Introduction

An ideal low-pass filter has a frequency response characteristics

\[ H(\omega) = \begin{cases} 
1, & |\omega| < \omega_c \\
0, & \omega_c < \omega \leq \pi 
\end{cases} \]

The impulse response of this filter

\[ h(n) = \begin{cases} 
\frac{\omega_c}{\pi}, & n = 0 \\
\frac{\sin \omega_c n}{\pi n}, & n \neq 0 
\end{cases} \]

As we can see \( h(n) \) is infinite and hence we need to truncate the higher coefficients.
**Paely-Weiner Theorem**

For practical implementation, the filter function should be causal. Paely - Weiner theorem summarize the conditions that should be satisfied to guarantee causality which are:

1. The frequency response $H(\omega)$ cannot be zero except at a finite set of points at frequency.
2. The magnitude $H(\omega)$ cannot be constant in any finite range of frequencies.
3. The Transition between the passband and stopband cannot be infinitely sharp.
4. Real part and Imaginary part of $H(\omega)$ are interdependent by Hilbert Transform.

**Practical Filter**

- $\delta_1$ ~ Passband ripple
- $\delta_2$ ~ Stopband ripple
- $\omega_p$ ~ Passband edge frequency
- $\omega_s$ ~ Stopband edge frequency

<table>
<thead>
<tr>
<th>$H(\omega)$</th>
<th>Transition band</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 + \delta_1$</td>
<td>Passband ripple</td>
</tr>
<tr>
<td>$1 - \delta_1$</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>$\omega_p$</td>
</tr>
<tr>
<td>$\pi$</td>
<td>$\omega_s$</td>
</tr>
<tr>
<td></td>
<td>Stopband</td>
</tr>
</tbody>
</table>
FIR Filter

A FIR system can be described by the following difference equation
\[ y(n) = \sum_{k=0}^{M-1} b(k)x(n-k) \]
The transfer function is given by
\[ H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-(M-1)} \]
The impulse response is
\[ h(n) = \{b_0, b_1, b_2, \ldots, b_M\} \]
Then \( H(z) \) can be expressed as
\[ H(z) = \sum_{k=0}^{M-1} h(k)z^{-k} \]

Type I Symmetric FIR filter

In this type, the coefficients of impulse response are symmetric
\[ h(n) = h(M-1-n) \]
And \( N = M-1 \) is even
Example: consider
\[ H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} \]
But as \( h(0) = h(6), h(1) = h(5), h(2) = h(4) \)
\[ H(z) = h(0)(1 + z^{-6}) + h(1)(z^{-1} + z^{-5}) + h(2)(z^{-2} + z^{-4}) + h(3)z^{-3} \]
Type I Symmetric FIR filter

Or

\[ H(z) = z^{-3} \left[ h(0)(z^{-3} + z^{-3}) + h(1)(z^2 + z^{-2}) + h(2)(z^4 + z^{-4}) + h(3) \right] \]

Converting to DTFT

\[ H(\omega) = e^{-j3\omega} \left[ 2h(0)\cos(3\omega) + h(1)\cos(2\omega) + h(2)\cos(\omega) + h(3) \right] \]

\[ H(\omega) = e^{-j3\omega} H_R(\omega) \]

\( H_R(\omega) \) is a real-valued function but it can be positive or negative, then the phase is given by

\[
\theta(\omega) = \begin{cases} 
-3\omega & H_R(\omega) \geq 0 \\
-3\omega + \pi & H_R(\omega) < 0 
\end{cases}
\]

The result can be generalized

\[ H(\omega) = e^{-j(N/2)\omega} \left[ 2h(0)\cos\left(\frac{N}{2}\omega\right) + h(1)\cos\left(\frac{N}{2} - 1\omega\right) + h(2)\cos\left(\frac{N}{2} - 2\right) + \cdots + h(N/2) \right] \]

\[ H(\omega) = e^{-j(N/2)\omega} H_R(\omega) \]

the phase is given by

\[
\theta(\omega) = \begin{cases} 
-(N/2)\omega & H_R(\omega) \geq 0 \\
-(N/2)\omega + \pi & H_R(\omega) < 0 
\end{cases}
\]
Type II Symmetric FIR filter

In this type, the coefficients of impulse response are symmetric

\[ h(n) = h(M - 1 - n) \]

And \( N = M - 1 \) is odd

After mathematic manipulation, we get

\[
H(\omega) = e^{-j(N/2)\omega} \left[ \sum_{n=1}^{(N+1)/2} 2h \left( \frac{N + 1}{2} - n \right) \cos \left( (n - 0.5)\omega \right) \right]
\]

The phase is given by

\[
\theta(\omega) = \begin{cases} 
-\left( \frac{N}{2} \right)\omega & H_R(\omega) \geq 0 \\
-\left( \frac{N}{2} \right)\omega + \pi & H_R(\omega) < 0
\end{cases}
\]

Type III Symmetric FIR filter

In this type, the coefficients of impulse response are anti-symmetric

\[ h(n) = -h(M - 1 - n) \]

And \( M = N - 1 \) is even

After mathematic manipulation, we get

\[
H(\omega) = e^{-j(N\omega - \pi)/2} \left[ 2 \sum_{n=1}^{N/2} h \left( \frac{N}{2} - n \right) \sin(n\omega) \right]
\]

The phase is given by

\[
\theta(\omega) = \begin{cases} 
-\left( \frac{N\omega - \pi}{2} \right) & H_R(\omega) \geq 0 \\
-\left( \frac{N\omega - \pi}{2} \right) + \pi & H_R(\omega) < 0
\end{cases}
\]
**Type IV Symmetric FIR filter**

In this type, the coefficients of impulse response are antisymmetric

\[ h(n) = -h(M-1-n) \]

And \( M=N-1 \) is even

After mathematic manipulation, we get

\[ H(\omega) = e^{-j((N\omega-\pi)/2)} \left[ 2 \sum_{n=1}^{(N+1)/2} h\left(\frac{N+1}{2} - n\right) \sin((n-0.5)\omega) \right] \]

The phase is given by

\[ \theta(\omega) = \begin{cases} -\left(\frac{(N\omega-\pi)}{2}\right) & H_R(\omega) \geq 0 \\ -\left(\frac{(N\omega-\pi)}{2}\right) + \pi & H_R(\omega) < 0 \end{cases} \]

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**Properties of Linear Phase FIR Filters**

All the four type have constant group delay (linear phase)

\[ \phi = \frac{d\theta(\omega)}{d\omega} = -N/2 \]

1. Type I FIR filters have either an even number of zeros or no zeros at \( z = 1 \) and \( z = -1 \).

2. Type II FIR filters have an even number of zeros or no zeros at \( z = 1 \) and an odd number of zeros at \( z = -1 \).

3. Type III FIR filters have an odd number of zeros at \( z = 1 \) and \( z = -1 \).

4. Type IV FIR filters have an odd number of zeros at \( z = 1 \) and either an even or odd number of zeros at \( z = -1 \).
Properties of Linear Phase FIR Filters

- Type I and II are suitable only for lowpass filters
- Type III is suitable for designing bandpass filters
- Type IV is used mainly for highpass and bandpass filters

Window method

The magnitude responses of four ideal classical types of digital filters are shown in Figure.
Windows method

An ideal low-pass filter has a frequency response characteristics:

\[ h_{LP}(n) = \begin{cases} \frac{\omega_0}{\pi}, & n = 0 \\ \sin(\omega_0 n) & n \neq 0 \end{cases} \]

To get FIR filter:

\( h_{LP}(n) \) is truncated to be defined only between \( n = -N \) and \( n = N \)

This is equivalent to multiplying \( h(n) \) by \( w(n) \). Where

\[ w(\omega) = \begin{cases} 1, & -N \leq n \leq N \\ 0, & \text{otherwise} \end{cases} \]

Rectangular window

The resultant functions is given by

\[ h_d(n) = w(n)h_{LP}(n) \]

The new \( H_d(\omega) \), will not be the same as the ideal \( H(\omega) \). Gibbs phenomena is raised by this truncation of \( h(n) \). The overshot can be reduced by increasing \( M \) but the oscillation increases.
other windows

To reduce gibbs phenomena, researchers have use different type of windows such as

1) Bartlett windows

\[ w(n) = 1 - \frac{2n - M - 1}{M - 1}, \quad 0 \leq n \leq M - 1 \]

2) Hann Windows

\[ w(n) = \frac{1}{2} \left[ 1 - \cos \left( \frac{2\pi n}{M - 1} \right) \right], \quad 0 \leq n \leq M - 1 \]

3) Hamming windows

\[ w(n) = 0.54 - 0.46 \cos \left( \frac{2\pi n}{M - 1} \right), \quad 0 \leq n \leq M - 1 \]

4) Blackman window

\[ w(n) = 0.42 - 0.5 \cos \left( \frac{2\pi n}{M - 1} \right) + 0.08 \cos \left( \frac{4\pi n}{M - 1} \right), \quad 0 \leq n \leq M - 1 \]

5) Kaiser window

\[ I_0 \left[ \frac{\alpha \sqrt{\left( \frac{M - 1}{2} \right)^2 - \left( \frac{n - M - 1}{2} \right)^2}}{I_0 \left[ \frac{\alpha}{\frac{M - 1}{2}} \right]} \right] \]

6) Lanczos window

\[ \begin{cases} \sin \left[ 2\pi \left( \frac{n - M - 1}{2} \right)/ (M - 1) \right] \left( \frac{2\pi}{M - 1} \right)^L, & L > 0 \\ \sin \left( \frac{2\pi}{M} \right) \left( \frac{2\pi}{M} \right)/ \left( \frac{M - 1}{2} \right)^L, & L = 0 \\ 1, & \left| n - \frac{M - 1}{2} \right| \leq \alpha \frac{M - 1}{2}, \quad 0 < \alpha < 1 \end{cases} \]
4) Tukey window

\[
\frac{1}{2} \left[ 1 + \cos \left( \frac{n - (1 + \alpha)(M - 1)/2}{(1 - \alpha)(M - 1)/2} \pi \right) \right]
\]

where \(\alpha = \alpha(M - 1)/2 \leq |n - \frac{M - 1}{2}| \leq \frac{M - 1}{2}\)

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**Impulse response for HP, BP and BS Filers**

The impulse response of the highpass, bandpass and bandstop are derived by the same method

\[
h(n) = \begin{cases} 
\frac{\pi - \omega_c}{\pi} & n = 0 \\
\frac{-\sin(\omega_c n)}{\pi n} & |n| > 0 
\end{cases}
\]

\[
h(n) = \begin{cases} 
\frac{\omega_{c2} - \omega_{c1}}{\pi} & n = 0 \\
\frac{1}{\pi n} \left( \sin(\omega_c n) - \sin(\omega_{c2} n) \right) & |n| > 0 
\end{cases}
\]

\[
h(n) = \begin{cases} 
\frac{\pi - \omega_{c2} + \omega_{c1}}{\pi} & n = 0 \\
\frac{1}{\pi n} \left( \sin(\omega_c n) - \sin(\omega_{c2} n) \right) & |n| > 0 
\end{cases}
\]
Examples

Example 1: Design a bandpass filter using hamming window of length 11, given that $\omega_{c1} = 0.2\pi$ and $\omega_{c2} = 0.6\pi$

Solution: $M = 11 = 2N + 1$

$N = 5.$

substituting in $h(n)$ for bandpass filter and $w(n)$ for hamming window and shifting the result by 5

$h(n) = \begin{cases} 
0.4, & n = 0 \\
\frac{1}{\pi n} (\sin(0.6\pi n) - \sin(0.4\pi n)) & |n| > 0 
\end{cases}$

$h(n) = \begin{bmatrix} 
0.0289, -0.1633, -0.2449, 0.1156, 0.0, 0.1156, \\
-0.2449, -0.1633, 0.0289, 0 
\end{bmatrix}$

$w(n) = 0.54 - 0.46 \cos \left( \frac{2\pi n}{10} \right) \quad 0 \leq n \leq 9$

The result is achieved by $h_w(n) = w(n)h(n)$

$h_w(n) = \begin{bmatrix} 
0, 0.0049, 0.0650, -0.1671, 0.1055, 0.4, 0.1055, -0.1671, \\
-0.0650, 0.0049, 0 
\end{bmatrix}$
Hamming Window Examples

Example 2

a) Calculate the filter coefficients for a 3-tap FIR lowpass filter with a cutoff frequency of 800 Hz and a sampling rate of 8,000 Hz using Hamming window

b) Determine the transfer function and difference equation of the designed FIR system.

c) Plot the magnitude frequency response

\[ \omega_c = 2\pi f_c T_s = 2\pi \times 800 / 8000 = 0.2\pi \]

The coefficients of the lowpass filter has been calculated before

\[ h(0) = 0.2, \ h(-1) = h(1) = 0.1871 \]

For Hamming window with \( M = 3 \)

\[ w(n) = 0.54 + 0.46 \cos(n\pi) \]

and for \( n = 0, 1, 2 \)

\[ w_{ham}(0) = 1, \ w_{ham}(1) = w_{ham}(2) = 0.08 \]

Then the windowed impulse response coefficients are

\[ h_w(0) = 0.2, \ h_w(1) = h_w(2) = 0.01497 \]

convert to z-domain

\[ H(z) = 0.01497 + 0.2z^{-1} + 0.1497z^{-2} \]

Then the difference equation is

\[ y(n) = 0.01497x(n) + 0.2x(n-1) + 0.01497x(n-2) \]
Hamming Window Examples

\[ H(\omega) = 0.01497 + 0.2e^{-j\omega} + 0.01497e^{-j2\omega} \]
\[ H(\omega) = e^{-j\omega} (0.01497e^{j\omega} + 0.2 + 0.01497e^{-j\omega}) \]
\[ H(\omega) = e^{-j\omega} (0.2 + 0.02994 \cos \omega) \]
\[ |H(\omega)| = 0.2 + 0.02994 \cos \omega \]
\[ \angle H(\omega) = \begin{cases} -\omega & \text{if } 0.2 + 0.02994 \cos \omega > 0 \\ -\omega + \pi & \text{if } 0.2 + 0.02994 \cos \omega < 0 \end{cases} \]
Hamming Window Examples

Example 3:

a) Design a 5-tap FIR band reject filter with a lower cutoff frequency of 2,000 Hz, an upper cutoff frequency of 2,400 Hz, and a sampling rate of 8,000 Hz using the Hamming window method.

b) Determine the transfer function.

\[ 2N+1 = 5 \text{ which leads that } N = 2, \quad h(n) \text{ is given by} \]

\[ h(n) = \begin{cases} \frac{\pi - \omega_u + \omega_l}{\pi}, & n = 0 \\ \frac{-\sin(\omega_u n)}{\pi n} + \frac{\sin(\omega_l n)}{\pi n}, & n \neq 0 \end{cases} \]

Then the five samples of \( h(n) \) shifted by 2 is equal to

\[ h(2) = 0.9, \]
\[ h(1) = h(3) = 0.01558 \]
\[ h(0) = h(4) = 0.09355 \]

For Hamming window with \( M = 5 \)

\[ w(n) = 0.54 - 0.46 \cos \left( \frac{n\pi}{2} \right) \]

and for \( n = 0, 1, 2, 3, 4 \)

\[ w_{\text{win}}(2) = 1, \]
\[ w_{\text{win}}(1) = w_{\text{win}}(3) = 0.54, \]
\[ w_{\text{win}}(0) = w_{\text{win}}(4) = 0.08, \]

Then the windowed impulse response coefficients are

\[ h_{\text{win}}(2) = 0.9, \]
\[ h_{\text{win}}(1) = h_{\text{win}}(3) = 0.00841 \]
\[ h_{\text{win}}(0) = h_{\text{win}}(4) = 0.00748 \]

Convert to z-domain

\[ H(z) = 0.00748 + 0.00841z^{-1} + 0.9z^{-2} + 0.00841z^{-3} + 0.00748z^{-4} \]