACI Code 11.5.4.3* requires that longitudinal torsion reinforcement be developed at both ends. If high torsion acts near the end of a beam, the longitudinal torsion reinforcement should be adequately anchored. Sufficient development length should be provided outside the inner face of the support to develop the needed tension force in the bars. This may require hooks or horizontal U-shaped bars lapped with the longitudinal torsion reinforcement.

*ACI Code 11.5.6.3* requires that torsion reinforcement be provided for a distance of at least  $(b_t + d)$  beyond the point theoretically required for torsional reinforcement, where  $b_t$  is width of that part of cross section containing the closed stirrups, and  $d$  is the effective depth of section. This requirement is dictated because torsional cracks develop in a spiral form.

## 5.7.8 Spacing of Torsion Reinforcement

### 5.7.8.1 Transverse Reinforcement

According to *ACI Code 11.5.6.1*, the spacing of transverse torsion reinforcement center-to-center is not to exceed the smaller of  $\frac{P_h}{8}$  or 30 cm. The spacing of the stirrup is limited to ensure the development of the ultimate torsional strength of the beam and to control crack widths.

### 5.7.8.2 Longitudinal Reinforcement

*ACI Code 11.5.6.2* requires that longitudinal torsion reinforcement be distributed around the perimeter of closed stirrups with a maximum spacing of 30 cm. One bar must be positioned in each corner of the stirrups to provide anchorage for the legs of the stirrups. The least longitudinal bar diameter to be used is the larger of 0.042 times the stirrup spacing, or 10 mm. See Figure 5.12 for reinforcement detail.

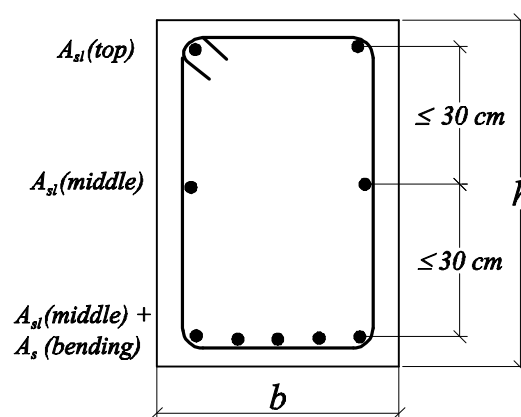


Figure 5.12: Longitudinal reinforcement for flexure and torsion

## 5.8 Equilibrium and Compatibility Torsion

In designing for torsion in reinforced concrete structures, two cases may be identified:

### 5.8.1 Equilibrium Torsion

The torsional moment cannot be reduced by redistribution of internal forces. This is referred to as equilibrium torsion since the torsional moment is required to keep the structure in equilibrium.

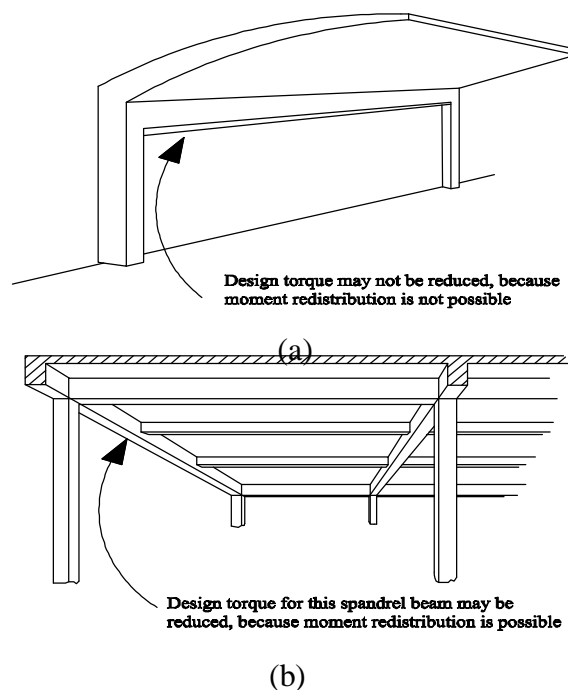
According to *ACI Code 11.5.2.1*, if the factored torsional moment  $T_u$  in a member is required to maintain equilibrium and exceed the minimum value given by Eq. (5.16), the member is to be designed to carry that torsional moment. An example for equilibrium torsion is shown in Figure 5.13.a.

### 5.8.2 Compatibility Torsion

The torsional moment can be reduced by redistribution of internal forces after cracking if the torsion arises from the member twisting to maintain compatibility of deformations. This type of torsion is referred to as compatibility torsion, an example of which is shown in Figure 5.13.b.

According to *ACI Code 11.5.2.2*, for statically indeterminate structures where reduction of the torsional moment in a member can occur due to redistribution of internal forces upon cracking, the maximum factored torsional moment  $T_u$  is permitted to be reduced

$$\text{to } \frac{fI \sqrt{f'_c} (A^2_{cp})}{P_{cp}}.$$



**Figure 5.13: Equilibrium vs. compatibility torsion: (a) design torque may not be reduced; (b) design torque may be reduced**

## 5.9 Summary of Design Procedure for Members Subjected to Bending Moment, Shear and Torsion

1. Draw the shear force, bending moment, and torque diagrams.
2. Select cross-sectional dimensions  $b$  and  $h$  based on factored bending moment, and determine the required area of reinforcement.
3. Check if torsion may be neglected. Torsion may be neglected if

$$T_u \leq f 0.27 I \sqrt{f'_c} \frac{(A^2_{cp})}{P_{cp}} \quad (5.17)$$

If this is the case, proceed on with shear design as done in Chapter 4, and choose flexural and shear reinforcement.

If  $T_u > f 0.27 I \sqrt{f'_c} \frac{(A^2_{cp})}{P_{cp}}$ , torsion must be considered as shown in the following steps.

The critical section for torsion is located at distance  $d$  from the face of the support if no torques are applied within this distance. If torques are applied within distance  $d$  from face of support, critical torsion is located at face of the support.

4. Check the adequacy of the size of the cross section in terms of preventing brittle mode of failure resulting from diagonal compressive stresses due to shear and torsion combined.

For a solid cross section to be adequate,

$$\sqrt{\left(\frac{V_u}{b_w d}\right)^2 + \left(\frac{T_u P_h}{1.7 A^2_{oh}}\right)^2} \leq f \left(\frac{V_c}{b_w d} + 2\sqrt{f'_c}\right) \quad (5.18)$$

For a hollow cross section to be adequate,

$$\left(\frac{V_u}{b_w d}\right) + \left(\frac{T_u P_h}{1.7 A^2_{oh}}\right) \leq f \left(\frac{V_c}{b_w d} + 2\sqrt{f'_c}\right) \quad (5.19)$$

If Eq. (5.18) or (5.19) is not satisfied, cross sectional dimensions need to be increased.

5. Determine the area of stirrups required for shear. To facilitate the addition of stirrups for shear and torsion, the area of shear reinforcement is expressed in terms

of  $\frac{A_v}{S}$ ,

$$\frac{A_v}{S} = \frac{V_s}{f_{yt} d} \quad (4.15)$$

If  $V_s > 2.2 \sqrt{f'_c} b_w d$ , the cross section needs to be enlarged. Also, determine maximum stirrup spacing based on shear.

If  $V_s \leq \sqrt{f'_c} b_w d$ , maximum stirrup spacing  $S_{max}$  should not exceed the smaller of  $\frac{d}{2}$  or 60 cm. If  $2.2 \sqrt{f'_c} b_w d > V_s > \sqrt{f'_c} b_w d$ , maximum stirrup spacing is limited to the smaller of  $\frac{d}{4}$  or 30 cm.

6. Determine the required area of stirrups for torsion in terms of  $\frac{A_t}{S}$ .

Since  $T_u = \Phi T_n$ , Eq. (5.26) when  $q$  is taken as  $45^\circ$  can be written as

$$\frac{A_t}{S} = \frac{T_n}{2A_o f_{yt}}$$

Besides, compute maximum stirrup spacing based on torsion. Maximum stirrup spacing is limited to the smaller of  $p_h/8$  or 30 cm.

7. Determine combined area of stirrups required for shear and torsion.  $\frac{A_t}{S}$  is expressed in terms of  $\frac{A_v}{S}$ .

For two-legged closed stirrups,

$$\left(\frac{A_v}{S}\right)_r = \left(\frac{A_v}{S}\right) + \left(\frac{2A_t}{S}\right)$$

where  $\left(\frac{A_v}{S}\right)_r$  should be equal or larger than  $\frac{0.2\sqrt{f'_c} b_w S}{f_{yt}} \geq \frac{3.5b_w S}{f_{yt}}$ .

8. Select stirrup size, and compute stirrup spacing based on the amount determined in step 7. Maximum stirrup spacing must not exceed the smaller of the two values evaluated in steps 5 and 6.

9. Calculate the longitudinal reinforcement required for torsion.

$$A_l = \left(\frac{A_t}{S}\right) p_h \left(\frac{f_{yt}}{f_y}\right) \cot^2 q \quad (5.34)$$

If  $f_{yt} = f_y$ , and  $q$  is taken as  $45^\circ$ ,  $A_l = \left(\frac{A_t}{S}\right) p_h$

The longitudinal reinforcement  $A_{l, \min}$  is not to be less than the value given by

$$A_{l, \min} = \frac{1.33 \sqrt{f'_c} A_{cp}}{f_y} - \left( \frac{A_t}{S} \right) p_h \frac{f_{yt}}{f_y} \quad (5.35)$$

where  $\frac{A_t}{S} \geq 0.175 \frac{b_w}{f_{yt}}$

10. Size combined longitudinal reinforcement. The longitudinal reinforcement is to be uniformly distributed around the perimeter of the cross section, and must have a diameter not less than 0.042 of stirrup spacing or 10 mm, whichever is larger.

Flexural reinforcement determined in step 2 is to be added to the longitudinal reinforcement required for torsion at the flexural steel level, and the reinforcement is chosen at the tension side of the section. Also, the reinforcement is chosen at the compression side of the section. If the vertical distance, center-to-center, between the top and bottom reinforcement exceeds 30 cm, middle layers are added until the spacing is satisfied.

11. Torsion reinforcement must be extended at least  $(b_t + d)$  beyond the point where no torsion reinforcement is required.

12. Draw neat sketches showing reinforcement details.

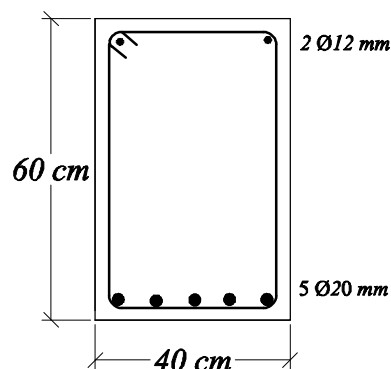
**Example (5.1):**

For the cross section shown in Figure 5.14.a, design the torsional reinforcement, if required, given that:

$$T_u = 4 \text{ t.m}, \quad V_u = 16.0 \text{ tons},$$

$$f'_c = 250 \text{ kg/cm}^2,$$

$$\text{and } f_y = 4200 \text{ kg/cm}^2.$$



**Figure 5.14.a: cross section**

**Solution:**

**1- Draw the shear force, bending moment, and torque diagrams.**

The section is already designed for bending moment. The magnitudes of shear force and torsion are given.

**2- Select cross sectional dimensions and determine area of flexural reinforcement.**

Cross sectional dimensions in addition to flexural reinforcement are given.

**3- Check if torsion may be neglected.**

Torsion may be neglected if  $T_u$  is smaller than

$$T_u \leq f 0.27 I \sqrt{f_c'} \frac{(A_{cp}^2)}{p_{cp}} = \frac{0.75 \times 0.27 \sqrt{250} (2400)^2}{(200) \times 10^5} = 0.922 \text{ t.m} < 4.0 \text{ t.m}$$

Thus, torsion reinforcement must be considered.

**4- Check for probability of brittle failure of concrete.**

The dimensions of the cross section are adequate to prevent a brittle mode of failure if the following equation is satisfied

$$\sqrt{\left(\frac{V_u}{b_w d}\right)^2 + \left(\frac{T_u p_h}{1.7 A_{oh}^2}\right)^2} \leq f \left(\frac{V_c}{b_w d} + 2\sqrt{f_c'}\right)$$

Assuming that  $f 8 \text{ mm}$  stirrups are used,

$$d = 60 - 4 - 0.8 - 1.0 = 54.20 \text{ cm}$$

$$x_o = 40 - 2 \times 4 - 0.80 = 31.20 \text{ cm}$$

$$y_o = 60 - 2 \times 4 - 0.80 = 51.2 \text{ cm}$$

$$p_h = 2(31.2 + 51.2) = 164.80 \text{ cm}$$

$$A_{oh} = (31.2 \times 51.2) = 1597.44 \text{ cm}^2$$

$$\begin{aligned} & \sqrt{\left(\frac{V_u}{b_w d}\right)^2 + \left(\frac{T_u p_h}{1.7 A_{oh}^2}\right)^2} \\ &= \sqrt{\left(\frac{16000}{40(54.20)}\right)^2 + \left(\frac{4.0 \times 10^5 \times 164.80}{1.70(1597.44)^2}\right)^2} = 16.89 \text{ kg/cm}^2 \end{aligned}$$

$$f \left(\frac{V_c}{b_w d} + 2\sqrt{f_c'}\right) = 2.53 f \sqrt{f_c'} = 2.53(0.75)\sqrt{250} = 30.0 \text{ kg/cm}^2$$

i.e. cross sectional dimensions are adequate for resisting brittle failure.



**5- Determine the required area of stirrup reinforcement for shear.**

$$V_c = 0.53 \sqrt{f'_c} b_w d$$

$$V_s = \frac{V_u}{\Phi} - V_c$$

$$V_s = \frac{16.0}{0.75} - \frac{0.53 \sqrt{250} (40) (54.20)}{1000} = 3.165 \text{ tons}$$

$$\frac{A_v}{S} = \frac{V_s}{f_{yt} d} = \frac{3.165 (1000)}{4200 (54.2)} = 0.0139 \text{ cm}^2 / \text{cm}$$

Since  $V_s \leq \sqrt{f'_c} b_w d$ , maximum stirrup spacing  $S_{\max}$  should not exceed the smaller of

$$\frac{d}{2} = \frac{54.20}{2} = 27.10 \text{ cm} \text{ or } 60 \text{ cm, thus } S_{\max} \text{ is not to exceed } 27 \text{ cm.}$$

**6- Determine the required area of stirrup reinforcement for torsion.**

The torsional stirrup reinforcement per unit length  $\frac{A_t}{S}$ , is given by

$$\frac{A_t}{S} = \frac{T_u}{2F f_{yt} A_o \cot^2(45^\circ)} = \frac{4.0 \times 10^5}{2 (0.75) (4200) (0.85 \times 1597.44)} = 0.047 \text{ cm}^2 / \text{cm}$$

$S_{\max}$  is the smaller of  $p_h / 8$  or 30 cm, where  $p_h / 8 = 164.80 / 8 = 20.60 \text{ cm}$ , thus  $S_{\max}$  is not to exceed 20 cm.

**7- Determine the combined area of stirrup reinforcement for shear and torsion.**

$$\left( \frac{A_v}{S} \right)_{\text{total}} = 0.0139 + 2 (0.047) = 0.1079 \text{ cm}^2 / \text{cm}$$

$$\frac{3.5 b_w}{f_{yv}} = \frac{3.5 (40)}{4200} = 0.033 \text{ cm}^2 / \text{cm} < 0.1079 \text{ cm}^2 \quad O.K$$

**8- Select stirrup size and spacing.**

Try f8 mm closed stirrups

$$\frac{2 (0.50)}{S} = 0.1079 \text{ and } S = 9.26 \text{ cm.}$$

Try  $f10$  mm closed stirrups

$$\frac{2(0.785)}{S} = 0.1079 \text{ and } S = 14.55 \text{ cm.}$$

Therefore, maximum stirrup spacing based on shear and torsion combined is  $20$  cm, which is larger than  $14$  cm. Use two-legged,  $f10$  mm closed stirrups spaced at  $14.0$  cm.

### 9- Calculate the longitudinal reinforcement required for torsion.

Since  $f_{yt} = f_y$  and for  $q = 45^\circ$ ,

$$A_l = \left( \frac{A_t}{S} \right) p_h = (0.047)(164.80) = 7.75 \text{ cm}^2$$

The longitudinal reinforcement is not to be less than the value given by the following equation

$$A_{l,min} = \frac{1.33 \sqrt{f'_c} A_{cp}}{f_y} - \left( \frac{A_t}{S} \right) p_h \frac{f_{yt}}{f_y}$$

$$\text{where } \frac{A_t}{S} \geq \frac{(0.175) b_w}{f_{yv}}$$

$$\frac{(0.175) b_w}{f_{yt}} = \frac{0.175 (40)}{4200} = 0.00167 \text{ O.K.}$$

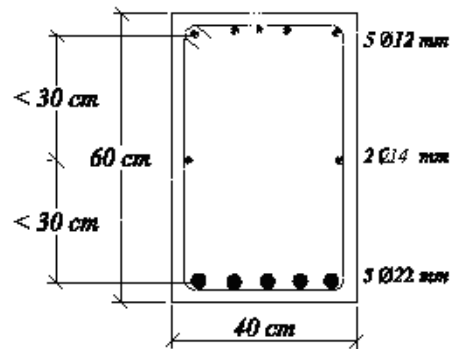
$$A_{l,min} = \frac{1.33 \sqrt{250} (40)(60)}{4200} - 7.75 = 4.27 \text{ cm}^2 \text{ O.K.}$$

### 10-Size combined longitudinal reinforcement.

Distance between the centerlines of tension and compression reinforcement exceeds  $30$  cm, so a middle layer of longitudinal reinforcement is needed.

$$\frac{A_l}{3} = \frac{7.75}{3} = 2.58 \text{ cm}^2$$

Minimum bar diameter is  $0.042 S = 0.042 (140) = 5.88$  mm or  $10$  mm, whichever is larger, and  $10$  mm is the least diameter to be used.



**Figure 5.14.b: Designed cross section**

Total area of top reinforcement =  $2.26 + 2.58 = 4.84 \text{ cm}^2$ , use 5 f 12 mm.

Area of bottom reinforcement =  $2.58 + 15.71 = 18.29 \text{ cm}^2$ , use 5 f 22 mm.

Area of middle reinforcement =  $2.58 \text{ cm}^2$ , use 2 f 14 mm.

#### **11-Develop torsional reinforcement.**

Torsion reinforcement must be extended at least  $(40 + 54.2) \cong 95.0 \text{ cm}$  beyond the section where no torsion reinforcement is required.

#### **12-Draw neat sketches showing reinforcement details.**

Figure 5.14.b shows cross sectional dimensions and reinforcement details.

#### **Example (5.2):**

The cantilever beam shown in 5.a supports a factored concentrated load of 8 tons applied at point C. Neglecting own weights of members AB and BC, design member AB for moment, shear, and torsion. خطأ! لم يتم العثور على مصدر المرجع.

Use  $f'_c = 300 \text{ kg/cm}^2$ , and  $f_y = 4200 \text{ kg/cm}^2$ .

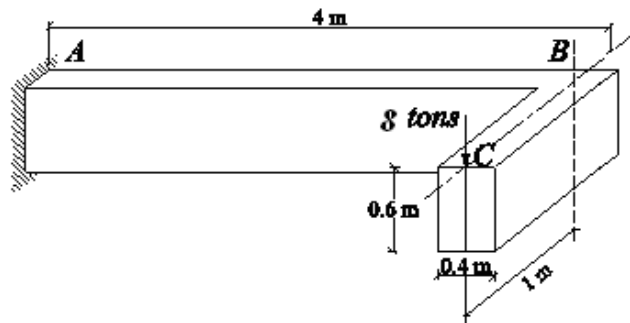


Figure 5.15.a: Beam and loads

**Solution:**

1- Draw the shear force, bending moment, and torque diagrams.

These diagrams are shown in Figure 5.15.b.

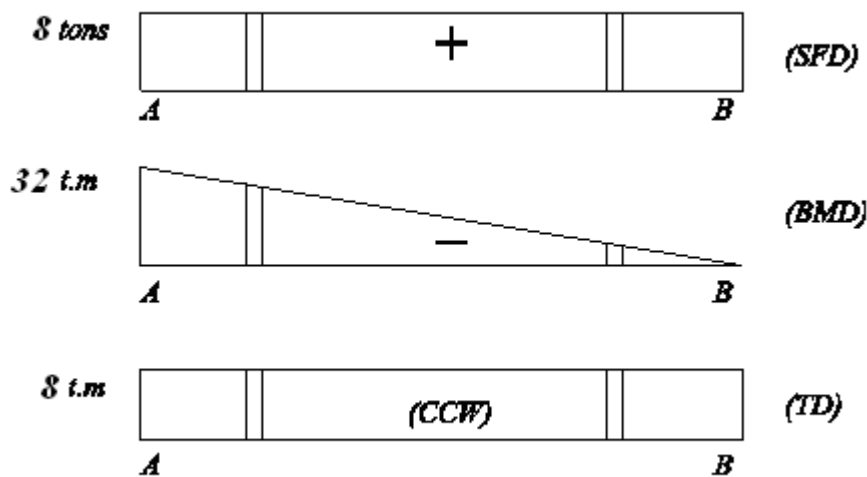


Figure 5.15.b: Shear force and bending moment diagrams

$$M_{u \max} = 32 \text{ t.m} , V_{u \max} = 8 \text{ tons} , \text{ and } T_{u \max} = 8 \text{ t.m} .$$

2- Select cross sectional dimensions and determine area of flexural reinforcement.

Minimum depth of member AB based on deflection requirement is equal to

$$\frac{l}{8} = \frac{400}{8} = 50 \text{ cm} . \text{ Use a } 40 \times 60 \text{ cm cross section.}$$

$d = 60 - 4 - 0.80 - 1.0 = 54.20 \text{ cm}$ , assuming  $f$  20 mm reinforcing bars, and  $f$  8 mm stirrups.

The reinforcement ratio at section of maximum factored moment is given by

$$= \frac{0.85(300)}{4200} \left[ 1 - \sqrt{1 - \frac{2.353(10)^5(32)}{0.9(40)(54.2)^2(300)}} \right] = 0.00769$$

$$A_s = 0.00769(40)(54.20) = 16.67 \text{ cm}^2$$

### 3- Check if torsion may be neglected.

Torsion may be neglected if  $T_u$  is smaller than

$$T_u \leq f 0.27 I \sqrt{f_c'} \frac{(A_{cp}^2)}{P_{cp}} = \frac{0.27(0.75)\sqrt{300}(2400)^2}{2(40+60)(10)^5} = 1.01 \text{ t.m} < T_u$$

Therefore, torsion must be considered.

### 4- Check for probability of brittle failure of concrete.

The dimensions of the cross section are adequate to insure a ductile mode of failure if this condition is satisfied

$$\sqrt{\left(\frac{V_u}{b_w d}\right)^2 + \left(\frac{T_u p_h}{1.7 A_{oh}^2}\right)^2} \leq f \left(\frac{V_c}{b_w d} + 2\sqrt{f_c'}\right)$$

where

$$x_o = 40 - 2 \times 4 - 0.80 = 31.2 \text{ cm}$$

$$y_o = 60 - 2 \times 4 - 0.80 = 51.2 \text{ cm}$$

$$p_h = 2(31.2 + 51.2) = 164.80 \text{ cm}$$

$$A_{oh} = (31.2 \times 51.2) = 1597.44 \text{ cm}^2$$

$$\sqrt{\left(\frac{V_u}{b_w d}\right)^2 + \left(\frac{T_u p_h}{1.70 A_{oh}^2}\right)^2} = \sqrt{\left(\frac{8000}{40(54.20)}\right)^2 + \left(\frac{8.0 \times 10^5(164.80)}{1.70(1597.44)^2}\right)^2} = 38.27 \text{ kg/cm}^2$$

$$f \left(\frac{V_c}{b_w d} + 2\sqrt{f_c'}\right) = 2.53 f \sqrt{f_c'} = 2.53(0.75)\sqrt{300} = 32.86 \text{ kg/cm}^2$$

i.e. cross sectional dimensions are adequate in terms of ensuring ductile failure.

**5- Determine the required area of stirrup reinforcement for shear.**

$$V_c = 0.53 \sqrt{f'_c} b_w d$$

$$V_s = \frac{V_u}{\Phi} - V_c$$

$$V_s = \frac{8.0}{0.75} - \frac{0.53 \sqrt{300} (40)(54.20)}{1000} = -9.24 \text{ tons}$$

This means that shear reinforcement is not required, and  $\frac{A_v}{S} = 0.0$ .

**6- Determine the required area of stirrup reinforcement for torsion.**

The torsional stirrup reinforcement per unit length  $\frac{A_t}{S}$ , is given as

$$\frac{A_t}{S} = \frac{T_u}{2F f_{yt} A_o \cot(45^\circ)} = \frac{8.0 \times 10^5}{2 (0.75)(4200)(0.85 \times 1597.44)} = 0.0935 \text{ cm}^2 / \text{cm}$$

$S_{\max}$  is the smaller of  $p_h / 8$  or  $30 \text{ cm}$ , where  $p_h / 8 = 164.80 / 8 = 20.60 \text{ cm}$ , thus  $S_{\max}$  is not to exceed  $20 \text{ cm}$ .

**7- Determine the combined area of stirrup reinforcement for shear and torsion.**

$$\left( \frac{A_v}{S} \right)_{\text{total}} = 0.0 + 2(0.0935) = 0.187 \text{ cm}^2 / \text{cm}$$

$$\frac{3.5 b_w}{f_{yv}} = \frac{3.5 (40)}{4200} = 0.033 \text{ cm}^2 / \text{cm} < 0.187 \text{ cm}^2 \quad O.K$$

**8- Select stirrup size and spacing.**

Try  $f$  10 mm closed stirrups

$$\frac{2(0.785)}{S} = 0.187 \quad \text{and} \quad S = 8.40 \text{ cm. This spacing is rather small, so try } f \text{ 12 mm closed}$$

stirrups,

$$\frac{2(1.13)}{S} = 0.187 \text{ and } S = 12.09 \text{ cm.}$$

Use  $f$  12 mm closed stirrups spaced at 12.0 cm.

**9- Calculate the longitudinal reinforcement required for torsion.**

Since  $f_{yt} = f_y$  and for  $q = 45^\circ$ ,

$$A_l = \left( \frac{A_t}{S} \right) p_h = (0.0935)(164.80) = 15.41 \text{ cm}^2$$

The longitudinal reinforcement is not to be less than the value given by the following equation

$$A_{l,min} = \frac{1.33 \sqrt{f'_c} A_{cp}}{f_y} - \left( \frac{A_t}{S} \right) p_h \frac{f_{yt}}{f_y}$$

$$\text{where } \frac{A_t}{S} \geq \frac{(0.175) b_w}{f_{yv}}$$

$$\frac{(0.175) b_w}{f_{yt}} = \frac{0.175 (40)}{4200} = 0.00167 \text{ O.K}$$

$$A_{l,min} = \frac{1.33 \sqrt{300} (40)(60)}{4200} - 15.41 = -2.35 \text{ cm}^2 \text{ (not applicable)}$$

**10-Size combined longitudinal reinforcement.**

Distance between centerlines of tension and compression reinforcement exceeds 30 cm, so a middle layer of longitudinal reinforcement is needed.

$$\frac{A_l}{3} = \frac{15.41}{3} = 5.14 \text{ cm}^2$$

Minimum bar diameter is  $0.042 S = 0.042(120) = 5.04 \text{ mm}$  or 10 mm, whichever is larger, and 10 mm is the least diameter to be used.

Total area of top reinforcement =  $16.67 + 5.614 = 21.81 \text{ cm}^2$ , use 5  $f$  25 mm.

Area of bottom reinforcement =  $5.14 \text{ cm}^2$ , use 4  $f$  14 mm.

Area of middle reinforcement =  $5.14 \text{ cm}^2$ , use 4  $f$  14 mm.

### 11-Develop torsional reinforcement.

Torsional reinforcement is needed for the entire length of part *AB*, since the torque is constant in magnitude.

### 12-Draw neat sketches showing reinforcement details.

Figure 5.15.c shows cross sectional dimensions and designed reinforcement.

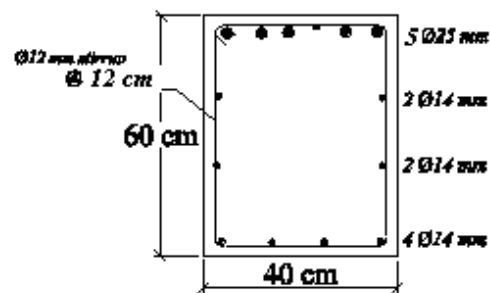


Figure 5.15.c: Designed cross section

## 5.10 Problems

**P5.11.1** A cantilever beam 3.0 m long and 40 cm wide, shown in Figure P5.11.1 supports its own weight plus a concentrated service load located 0.50 m from the end of the beam and 0.50 m away from the vertical axis of the beam. The concentrated load consists of 7.0 ton dead load and 8.0 ton live load. Design the reinforcement for flexure, shear, and torsion.

Use  $f'_c = 300 \text{ kg/cm}^2$  and  $f_{yt} = f_y = 4200 \text{ kg/cm}^2$ .

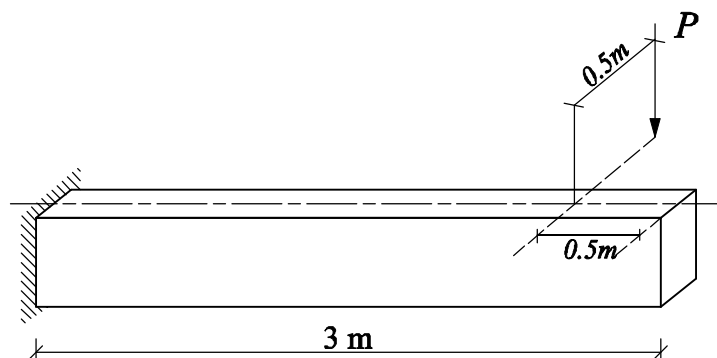
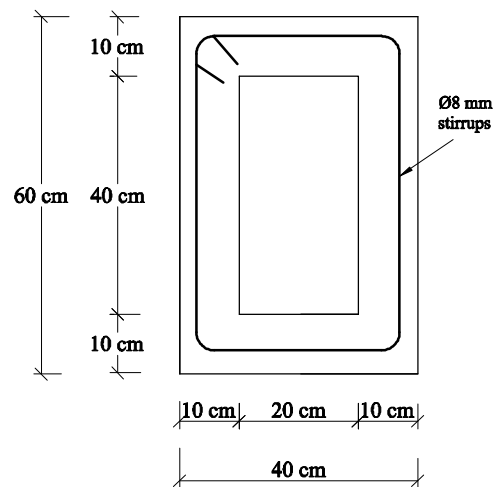


Figure P5.11.1



**P5.11.2** A shear force  $V_u = 20 \text{ tons}$  and a torque  $T_u = 8.0 \text{ t.m}$  act on the cross section of the box-beam in Figure P 5.11.2. Determine the spacing of the stirrups and the area of longitudinal steel required to carry the given shear and torsion.

Use  $f'_c = 280 \text{ kg/cm}^2$  and  $f_{yt} = f_y = 4200 \text{ kg/cm}^2$ .



**Figure P5.11.2**