Chapter 3 Inner Product Spaces. Hilbert Spaces

3.1 Inner Product Spaces. Hilbert Spaces

3.1-2 Definition. An inner product space is a vector space X with an inner product defined on X. A Hilbert space is a complete inner product space. An inner product on X is a mapping of X × X into the scalar field k of X such that for all x, y and z in X and any scalar α we have,

(Ip1) \(< x + y, z > = < x, z > + < y, z >.\)

(Ip2) \(< αx, y > = α < x, y >.\)

(Ip3) \(< x, y > = \overline{< y, x >}, \) the complex conjugate of \(< y, x >.\)

(Ip4) \(< x, x > ≥ 0, \) and \(< x, x > = 0 \) if and only if \(x = 0.\)

Notes. (1) An inner product on X defines a norm on X given by \(||x|| = \sqrt{< x, x >}\) and a metric on X given by \(d( x, y ) = || x − y || = \sqrt{< x − y, x − y >}.\)

(2) The inner product spaces are normed spaces, Hilbert spaces are Banach spaces.

(3) For all x, y and z in an inner product space and any scalar α,

a) \(< αx + y, z > = α < x, z > + < y, z >.\)

b) \(< x, αy > = \overline{α} < x, y >.\)

c) \(< x, αy + z > = \overline{α} < x, y > + < x, z >.\)

Then the inner product is sesquilinear (linear in the first factor and conjugate linear in the second).

4) Parallelogram equality. If X is an inner product space, then for all x, y \(\in X, \) \(|| x + y ||^2 + || x − y ||^2 = 2(|| x ||^2 + || y ||^2).\) (How?) (From 4) we have: if parallelogram equality is not satisfied in a normed space X, then X is not an inner product space.)

3.1-3 Definition. An element x in an inner product space X is said to be orthogonal to y \(\in X\) if \(< x, y > = 0\) an we write \(x ⊥ y.\) If A, B are subsets of X, then \(x ⊥ A\) if \(x ⊥ a\) for all \(a ∈ A,\) and \(A ⊥ B\) if \(a ⊥ b\) for all \(a ∈ A\) and all \(b ∈ B.\)

Examples.

3.1-4 Euclidean space \(\mathbb{R}^n\) is a Hilbert spaces with inner product defined by \(< x, y > = \sum_{i=1}^{n} \xi_i \gamma_i.\) This inner product induces the norm \(|| x || = (\sum_{i=1}^{n} \xi_i^2)^{\frac{1}{2}}\) and the metric \(d( x, y ) = (\sum_{i=1}^{n} (\xi_i − \gamma_i)^2)^{\frac{1}{2}}, \) where \(x = (\xi_1, \xi_2, \ldots, \xi_n)\) and \(y = (\gamma_1, \gamma_2, \ldots, \gamma_n).\)
3.1-5 \textbf{Unitary space $C^n$} is a Hilbert spaces with inner product defined by \\
\[< x, y > = \sum_{i=1}^{n} \xi_i \overline{\gamma_i}. \]
This inner product induces the norm $\| x \| = \left( \sum_{i=1}^{n} |\xi_i|^2 \right)^{\frac{1}{2}}$ \\
and the metric $d( x, y ) = \left( \sum_{i=1}^{n} |\xi_i - \gamma_i|^2 \right)^{\frac{1}{2}}$, where $x = (\xi_1, \xi_2, \ldots, \xi_n)$ 
and $y = (\gamma_1, \gamma_2, \ldots, \gamma_n)$.

3.1-6 \textbf{Space $L^2[a, b]$}. The completion of the space of all continuous real-valued functions on $[a, b]$ with norm defined by $\| x \| = \left( \int_{a}^{b} x^2(t) dt \right)^{\frac{1}{2}}$ and the inner product $< x, y > = \int_{a}^{b} x(t) y(t) dt$.

\textbf{Note.} The function $x(t)$ can be extended to be complex valued on $[a, b]$ and the corresponding inner product is $< x, y > = \int_{a}^{b} x(t) \overline{y(t)} \, dt$ and the norm is $\| x \| = \left( \int_{a}^{b} |x(t)|^2 \, dt \right)^{\frac{1}{2}}$. Then $L^2[a, b]$ becomes as the completion of the space of all continuous complex-valued functions on $[a, b]$ corresponding to this norm. In each case real or complex, $L^2[a, b]$ is a Hilbert space.

3.1-7 \textbf{Hilbert sequence space $\ell^2$}. This space is a Hilbert spaces with inner product defined by $< x, y > = \sum_{i=1}^{\infty} \xi_i \overline{\gamma_i}$. This inner product induces the norm $\| x \| = \left( \sum_{i=1}^{\infty} |\xi_i|^2 \right)^{\frac{1}{2}}$, where $x = (\xi_i)$ and $y = (\gamma_i)$.

3.1-8 \textbf{Space $\ell^p$}. This space with $p \neq 2$ is not an inner product space, hence is not a Hilbert spaces. 
\textbf{Proof.} Consider $x = (1, 1, 0, 0, 0, \ldots)$ and $y = (1, -1, 0, 0, 0, \ldots)$.
Then $x, y \in \ell^p$ and $\| x + y \|^2 + \| x - y \|^2 = \| (2, 0, 0, 0, 0, \ldots)\|^2 + \| (0, 2, 0, 0, 0, \ldots)\|^2 = 8$. But $2(\| x \|^2 + \| y \|^2) = 2(\| (1, 1, 0, 0, 0, 0, \ldots)\|^2 + \| (1, -1, 0, 0, 0, \ldots)\|^2) = 4(2^\frac{2p}{p}) = 4(2^2) \neq 8$ when $p \neq 2$. Therefore, $\| x + y \|^2 + \| x - y \|^2 \neq 2(\| x \|^2 + \| y \|^2)$. 
Hence by note 4) above, the space $\ell^p$ with $p \neq 2$ is not an inner product space, hence is not a Hilbert spaces.
3.1-9 **Space C[a, b]**. This space is not an inner product space, hence is not a Hilbert spaces.

**Proof.** Consider \( x(t) = 1 \) and \( y(t) = \frac{t-a}{b-a} \) on the closed interval \( J = [a, b] \). Then \( x, y \in C[a, b] \) and \( \| x \| = 1 \), \( \| y \| = \max_{t \in J} \left| \frac{t-a}{b-a} \right| = 1 \), \( \| x + y \| = \max_{t \in J} \left| 1 + \frac{t-a}{b-a} \right| = 2 \), and \( \| x - y \| = \max_{t \in J} \left| 1 - \frac{t-a}{b-a} \right| = \max_{t \in J} \left| \frac{b-t}{b-a} \right| = 1 \).

Hence \( 2(\| x \|^2 + \| y \|^2) = 4 \), but \( \| x + y \|^2 + \| x - y \|^2 = 5 \). Then, \( \| x + y \|^2 + \| x - y \|^2 \neq 2(\| x \|^2 + \| y \|^2) \). Hence by note 4) above, the space \( C[a, b] \) is not an inner product space, hence is not a Hilbert spaces.

3.1-10 **Remarks.** (1) For any \( x, y \) in a real inner product space, 
\[ < x, y > = \frac{1}{4} (\| x + y \|^2 - \| x - y \|^2). \]
(2) For any \( x, y \) in a complex inner product space,
a) \( \text{Re} < x, y > = \frac{1}{4} (\| x + y \|^2 - \| x - y \|^2) \).
a) \( \text{Im} < x, y > = \frac{1}{4} (\| x + iy \|^2 - \| x - iy \|^2) \).

**Proof.** Left to the reader.

\[ H. W. 1-9, 11. H.W.* 5, 6, 8. \]
3.2 Further Properties of Inner Product Spaces.

3.2-1 Lemma (Schwarz inequality, triangle inequality). If \( x \) and \( y \) are elements in an inner product space, then

a) \( | < x, y > | \leq \| x \| \| y \| \) (Schwarz inequality)

where the equality sign holds if and only if \( \{ x, y \} \) is a linearly dependent set.

b) \( \| x + y \| \leq \| x \| + \| y \| \) (Triangle inequality)

where the equality sign holds if and only if \( y = 0 \) or \( x = cy \) for some \( c \geq 0 \).

Proof. a) If \( y = 0 \), then \( < x, y > = 0 \) and the result is trivial. Let \( y \neq 0 \) and \( \alpha \) be a scalar, we have \( 0 \leq \| x - \alpha y \|^2 = < x - \alpha y, x - \alpha y > = < x, x > - \alpha < x, y > - \alpha < y, x > + \alpha^2 < y, y > \)

\[ \frac{|< x, y >|^2}{\| y \|^2} \]

we have, \( 0 \leq < x, x > - \frac{< y, x >}{< y, y >} < x, y > = \| x \|^2 - < y, y > \)

Hence \( |< x, y >|^2 \leq \| x \|^2 \| y \|^2 \), and then the result follows.

If \( \{ x, y \} \) is linearly dependent, then there is a scalar \( \alpha \) such that \( x = \alpha y \) and \( |< x, y >| = |< \alpha y, y >| = |\alpha| \| y \|^2 = \| \alpha y \| \| y \| = \| x \| \| y \| \).

Conversely, if equality holds, then by (1) either \( y = 0 \) or \( x - \alpha y = 0 \).

Hence \( \{ x, y \} \) is linearly dependent.

b) By using Schwarz inequality we have, \( \| x + y \|^2 = < x + y, x + y > = \| x \|^2 + < x, y > + < y, x > + \| y \|^2 \leq \| x \|^2 + |< x, y >| + |< y, x >| + \| y \|^2 = (\| x \| + \| y \|)^2 \). Hence \( \| x + y \| \leq \| x \| + \| y \| \).

If equality holds, then \( < x, y > + < y, x > = 2\| x \| \| y \| \). However, \( < x, y > + < y, x > = 2 \text{Re}< x, y > \). Then \( \text{Re}< x, y > = \| x \| \| y \| \geq |< x, y >| \). Since the real part can’t exceed the modulus, then \( \text{Re}< x, y > = \| x \| \| y \| \geq |< x, y >| \).

Conversely, if \( y = 0 \) or \( x = cy \), then it is an easy calculation for getting \( \| x + y \| = \| x \| + \| y \| \).

3.2-2 Lemma (Continuity of inner product). If in an inner product space, \( x_n \rightarrow x \) and \( y_n \rightarrow y \) then \( < x_n, y_n > \rightarrow < x, y > \).

Proof. Left to the reader.

3.3 Orthogonal Complements and Direct Sums

**Remark 1.** A subset M of a vector space X is said to be convex if for all \( x, y \in M \) and all \( \alpha \in [0, 1) \), \( \alpha x + (1-\alpha)y \in M \). Hence, every subspace of X is convex and the intersection of convex sets is convex.

3.3-1 **Theorem.** Let X be an inner product space and M a non empty convex subset which is complete. Then for every \( x \in X \) there exists a unique \( y \in M \) such that \( \delta = \inf_{y \in M} || x - y || = || x - y ||. \)

**Proof.**

a) **Existence.** Since \( \delta = \inf_{y \in M} || x - y || \), then there is a sequence \( (y_n) \) in M such that \( \delta_n \to \delta \), where \( \delta_n = || x - y_n || \). We show that \( (y_n) \) is Cauchy. Let \( y_n - x = v_n \). Then \( \delta_n = || v_n || \) and \( || v_n + v_m || = || y_n + y_m - 2x || = 2 || \frac{1}{2} (y_n + y_m) - x || \geq 2 \delta \), where \( \frac{1}{2} (y_n + y_m) \in M \) because M is convex. Furthermore, \( y_n - y_m = v_n - v_m. \) By using parallelogram equality we have \( || y_n - y_m ||^2 = || v_n - v_m ||^2 = -|| v_n + v_m ||^2 + 2(|| v_n ||^2 + || v_m ||^2) \leq - (2 \delta)^2 + 2 (\delta_n^2 + \delta_m^2) \). As \( n, m \to \infty \), \( || y_n - y_m ||^2 \to 0 \). Hence for every \( \epsilon > 0 \), there is \( k \) sufficiently large such that \( || y_n - y_m || < \epsilon \) for all \( n, m > k \). Hence \( (y_n) \) is a Cauchy sequence in M. But M is complete, then \( (y_n) \) converges, say \( y_n \to y \in M \). Then \( || x - y || \geq \delta \). However, \( || x - y || \leq || x - y_n || + || y_n - y || = \delta_n + || y_n - y || \to \delta. \) Then \( || x - y || \leq \delta. \) Therefore, \( || x - y || = \delta \).

b) **Uniqueness.** Suppose that there are \( y, y_0 \in M \) with \( || x - y || = \delta \) and \( || x - y_0 || = \delta \). By parallelogram equality, \( || y - y_0 ||^2 = || (y - x) - (y_0 - x) ||^2 + 2 || y_0 - x ||^2 - || (y - x) + (y_0 - x) ||^2 = 2 \delta^2 + 2 \delta^2 - 4 || \frac{1}{2} (y + y_0) - x ||^2 \)

\[\text{(1)}\]

Since M is convex, then \( \frac{1}{2} (y + y_0) \in M \) which implies that \( || \frac{1}{2} (y + y_0) - x || \geq \delta \). By this and \( (1) \), \( || y - y_0 ||^2 \leq 2 \delta^2 + 2 \delta^2 - 4 \delta^2 = 0. \) Hence \( || y - y_0 || = 0 \), that is \( y = y_0 \). This proves the uniqueness.

3.3-2 **Lemma.** In Theorem 3.3-1, let M be a complete subspace Y and \( x \in X \) fixed. Then \( z = x - y \) is orthogonal to Y.

**Proof.** Suppose that \( z = x - y \) is not orthogonal to Y. Then there is \( y_1 \in Y \) such that \( < z, y_1 > = \beta \neq 0 \). Clearly \( y_1 \neq 0 \), otherwise \( < z, y_1 > = 0 \).

Furthermore, for any scalar \( \alpha \), \( || z - \alpha y_1 ||^2 = < z, z > - \alpha < z, y_1 > - \alpha [ < y_1, z > - \bar{\alpha} < y_1, y_1 > ] = < z, z > - \alpha \beta - \alpha [ \bar{\beta} - \bar{\alpha} < y_1, y_1 > ] \)

Choose \( \bar{\alpha} = \frac{\beta}{< y_1, y_1 >} \) to get \( || z - \alpha y_1 ||^2 = || z ||^2 - \frac{|| \beta ||^2}{|| y_1 ||^2} \). However, \( || z || = || x - y || = \delta \) and \( \beta \neq 0 \), then \( || z - \alpha y_1 ||^2 < || z ||^2 = \delta^2 \) \[\text{.........(2)}\]

Since Y is a subspace, then \( (y - \alpha y_1) \in Y \) and so \( || z - \alpha y_1 || = || x - (y - \alpha y_1) || \geq \delta. \) This is a contradiction with \( (2) \). Therefore, \( z = x - y \) is orthogonal to Y.
3.3-3 Definition. A vector space \( X \) is said to be the direct sum of two subspaces \( Y \) and \( Z \) of \( X \), written \( X = Y \oplus Z \), if each \( x \in X \) has a unique representation \( x = y + z \), \( y \in Y \) and \( z \in Z \). Then \( Z \) is called the algebraic complement of \( Y \) in \( X \).

3.3-4 Theorem (Projection Theorem). Let \( Y \) be any closed subspace of a Hilbert space \( H \). Then \( H = Y \oplus Z \), where \( Z = Y^\perp = \{ z \in H : z \perp Y \} \).

Proof. Since \( Y \) is a closed subspace of a Hilbert space \( H \), then \( Y \) is complete. Since \( Y \) is convex then Theorem 3.3-1 and lemma 3.3-2 imply that for any \( x \in X \) there is \( y \in Y \) such that \( z = x - y \perp Y \). Then \( x = y + z \), \( y \in Y \) and \( z \in Y^\perp = Z \) …………………………………………………..(2)

Suppose that there are \( y_1, y_2 \in Y \) and \( z_1, z_2 \in Z \) with \( x = y_1 + z_1 = y_2 + z_2 \). Then \( y_1 - y_2 = z_2 - z_1 \). However, \( y_1 - y_2 \in Y \) and \( z_2 - z_1 \in Z \). Then \( y_1 - y_2 \in Y \cap Y^\perp = \{0\} \). Hence \( y_1 = y_2 \) and \( z_2 = z_1 \). This proves the uniqueness.

Remark 2. From (2) above we can define a mapping (called the orthogonal projection of \( H \) onto \( Y \)) \( P : H \to Y \) defined by \( P_x = y \), where \( x = y + z \), \( y \in Y \) and \( z \in Z \). This mapping has the following properties:

(1) \( P \) is linear and bounded.
(2) \( P \) maps \( H \) onto \( Y \).
(3) \( P(Y) = Y \).
(4) \( P(Y^\perp) = \{0\} \).
(5) \( P^2 = P \) and the restriction of \( P \) on \( Y \) is the identity operator on \( Y \).

Proof. Left to the reader.

3.3-5 Lemma. The orthogonal complement \( Y^\perp \) of a closed subspace \( Y \) of a Hilbert space \( H \) is the null space \( N(P) \) of the orthogonal projection \( P \) of \( H \) onto \( Y \).

Proof. Left to the reader.

Remark 3. If \( M \) is a nonempty subset of an inner product space \( X \), then \( M^\perp \) is a closed vector subspace of \( X \) and \( M \) is a subset of \( M^\perp \perp \).

Proof. Left to the reader.

3.3-6 Lemma. If \( Y \) is a closed subspace of a Hilbert space \( H \), then \( Y = Y^\perp \perp \).

Proof. By Remark 3, \( Y \subseteq Y^\perp \perp \). Conversely, let \( x \in Y^\perp \perp \). By Theorem 3.3-4 there is a unique \( y \in Y \) such that \( x = y + z \). However, \( Y \subseteq Y^\perp \perp \) and \( Y^\perp \perp \) is a vector space, then \( z = x - y \in Y^\perp \perp \), that is \( z \perp Y \). Since \( Y \) is a closed subspace of a Hilbert space then it is complete and so by Lemma 3.3-2 \( z \in Y^\perp \). Hence \( z \perp z \) and so \( z = 0 \). So that \( x = y \) and \( x \in Y \).

Therefore, \( Y^\perp \perp \subseteq Y \). Hence \( Y^\perp \perp = Y \bullet \).

Remark 4. If \( Y \) is a closed subspace of a Hilbert space \( H \) and \( Z = Y^\perp \) then \( Z^\perp = Y^\perp \perp = Y \), \( H = Z + Z^\perp \) and \( P_z x = z \) defines a projection \( P_z : H \to Z \).

Proof. Left to the reader.
**3.3-7 Lemma.** For any non empty subset M of a Hilbert space H, the span of M is dense if and only if $M^\perp = \{0\}$.

**Proof.** Let $x \in M^\perp$ and assume that $V = \text{span} \ M$ to be dense in H. Then $x \in \overline{V} = H$, then there is a sequence $(x_n)$ of elements in V such that $x_n \to x$. Since $x \in M^\perp$ then for all $m \in M$, $\langle x, m \rangle = 0$ which implies that $\langle x, v \rangle = 0$ for all $v \in V$, in particular $\langle x, x_n \rangle = 0$ for all $n \in \mathbb{N}$. By the continuity of the inner product, $\langle x, x_n \rangle \to \langle x, x \rangle$. Hence $\langle x, x \rangle = 0$ and so $x = 0$. Therefore, $M^\perp = \{0\}$.

Conversely, suppose that $M^\perp = \{0\}$. Since $M \subseteq \overline{V}$, then $\overline{V}^\perp \subseteq M^\perp = \{0\}$. Hence $\overline{V}^\perp = \{0\}$. By Theorem 3.3-4, $H = \overline{V} \oplus \overline{V}^\perp = \overline{V} \oplus \{0\} = \overline{V}$.

**H. W. 6-10. H.W.* 8**, Remarks 2, 3 and Lemma 3.3-5.
### 3.4 Orthonormal Sets and Sequences

**3.4-1 Definition.** An orthogonal set \( M \) in an inner product space \( X \) is a subset \( M \) of \( X \) whose elements are pairwise orthogonal. An orthonormal subset \( M \) of \( X \) is an orthogonal set in \( X \) whose elements have norm 1. That is, for all \( x, y \in M \),

\[
\langle x, y \rangle = \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y \end{cases}.
\]

**Notes.**

a) If an orthogonal or an orthonormal set \( M \) is countable, we can arrange it in a sequence \( (x_n) \) and call it an orthogonal or an orthonormal sequence, respectively.

b) A family \( (x_\alpha) \), \( \alpha \in I \) is called orthogonal if \( x_\alpha \perp x_\beta \) for all \( \alpha, \beta \in I \) and \( \alpha \neq \beta \). The family is orthonormal if it is orthogonal and all \( x_\alpha \) have norm 1, so that for all \( \alpha, \beta \in I \),

\[
\langle x_\alpha, x_\beta \rangle = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ 1 & \text{if } \alpha = \beta \end{cases}.
\]

**Remark 1.** For an orthogonal elements \( x \) and \( y \) we have,

\[
\| x + y \|_2^2 = \| x \|_2^2 + \| y \|_2^2 \quad \text{(Pythagorean Relation)}.
\]

In general, if \( \{x_1, x_2, \ldots, x_n\} \) is an orthogonal set then

\[
\| x_1 + x_2 + \ldots + x_n \|_2^2 = \| x_1 \|_2^2 + \| x_2 \|_2^2 + \ldots + \| x_n \|_2^2.
\]

**Proof.** Left to the reader.

**3.4-2 Lemma.** An orthonormal set is linearly independent.

**Proof.** Left to the reader.

#### Examples:

**3.4-3 Euclidean space \( \mathbb{R}^n \).** The standard basis of \( \mathbb{R}^n \) forms an orthonormal set. (How?)

**3.4-4 Hilbert sequence space \( \ell^2 \).** The Schauder basis \( (e_n) \) of \( \ell^2 \) forms an orthonormal sequence in \( \ell^2 \). (How?)

**3.4-5 Continuous functions.** Let \( X \) be the inner product space of all continuous real-valued functions on \([0, 2\pi]\) with \( \langle x, y \rangle = \int_0^{2\pi} x(t)y(t)\,dt \). Then \( (\cos n t), \) \( n = 0, 1, 2, \ldots \) and \( (\sin n t), n \in \mathbb{N} \) are orthogonal sequences. Moreover,

\[
\left( \frac{1}{\sqrt{2\pi}}, \frac{\cos t}{\sqrt{\pi}}, \frac{\cos 2t}{\sqrt{\pi}}, \ldots \right) \text{ and } \left( \frac{\sin nt}{\sqrt{\pi}}, n \in \mathbb{N} \right)
\]

are orthonormal sequences.

**Proof.** Left to the reader.
3.4-6 Theorem (Bessel's Inequality). Let \((e_n)\) be an orthonormal sequence in an inner product space \(X\). Then for every \(x \in X\), \(\sum_{k=1}^{\infty} |<x, e_k>|^2 \leq \|x\|^2\).

Proof. Consider the finite subset \(\{e_1, e_2, \ldots, e_N\}\) from \((e_n)\). Then

\[
0 \leq \|x - \sum_{i=1}^{N} <x, e_i>e_i\|^2 = <x - \sum_{i=1}^{N} <x, e_i>e_i, x - \sum_{k=1}^{N} <x, e_k>e_k>
\]

\[
= \|x\|^2 - \sum_{k=1}^{N} <x, e_k>e_k - \sum_{i=1}^{N} <x, e_i>e_i + \sum_{i=1}^{N} <x, e_i>e_i + \sum_{k=1}^{N} <x, e_k>e_k
\]

\[
= \|x\|^2 - \sum_{k=1}^{N} |<x, e_k>|^2 + \sum_{k=1}^{N} |<x, e_k>|^2 = \|x\|^2 - \sum_{k=1}^{N} |<x, e_k>|^2.
\]

Hence \(\sum_{k=1}^{\infty} |<x, e_k>|^2 \leq \|x\|^2\).

By letting \(N \to \infty\) we get, \(\sum_{k=1}^{\infty} |<x, e_k>|^2 \leq \|x\|^2\).

Notes. a) The inner product \(<x, e_k>\) above is called the Fourier coefficients of \(x\) with respect to the orthonormal sequence \((e_k)\).

b) If \(\text{dim}(X)\) is finite, then any orthonormal set in \(X\) must be finite because it is linearly independent.

Gram-Schmidt process for orthonormalizing a linearly independent sequence \((x_j)\) in an inner product space \(X\).

1st step. The first element of \((e_k)\) is \(e_1 = \frac{x_1}{\|x_1\|}\).

2nd step. Let \(v_2 = x_2 - <x_2, e_1>e_1\). Since \((x_j)\) is linearly independent, then \(v_2 \neq 0\). Also \(v_2 \perp e_1\), where \(<v_2, e_1> = <x_2, e_1> - <x_2, e_1><e_1, e_1> = 0\). So we can take \(e_2 = \frac{v_2}{\|v_2\|}\).

3rd step. Let \(v_3 = x_3 - <x_3, e_1>e_1 - <x_3, e_2>e_2\). As above \(v_3 \neq 0\), \(v_3 \perp e_1\) and \(v_3 \perp e_2\) (how?). So we can take \(e_3 = \frac{v_3}{\|v_3\|}\).

nth step. Let \(v_n = x_n - \sum_{k=1}^{n-1} <x_n, e_k>e_k\). As above \(v_n \neq 0\) and \(v_n \perp e_i\) for all \(i = 1, 2, \ldots, n-1\). So we can take \(e_n = \frac{v_n}{\|v_n\|}\). Therefore, we have \((e_k)\) as an orthonormal sequence.

3.5 Series Related to Orthonormal Sets and Sequences

3.5-1 Theorem. Let \( (e_k) \) be an orthonormal sequence in a Hilbert space \( H \). Then for any scalars \( \alpha_1, \alpha_2, \alpha_3, \ldots \)

a) \( \sum_{k=1}^{\infty} \alpha_k e_k \) converges (in the norm of \( H \)) if and only if \( \sum_{k=1}^{\infty} |\alpha_k|^2 \) converges.

b) If \( \sum_{k=1}^{\infty} \alpha_k e_k \) converges then \( \alpha_k \)'s are the Fourier coefficients \( <x,e_k>\),

where \( x \) denotes the sum \( x = \sum_{k=1}^{\infty} \alpha_k e_k \) and so \( \sum_{k=1}^{\infty} \alpha_k e_k = x = \sum_{k=1}^{\infty} <x,e_k>e_k \).

c) For any \( x \in H \), \( \sum_{k=1}^{\infty} <x,e_k>e_k \) converges.

Proof. a) Let \( S_n = \alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n \) and \( \sigma_n = |\alpha_1|^2 + |\alpha_2|^2 + \ldots + |\alpha_n|^2 \). Since \( (e_k) \) is orthonormal then for \( n > m \), \( \| S_n - S_m \|^2 = \| \alpha_{m+1} e_{m+1} + \alpha_{m+2} e_{m+2} + \ldots + \alpha_n e_n \|^2 = <\alpha_{m+1} e_{m+1} + \alpha_{m+2} e_{m+2} + \ldots + \alpha_n e_n, \alpha_{m+1} e_{m+1} + \alpha_{m+2} e_{m+2} + \ldots + \alpha_n e_n> = |\alpha_{m+1}|^2 \| e_{m+1} \|^2 + |\alpha_{m+2}|^2 \| e_{m+2} \|^2 + \ldots + |\alpha_n|^2 \| e_n \|^2 = \sigma_n - \sigma_m \). Therefore, \( (S_n) \) is Cauchy in \( H \) if and only if \( (\sigma_n) \) is Cauchy in \( R \). However, both \( H \) and \( R \) are complete then, \( (S_n) \) converges if and only if \( (\sigma_n) \) converges. Hence, \( \sum_{k=1}^{\infty} \alpha_k e_k \) converges (in the norm of \( H \)) if and only if \( \sum_{k=1}^{\infty} |\alpha_k|^2 \) converges in \( R \).

b) Let \( S_n \rightarrow x; \) that is \( x = \sum_{k=1}^{\infty} \alpha_k e_k \). Let \( n \) be fixed, then for a fixed \( j = 1, 2, \ldots \), \( k \) and \( k \leq n \) we have, \( <S_n, e_j> = <\alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n, e_j> = <\alpha_j e_j, e_j> = \alpha_j \). By the continuity of the inner product we have, \( \alpha_j = <S_n, e_j> \rightarrow <x, e_j> \) as \( n \rightarrow \infty \). Then we can take \( k (\leq n) \) as large as we please. Hence \( \alpha_j = \lim_{n \rightarrow \infty} \alpha_j = <x, e_j> \) for all \( j = 1, 2, 3, \ldots \). Therefore, \( x = \sum_{k=1}^{\infty} \alpha_k e_k = \sum_{k=1}^{\infty} <x, e_k>e_k \).

b) Let \( x \) be any element in \( H \). By Bessel’s inequality \( \sum_{k=1}^{\infty} |<x, e_k>|^2 \leq \| x \|^2 \). Hence \( \sum_{k=1}^{\infty} |<x, e_k>|^2 \) converges and so by a) \( \sum_{k=1}^{\infty} |<x, e_k>|^2 \) converges.

3.5-2 Lemma. Any \( x \) in an inner product space \( X \) can have at most accountably many non zero Fourier coefficients \( <x,e_k> \) with respect to an orthonormal family \( (e_k)_{k \in I} \), in \( X \).

Proof. Left to the reader.

Remark 1. a) The Bessel inequality holds in the case of \( (e_k)_{k \in I} \) as an orthonormal family, \( \sum_k |<x, e_k>|^2 \leq \| x \|^2 \). If the equality holds we say that it is the Parseval relation.

3.6 Total Orthonormal Sets and Sequences

3.6-1 Definition. A total set (fundamental set) in a normed space X is a subset M of X whose span is dense in X. A total orthonormal set in X is an orthonormal set which is total.

Remark 1. a) In every Hilbert space \( H \neq \{0\} \) there exists a total orthonormal set.
b) All total orthonormal sets in a given Hilbert space \( H \neq \{0\} \) have the same cardinality, where the cardinality of the total orthonormal set in a Hilbert space \( H \neq \{0\} \) is called the Hilbert dimension or orthogonal dimension of H. If \( H = \{0\} \), the Hilbert dimension is defined to be zero.

3.6-2 Theorem. Let M be a subset of an inner product space X. Then
a) If M is total in X then \( x \in X \) and \( x \perp M \) implies \( x = 0 \).
b) If X is complete and if the condition ( \( x \in X \) and \( x \perp M \) implies \( x = 0 \) ) is satisfied then M is total in X.

Proof. a) Let H be the completion of X. Then X can be regarded as a subspace of H which is dense in H. However, M is total in X, then spanM is dense in X, then it is dense in H. Hence by Lemma3.3-7, \( M^\perp = \{0\} \). Therefore, \( x \in X \) and \( x \perp M \) implies \( x = 0 \).
b) Use Lemma3.3-7 and the definition of total set to get the result.

3.6-3 Theorem. An orthonormal set M in a Hilbert space H is total in H if and only if \( \sum_{k} |<x,e_k>|^2 = \|x\|^2 \) for all \( x \in H \).

Proof. If M is not total then by Theorem3.6-2 b), there is \( x \in H \), \( x \neq 0 \) and \( x \perp M \). Then \( <x,e_k> = 0 \) for all \( e_k \in M \). Hence \( \sum_{k} |<x,e_k>|^2 = 0 \neq \|x\|^2 \).

Therefore, if \( \sum_{k} |<x,e_k>|^2 = \|x\|^2 \) for all \( x \in H \), then M is total in H. Conversely, suppose that M is total in H. Let \( x \) be any element in H. and arrange all its nonzero Fourier coefficients in a sequence \( <x,e_1>, <x,e_2>, \ldots \) or written in some definite order if there are only finitely many of them.

Define \( y \) by \( y = \sum_{k} <x,e_k>e_k \) ……………………………………… (1)

Since M is orthonormal then for every \( e_j \) occurring in (1) we have, \( <x,y,e_j> = <x,e_j> - <y,e_j> = <x,e_j> - \sum_{k} <x,e_k> <e_k,e_j> = <x,e_j> - <y,e_j> = 0 \). But for all \( v \in M \) not contained in (1) we have \( <x,v> = 0 \).

So that \( <x-y,v> = <x,v> - \sum_{k} <x,e_k> <e_k,v> = 0 \). Hence \( x-y \perp M \).

However, M is total in H then by Lemma3.3-7 \( M^\perp = \{0\} \) and so \( x-y = 0 \), that is \( x = y \). Therefore, \( \|x\|^2 = <y,y> = <\sum_{k} <x,e_k>e_k, \sum_{m} <x,e_m>e_m> = \sum_{k} \sum_{m} <x,e_k><x,e_m><e_k,e_m> = \sum_{k} |<x,e_k>|^2 \).
3.6-4 **Theorem.** Let H be a Hilbert space. Then

a) If H is separable, then every orthonormal set in H is countable.

b) If H contains an orthonormal sequence which is total in H, then H is separable.

**Proof.** a) Let H be separable, B any dense set in H and M any orthonormal set. Since M is orthonormal then for any $x, y \in M$ with $x \neq y$ we have, $\| x - y \|^2 = \langle x - y, x - y \rangle = \langle x, x \rangle + \langle y, y \rangle = 2$. Hence spherical neighborhoods $N_x$ of x and $N_y$ of y of radius $\frac{\sqrt{2}}{2}$ are disjoint (why?). Since B is dense in H then for any $x \in M$, $N_x \cap B \neq \emptyset$. Hence there is $a \in N_x \cap B$ and $b \in N_y \cap B$. Therefore, $a \neq b$. If M is uncountable then we have unaccountably many pair wise disjoint spherical neighborhoods, so that B would be uncountable. Since B was any dense set this makes that H would not contain a countable dense set which contradicts the separability of H. Therefore, M must be countable.

b) Let $(e_m)$ be a total orthonormal sequence in H and $A = \{ \sum_{k=1}^{n} \gamma^{(n)}_{k} e_k : \gamma^{(n)}_{k} = a^{(n)}_{k} + i b^{(n)}_{k}, a^{(n)}_{k}, b^{(n)}_{k} \in \mathbb{Q}, n \in \mathbb{N} \}$. ($b^{(n)}_{k} = 0$ when H is real). A is countable (how?). We show that A is dense in H. Let x be any fixed element in H. Since $(e_m)$ is total in H, then $\overline{\text{span}(e_m)} = H$. Then for every $\varepsilon > 0$ there is $w \in \text{span}(e_m)$ such that $\| x - w \| < \frac{\varepsilon}{2}$. Hence $w \in Y_n = \text{span}\{ e_1, e_2, \ldots, e_n \}$ for some n. By Lemma 3.3-2 there is $y \in Y_n$ such that $x - y \perp Y_n$ and $\| x - y \| \leq \| x - w \| < \frac{\varepsilon}{2}$. By (8a&b) in 3.4[ see the text book] y can be written as $y = \sum_{k=1}^{n} \langle x, e_k \rangle e_k$. Then $\| x - \sum_{k=1}^{n} \langle x, e_k \rangle e_k \| < \frac{\varepsilon}{2}$. Since Q is dense in R then for any $\langle x, e_k \rangle$ there is $\gamma^{(n)}_{k} = a^{(n)}_{k} + i b^{(n)}_{k}, a^{(n)}_{k}, b^{(n)}_{k} \in \mathbb{Q}$ such that $\| \sum_{k=1}^{n} [\langle x, e_k \rangle - \gamma^{(n)}_{k}] e_k \| < \frac{\varepsilon}{2}$. Hence there is $v = \sum_{k=1}^{n} \gamma^{(n)}_{k} e_k \in A$ that satisfies $\| x - v \| = \| x - \sum_{k=1}^{n} \gamma^{(n)}_{k} e_k \| \leq \| x - \sum_{k=1}^{n} \langle x, e_k \rangle e_k \| + \| \sum_{k=1}^{n} \langle x, e_k \rangle e_k - \sum_{k=1}^{n} \gamma^{(n)}_{k} e_k \| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Hence $v \in B(x; \varepsilon) \cap A$. Thus A is dense in H.
3.6-5 Theorem. Two Hilbert spaces $H$ and $\tilde{H}$ both real or both complex are isomorphic if and only if they have the same Hilbert dimension.

Proof. Suppose that $H$ is isomorphic with $\tilde{H}$, then there is a bijective linear mapping $T : H \to \tilde{H}$ that satisfies $<Tx, Ty> = <x, y>$ for all $x, y \in H$. Hence orthonormal elements in $H$ have orthonormal images under $T$. However, $T$ is bijective then $T$ maps every total orthonormal set in $H$ onto a total orthonormal set in $\tilde{H}$ (how?) Therefore, $H$ and $\tilde{H}$ have the same Hilbert dimension.

Conversely, suppose that $H$ and $\tilde{H}$ have the same Hilbert dimension. The case that $H = \{0\}$ and $\tilde{H} = \{0\}$ is trivial. Let $H \neq \{0\}$, then $\tilde{H} \neq \{0\}$ and any total, orthonormal sets $M$ in $H$ and $\tilde{M}$ in $\tilde{H}$ have the same cardinality. So we can index them by the same index set $\{k\}$ and write $M = (e_k)$ and $\tilde{M} = (\tilde{e}_k)$. Now define $T : H \to \tilde{H}$ by $Tx = \sum_{k} <x, e_k> \tilde{e}_k$. This is well defined, because for all $x \in H$ we have $x = \sum_{k} <x, e_k> e_k$ and by Bessel’s inequality

$$\sum_{k} |<x, e_k>|^2$$

converges. Then by Theorem 3.5-2 $\sum_{k} <x, e_k> \tilde{e}_k$ converges so that $Tx \in \tilde{H}$. Let $x = \sum_{k} <x, e_k> e_k$ and $y = \sum_{k} <y, e_k> e_k$ be any elements in $H$ and $\alpha$ any scalar, $T(\alpha x + y) = \sum_{k} <\alpha x + y, e_k> \tilde{e}_k = \alpha \sum_{k} <x, e_k> \tilde{e}_k + \sum_{k} <y, e_k> \tilde{e}_k$

$= \alpha Tx + Ty$. Hence $T$ is linear. Since $(\tilde{e}_k)$ is orthonormal then $<Tx, Tx> = \sum_{k} <x, e_k> e_k \sum_{k} <x, e_k> e_k = \sum_{k} <x, e_k> <e_k, e_m> <\tilde{e}_k, \tilde{e}_m> = \sum_{k} |<x, e_k>|^2 = \|x\|^2$. For any $x, y \in H$ (if $H$ is real), $<Tx, Ty> = \frac{1}{2} (\|Tx + Ty\|^2 - \|Tx - Ty\|^2) = \frac{1}{2} (\|T(x + y)\|^2 - \|T(x - y)\|^2) = \frac{1}{2} (\|x + y\|^2 - \|x - y\|^2) = <x, y>$. Similarly for the complex case. Hence $T$ preserves the inner product which implies that $T$ is 1-1. Given any $\tilde{x} = \sum_{k} \alpha_k \tilde{e}_k \in \tilde{H}$.

By Bessel’s inequality $\sum_{k} |\alpha_k|^2$ converges (how?) and so $\sum_{k} \alpha_k e_k$ is a finite sum or a series which converges to $x \in H$ by Theorem 3.5-2, and $\alpha_k = <x, e_k>$ by the same theorem. Hence $\tilde{x} = \sum_{k} <x, e_k> \tilde{e}_k = Tx$. Thus $T$ is onto.

Therefore, $T$ is an isomorphism, so $H$ and $\tilde{H}$ are isomorphic.
3.8 Representation of Functionals on Hilbert Spaces

3.8-1 Riesz’s Theorem. Every bounded linear functional \( f \) on a Hilbert space \( H \) can be represented in terms of inner product, namely, \( f(x) = \langle x, z \rangle \), where \( z \) depends on \( f \) and is uniquely determined by \( f \) with norm \( \| z \| = \| f \| \).

**Proof.** The case \( f = 0 \) is a trivial case (why?). Let \( f \neq 0 \). Then \( N(f) \neq H \). However, \( N(f) \) is a closed subspace of \( H \), then by Theorem 3.3-4 \( H = N(f) \oplus N(f)^\perp \). Hence \( N(f)^\perp \neq \{0\} \). Let \( z_0 \neq 0 \) and \( z_0 \in N(f)^\perp \). Set \( v = \frac{f(z_0)}{\langle z_0, z_0 \rangle} \). Then \( f(v) = \frac{f(z_0)}{\langle z_0, z_0 \rangle} \langle z_0, v \rangle = \frac{f(z_0)}{\langle z_0, z_0 \rangle} \langle z_0, v \rangle = 0 \), hence \( v \in N(f) \). Hence \( \langle v, z_0 \rangle = 0 \). To prove the uniqueness of \( z \). Suppose that there are \( z_1 \) and \( z_2 \) such that \( f(x) = \langle x, z_1 \rangle = \langle x, z_2 \rangle \) for all \( x \in H \). Then \( \langle x, z_1 - z_2 \rangle = 0 \) for all \( x \in H \), and for \( x = z_1 - z_2 \) we have \( \langle z_1 - z_2, z_1 - z_2 \rangle = 0 \). Hence \( z_1 = z_2 \) and the uniqueness is proved.

3.8-2 Lemma. If \( \langle v_1, w \rangle = \langle v_2, w \rangle \) for all \( w \) in an inner product space \( X \), then \( v_1 = v_2 \). In particular \( \langle v_1, w \rangle = 0 \) for all \( w \in X \) implies that \( v_1 = 0 \).

**Proof.** Left to the reader.

3.8-3 Definition. Let \( X \) and \( Y \) be vector spaces over a field \( k \). A sesquilinear form (sesquilinear functional) \( h \) on \( X \times Y \) is a mapping \( h : X \times Y \rightarrow k \) such that for all \( x, x_1, x_2 \in X \), \( y, y_1, y_2 \in Y \) and all scalars \( \alpha, \beta \) it satisfies,

a) \( h(x_1 + x_2, y) = h(x_1, y) + h(x_2, y) \)

b) \( h(x, y_1 + y_2) = h(x, y_1) + h(x, y_2) \)

c) \( h(\alpha x, y) = \alpha h(x, y) \)

d) \( h(x, \beta y) = \bar{\beta} h(x, y) \)

**Notes.**

1) If \( k = R \), then \( h \) above is bilinear.

2) If \( X \) and \( Y \) are normed spaces and there is \( c > 0 \) such that for all \( x \in X \), and all \( y \in Y \), \( |h(x, y)| \leq c \| x \| \| y \| \), then \( h \) is bounded, and the number \( \| h \| = \sup \{ \frac{|h(x, y)|}{\| x \| \| y \|} : x \in X \setminus \{0\} \text{ and } y \in Y \setminus \{0\} \} = \sup \{ h(x, y) : \| x \| = \| y \| = 1 \} \) is called the norm of \( h \), then \( |h(x, y)| \leq \| h \| \| x \| \| y \| \) for all \( x, y \).

3) The inner product is sesquilinear and bounded (how?)
3.8-4 Theorem (Riesz's Representation). Let $H_1$ and $H_2$ be Hilbert spaces over a field $k$ and $h : H_1 \times H_2 \to k$ a bounded sesquilinear form. Then $h$ has a representation $h(x, y) = \langle Sx, y \rangle$ where $S : H_1 \to H_2$ is a bounded linear operator. Moreover, $S$ is uniquely determined by $h$ and has the norm $\| S \| = \| h \|$. 

Proof. Consider $h(x, y)$ which is linear in $y$. Now for any fixed $x \in H_1$, we apply Theorem 3.8-1 to get a unique $z \in H_2$ that depends on $x \in H_1$ such that $h(x, y) = \langle y, z \rangle$ for all $y \in H_2$. Hence $S : H_1 \to H_2$ given by $Sx = z$ is an operator and $h(x, y) = \langle Sx, y \rangle$. 

For all $x_1, x_2 \in H_1$, $y \in H_2$ and all scalars $\alpha$, 

$\langle \alpha x_1 + x_2, y \rangle = \alpha \langle x_1, y \rangle + \langle x_2, y \rangle = \alpha \langle Sx_1, y \rangle + \langle Sx_2, y \rangle = \alpha \langle Sx_1 + Sx_2, y \rangle$. Then by Lemma 3.8-2, $S(\alpha x_1 + x_2) = \alpha Sx_1 + Sx_2$, so $S$ is linear. If $S = 0$ then it is bounded. If $S \neq 0$ then 

$\| S \| = \sup \{ \| Sx \| : x \in H_1 \setminus \{0\} \} = \| S \|$. Hence $S$ is bounded and $\| h \| \geq \| S \|$. 

But, $\| h \| = \sup \{ \| h(x, y) \| : x \in H_1 \setminus \{0\} \} \leq \sup \{ \| Sx \| \| y \| \| : x \in H_1 \setminus \{0\} \} = \| S \|$. Therefore, $\| h \| = \| S \|$. To prove the uniqueness, suppose that there is a linear operator $T : H_1 \to H_2$ such that $h(x, y) = \langle Sx, y \rangle = \langle Tx, y \rangle$ for all $x \in H_1$ and $y \in H_2$. Then by Lemma 3.8-2 $Sx = Tx$ for all $x \in H_1$. This proves the uniqueness of $S$. 

3.9 Hilbert adjoint operator

3.9-1 Definition. Let $H_1$ and $H_2$ be two Hilbert spaces and $T:H_1 \to H_2$ a bounded linear operator. The Hilbert adjoint operator $T^*$ of $T$ is the operator $T^*:H_2 \to H_1$ such that for all $x \in H_1$ and all $y \in H_2$, $\langle Tx, y \rangle = \langle x, T^*y \rangle$.

3.9-2 Theorem. The Hilbert adjoint operator $T^*$ of $T$ in Definition 3.9 exists, is unique, is linear and is bounded with norm $\| T^* \| = \| T \|$. 

Proof. Consider $h(y, x) = \langle y, Tx \rangle$ which defines a sesquilinear form on $H_2 \times H_1$ (The details is left to the reader). By Schwarz inequality, $| h(y, x) | = | \langle y, Tx \rangle | \leq \| y \| \| Tx \| \| T \|$, this implies that $h$ is bounded and $\| h \| \leq \| T \|$. But $\| h \| = \text{sup} \{ \frac{| h(y, x) |}{\| y \| \| x \|} : y \in H_2 \setminus \{0\} \& x \in H_1 \setminus \{0\} \}$. Therefore, $\| h \| = \| T \|$. Since $h$ is a bounded sesquilinear form then by Theorem 3.8 there exists a bounded linear operator call it $T^*$ that is uniquely determined by $h$ and $h(y, x) = \langle T^*y, x \rangle$, $x > \| T^* \| = \| h \|$. Hence $\| T^* \| = \| T \|$. However, $h(y, x) = \langle y, Tx \rangle$, then $\langle y, Tx \rangle = \langle T^*y, x \rangle$ and so $\langle Tx, y \rangle = \langle x, T^*y \rangle$.

3.9-3 Lemma. Let $X$ and $Y$ be inner product spaces and $Q:X \to Y$ a bounded linear operator. Then

a) $Q = 0$ if and only if $\langle Qx, y \rangle = 0$ for all $x \in X$ and all $y \in Y$.
b) If $Q:X \to Y$, $X$ is complex and $\langle Qx, x \rangle = 0$ for all $x \in X$, then $Q = 0$.

Proof. a) Left to the reader.
b) From assumption, $\langle Qv, v \rangle = 0$ for all $v = \alpha x + y \in X$; that is $0 = \langle Q(\alpha x + y), \alpha x + y \rangle = |\alpha|^2 \langle Qx, x \rangle + \langle Qy, y \rangle + \overline{\alpha} \langle Qy, x \rangle + \overline{\alpha} \langle Qx, y \rangle$. Put $\alpha = 1$ and then $\alpha = i$ to get $\langle Qx, y \rangle + \langle Qy, x \rangle = 0$ and $\langle Qx, x \rangle = - \langle Qy, y \rangle$, respectively. By addition, $\langle Qx, y \rangle = 0$ and so $Q = 0$ follows from a) •

Remark. If $X$ is real then b) above need not holds. To see this consider the mapping $Q: \mathbb{R}^2 \to \mathbb{R}^2$ given by $Q(\alpha, \beta) = (\beta, -\alpha)$. It is clear that for any $x = (\alpha, \beta) \in \mathbb{R}^2$, $\langle Qx, x \rangle = (\beta, -\alpha)(\alpha, \beta) = 0$, but $Q \neq 0$ •

3.9-4 Theorem. Let $H_1$ and $H_2$ be Hilbert spaces, $S, T:H_1 \to H_2$ bounded linear operators and $\alpha$ any scalar. Then we have, a) $T^*y, x = \langle y, Tx \rangle$ for all $x \in H_1$ and $y \in H_2$.

b) $(S + T)^* = S^* + T^*$ and $(\alpha T)^* = \overline{\alpha} T^*$.

c) $(T^*)^* = T$, and in the case $H_1 = H_2$, $(ST)^* = T^*S^*$.

d) $\| T^*T \| = \| TT^* \| = \| T \|^2$. e) $T^*T = 0$ if and only if $T = 0$.

Proof. We prove d) and left the proof of the other parts to the reader. First note that $T^*T:H_1 \to H_1$ but $TT^*:H_2 \to H_2$. Then $\| Tx \|^2 = \langle Tx, Tx \rangle = \langle T^*Tx, x \rangle \leq \| T^*Tx \| \| x \| \leq \| T^*T \| \| x \|$. Then $\| T \|^2 = \sup \{ \| Tx \|^2 : \| x \| = 1, x \in H_1 \} \leq \| T^*T \| \leq \| T^*T \| \| T \| = \| T \|^2$. Hence $\| T^*T \| = \| T \|^2$. Replacing $T$ by $T^*$ to get $\| T^*T^* \| = \| T \|^2$. However, $\| T^*T \| \leq \| TT^* \|$ and $\| T \|^2 = \| T \|^2$. Therefore, $\| T^*T \| = \| TT^* \| = \| T \|^2$ •

3.10 Self-adjoint, Unitary and Normal Operators

3.10-1 Definition. A bounded linear operator $T : H \to H$ on a Hilbert space $H$ is said to be 1) self-adjoint or Hermitian if $T^* = T$. 2) Unitary if $T$ is bijective and $T^* = T^{-1}$. 3) Normal if $TT^* = T^*T$.

3.10-2 Remark. a) If $T$ is self-adjoint then $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x$ and $y$. b) If $T$ is self-adjoint or unitary then $T$ is normal. c) Normal operators need not be self-adjoint or unitary. To see this consider the identity operator $I : H \to H$ on a Hilbert space $H$. It is easy to see that $T = 2iI$ is normal but it is neither self-adjoint nor unitary.

3.10-3 Theorem. Let $T : H \to H$ be a bounded linear operator on a Hilbert space $H$. Then a) If $T$ is self-adjoint then $\langle Tx, x \rangle$ is real for all $x \in H$. b) If $H$ is complex and $\langle Tx, x \rangle$ is real for all $x \in H$ then $T$ is self-adjoint.

Proof. a) Suppose that $T$ is self-adjoint. Then for all $x \in H$,
$$\langle Tx, x \rangle = \langle x, Tx \rangle = \langle Tx, x \rangle.$$ Hence $\langle Tx, x \rangle$ is real.

b) Suppose that $H$ is complex and $\langle Tx, x \rangle$ is real for all $x \in H$. Then
$$\langle Tx, x \rangle = \overline{\langle Tx, x \rangle} = \langle x, T^*x \rangle = \langle T^*x, x \rangle,$$ and $0 = \langle Tx, x \rangle - \langle T^*x, x \rangle = \langle (T-T^*)x, x \rangle$. Then by Lemma 3.9-3(b) $T-T^* = 0$. Hence $T$ is self-adjoint.

3.10-4 Theorem. The product of two bounded self-adjoint linear operators $S$ and $T$ on a Hilbert space $H$ is self-adjoint if and only if $ST = TS$.

Proof. Left to the reader.

3.10-5 Theorem. Let $(T_n)$ be a sequence of bounded self-adjoint linear operators $T_n : H \to H$ on a Hilbert space $H$. Suppose that $T_n \to T$, that is $\| T_n - T \| \to 0$, where this norm is the norm on the space $B(H, H)$. Then $T$ is a bounded self-adjoint linear operator on $H$.

Proof. Left to the reader.

3.10-6 Theorem. Let the operators $U, V : H \to H$ be unitary on a Hilbert space $H$. Then:

a) $U$ is isometric, thus $\| Ux \| = \| x \|$ for all $x \in H$.

b) $\| U \| = 1$, provided that $H \neq \{0\}$. c) $U^{-1}$ and $UV$ are unitary.

d) $U$ is normal. Furthermore,

c) A bounded linear operator $T$ on a complex Hilbert space $H$ is unitary if and only if $T$ is isometric and onto.

Proof. Left to the reader.

3.10-7 Remark. Isometric operators need not be unitary.

Proof. Consider the bounded linear operator $T: \ell^2 \to \ell^2$ given by $T(\xi_1, \xi_2, \ldots) = (0, \xi_1, \xi_2, \ldots)$. It is easy to see that $T$ is an isometric, but $T$ is not onto, where there is $(\xi_1, \xi_2, \ldots) \in \ell^2$ and $\xi_1 \neq 0$ but there is no $x \in \ell^2$ with $Tx = (\xi_1, \xi_2, \ldots)$. Hence we have an isometric which is not unitary.