Chapter 10

Analysis of Statically Indeterminate Structures

Force Method of Analysis
Methods of Analysis

Two different Methods are available

- **Force method**
  - known as consistent deformation, unit load method, flexibility method, or the superposition equations method.
  - The primary unknowns in this way of analysis are forces

- **Displacement method**
  - Known as stiffness matrix method
  - The primary unknowns are displacements

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Methods of Analysis

Force method of analysis

The deflection or slope at any point on a structure as a result of a number of forces, including the reactions, is equal to the algebraic sum of the deflections or slopes at this particular point as a result of these loads acting individually.
Force Method of Analysis

- General Procedure
  - Indeterminate to the first degree
  - 1 Compatibility equation is needed
  - Choosing one of the support reaction as a redundant
  - The structure become statically determinate & stable
  - Downward displacement $\Delta_B$ at B calculated (load action)
  - $f_{BB}$ upward deflection per unit force at B
  - Compatibility equation
    
    $0 = \Delta_B + B_y f_{BB}$
  - Reaction $B_y$ known
  - Now the structure is statically determinate
Force Method of Analysis

• General Procedure
  – Indeterminate to the first degree
  – 1 Compatibility equation is needed
  – Choosing $M_A$ at A as a redundant
  – The structure become statically determinate & stable
  – Rotation $\theta_A$ at A caused by load P is determined
  – $\alpha_{AA}$ rotation caused by unit couple moment applied at A
  – Compatibility equation
    \[ 0 = \theta_A + M_A \alpha_{AA} \]
  – Moment $M_A$ known
  – Now the structure is statically determinate
Force Method of Analysis

• General Procedure
  – Indeterminate to the 2\textsuperscript{nd} degree
  – 2 Compatibility equations needed
  – Redundant reaction B & C
  – Displacement $\Delta_B$ & $\Delta_C$ caused by load $P_1$ & $P_2$ are determined
Force Method of Analysis

• General Procedure
  – $f_{BB}$ & $f_{BC}$: Deflection per unit force at B are determined
  – $f_{CC}$ & $f_{CB}$: Deflection per unit force at C are determined
  – Compatibility equations
    \[
    0 = \Delta_B + B_y f_{BB} + C_y f_{BC} \\
    0 = \Delta_C + B_y f_{CB} + C_y f_{CC}
    \]
  – Reactions at B & C are known
  – Statically determinate structure
Maxwell’s Theorem

• The displacement of a point B on a structure due to a unit load acting at a point A is equal to the displacement of point A when the unit load is acting at point B the is

\[ f_{BC} = f_{CB} \]

• The rotation of a point B on a structure due to a unit moment acting at a point A is equal to the rotation of point A when the unit moment is acting at point B the is

\[ \alpha_{BC} = \alpha_{CB} \]
Force Method of Analysis

• Procedure for Analysis
  – Determine the degree of statically indeterminacy
  – Identify the redundants, whether it’s a force or a moment, that would be treated as unknown in order to form the structure statically determinate & stable
  – Calculate the displacements of the determinate structure at the points where the redundants have been removed
  – Calculate the displacements at these same points in the determinate structure due to the unit force or moment of each redundants individually
  – Workout the compatibility equation at each point where there is a redundant & solve for the unknown redundants
  – Knowing the value of the redundants, use equilibrium to determine the remaining reactions
Beam Deflections and Slopes

\[ v = \frac{P}{6EI} \left( x^3 - 3Lx^2 \right) \]

at \( x = L \)

\[ v_{\text{max}} = -\frac{PL^3}{3EI} \]

\[ \theta_{\text{max}} = -\frac{PL^2}{2EI} \]

\[ v = \frac{M_0}{2EI} x^2 \]

at \( x = L \)

\[ v_{\text{max}} = \frac{M_0L^2}{2EI} \]

\[ \theta_{\text{max}} = \frac{M_0L}{EI} \]
Beam Deflections and Slopes

\[ v = -\frac{w}{24EI} \left( x^4 - 4Lx^3 + 6L^2x^2 \right) \]

at \( x = L \), \( v_{\text{max}} = -\frac{wL^4}{8EI} \)

\[ \theta_{\text{max}} = -\frac{wL^3}{6EI} \]

\[ v = \frac{P}{48EI} \left( 4x^3 - 3L^2x \right), \quad 0 \leq x \leq \frac{L}{2} \]

at \( x = \frac{L}{2} \), \( v_{\text{max}} = -\frac{PL^3}{48EI} \)

\[ \theta_{\text{max}} = \pm \frac{PL^2}{16EI} \]
Beam Deflections and Slopes

\[ v = -\frac{P_b x}{6LEI} \left( L^2 - b^2 - x^2 \right), \quad 0 \leq x \leq a \]

\[ \theta_L = -\frac{P_{ab} (L + b)}{6LEI} \]

\[ \theta_R = \frac{P_{ab} (L + a)}{6LEI} \]

\[ v = -\frac{wx}{24EI} \left( x^3 - 2Lx^2 + L^3 \right) \]

at \( x = \frac{L}{2} \)

\[ v_{\text{max}} = -\frac{5wL^4}{384EI} \]

\[ \theta_{\text{max}} = \pm \frac{wL^3}{24EI} \]
Beam Deflections and Slopes

\[ \nu = -\frac{wx}{384EI} \left(9L^3 - 24Lx^2 + 16x^3\right) \quad 0 \leq x \leq \frac{L}{2} \]

\[ \nu = -\frac{wL}{384EI} \left(8x^3 - 24Lx^2 + 17L^2x - L^3\right) \]

\[ \frac{L}{2} \leq x \leq L \]

\[ \theta_L = -\frac{3wL^3}{128EI} \quad \theta_R = \frac{7wL^3}{384EI} \]

\[ \nu = \frac{M_0x}{6EIL} \left(x^2 - 3Lx + 2L^2\right) \]

\[ \nu_{\text{max}} = -\frac{M_0L^2}{9\sqrt{2EI}} \]

\[ \theta_L = -\frac{M_0L}{6EI} \quad \theta_R = \frac{M_0L}{3EI} \]
Example 1

- Determine the reaction at B
  - Indeterminate to the 1st degree thus one additional equation needed
  - Let's take B as a redundant
  - Determine the deflection at point B in the absence of support B. Using the moment-area method

\[
\Delta_B = \frac{1}{EI} \left[ 300 \times \frac{6}{2} \left( \frac{2}{3} 6 + 6 \right) \right]
\]

\[
\Delta_B = \frac{9000 kN \cdot m^3}{EI}
\]

- Determine the deflection caused by the unit load at point B

\[
50 \times 6 = 300 kN \cdot m
\]
\[
50 kN
\]
\[
B \text{M.D}
\]

\[
\Delta_B \quad f_{BB}
\]
Example 1

\[ f_{BB} = \frac{1}{EI} \left( 12 \times \frac{12}{2} \times \frac{2}{3} \times 12 \right) \]

\[ f_{BB} = \frac{576 m^3}{EI} \uparrow \]

- Compatibility equation

\[ 0 = \Delta_B + B_y f_{BB} \]

\[ 0 = -\frac{9000}{EI} + B_y \frac{576}{EI} \]

\[ B_y = 15.6 \text{kN} \]

- The reaction at B is known now so the structure is statically determinate & equilibrium equations can be applied to get the rest of the unknowns.
Example 2

- Determine the moment at A
  - Indeterminate to the 1\textsuperscript{st} degree thus one additional equation needed
  - Lets take $M_A$ as a redundant
  - Determine the slope $\theta_A$ at point A ignoring the fixation at A. Using the moment-area method

\[
d_1 = \frac{1}{EI} \left( 20 \times \frac{10}{2} \times \frac{10}{3} \right) = \frac{333.33}{EI}
\]

\[
\theta_A = \frac{d_1}{L} = \frac{1}{EI} \cdot \frac{333.33}{10}
\]

\[
\theta_A = \frac{33.33k \cdot ft^2}{EI}
\]
Example 2

- Determine rotation caused by the unit moment applied at A

\[ d_2 = \frac{1}{EI} \left( \frac{1 \times 10}{2} \times \frac{2 \times 10}{3} \right) = \frac{33.33}{EI} \]

\[ \alpha_{AA} = \frac{d_2}{L} = \frac{1}{EI} \cdot \frac{33.33}{10} = \frac{3.33 \text{ ft}}{EI} \]

- Compatibility equation

\[ 0 = \theta_A + M_A \alpha_{AA} \]

\[ 0 = \frac{33.33}{EI} + M_A \frac{3.33}{EI} \]

- The moment at A is known now so the structure is statically determinate
Example 3

Draw the shear and moment diagrams for the beam shown in Fig. 10–10a. The support at B settles 1.5 in. Take $E = 29(10^3)$ ksi, $I = 750 \text{ in}^4$.

actual beam
(a)

primary structure
(b)

redundant $B_y$ applied
Example 3

**Compatibility Equation.** With reference to point B in Fig. 10–10b, using units of ft, we require

\[
\frac{1.5}{12} = \Delta_B + B_y f_{BB} \tag{1}
\]

\[
\mu + \sum M_{B'} = 0; \quad -M_{B'} + \frac{1440}{EI} (8) - \frac{1800}{EI} (24) = 0
\]

\[
M_{B'} = -\frac{31680}{EI} = \frac{31680}{EI} \gamma
\]
Example 3

Verify the calculations in Fig. 10–10d for calculating $f_{BB}$. Note that

$$\downarrow + \sum M_{B'} = 0; \quad -m_{B'} + \frac{144}{EI} (8) - \frac{144}{EI} (24) = 0$$

$$m_{B'} = -\frac{2304}{EI} = \frac{2304}{EI} \text{ ft}$$
Substituting these results into Eq. (1), we have

\[
\frac{1.5}{12} = \frac{31\,680}{EI} + B_y\left(\frac{2304}{EI}\right)
\]

Expressing the units of \(E\) and \(I\) in terms of \(k\) and \(ft\), we have

\[
\left(\frac{1.5}{12} \text{ft}\right)[29(10^3) \text{ k/in}^2((12)^2 \text{ in}^2/\text{ft}^2)][750 \text{ in}^4/(12)^4 \text{ in}^4)]
\]

\[
= 31\,680 + B_y(2304)
\]

\[B_y = -5.56\, k\]

*Equilibrium Equations.* The negative sign indicates that \(B_y\) acts upward on the beam. From the free-body diagram shown in Fig. 10–10e we have

\[\downarrow \sum M_A = 0; \quad -20(12) + 5.56(24) + C_y(48) = 0\]

\[C_y = 2.22\, k\]

\[\uparrow \sum F_y = 0; \quad A_y - 20 + 5.56 + 2.22 = 0\]

\[A_y = 12.22\, k\]

Using these results, verify the shear and moment diagrams shown in Fig. 10–10f.
Example 4

• Neglect the axial load
• The end moments at A&B will be considered as redundants
• From Table inside the front cover

\[
\theta_A = \frac{1}{EI} (375)
\]

\[
\theta_B = \frac{1}{EI} (291.7)
\]
Example 4

\[ \alpha_{AA} = \frac{6.67}{EI} \]

\[ \alpha_{BA} = \frac{3.33}{EI} \]

\[ \alpha_{BB} = \frac{6.67}{EI} \]

\[ \alpha_{AB} = \frac{3.33}{EI} \]

\[ 0 = \frac{375}{EI} + M_A \left( \frac{6.67}{EI} \right) + M_B \left( \frac{3.33}{EI} \right) \]

\[ 0 = \frac{291.7}{EI} + M_A \left( \frac{3.33}{EI} \right) + M_B \left( \frac{6.67}{EI} \right) \]

\[ M_A = -45.8\, k\, ft \quad M_B = -20.8\, k\, ft \]
Example 4
Example 5

Determine the reaction at the support for the beam shown, EI is Constant. Choose the internal moment at Internal support as the redundant

Compatibility Equations

\[ \theta_B + M_B \alpha_{BB} = 0 \]

where

\[ \theta_B = \theta'_B + \theta''_B \]

and

\[ \alpha_{BB} = \alpha'_{BB} + \alpha''_{BB} \]
Example 5

The slopes and angular flexibility coefficients can be determined from the table on the inside front cover, that is,

\[
\theta_B' = \frac{wL^3}{24EI} = \frac{120(12)^3}{24EI} = \frac{8640 \text{ lb} \cdot \text{ft}^2}{EI} \\
\theta_B'' = \frac{PL^2}{16EI} = \frac{500(10)^2}{16EI} = \frac{3125 \text{ lb} \cdot \text{ft}^2}{EI} \\
\alpha_{BB}' = \frac{ML}{3EI} = \frac{1(12)}{3EI} = \frac{4 \text{ ft}}{EI} \\
\alpha_{BB}'' = \frac{ML}{3EI} = \frac{1(10)}{3EI} = \frac{3.33 \text{ ft}}{EI}
\]

Thus

\[
\frac{8640 \text{ lb} \cdot \text{ft}^2}{EI} + \frac{3125 \text{ lb} \cdot \text{ft}^2}{EI} + M_B\left(\frac{4 \text{ ft}}{EI} + \frac{3.33 \text{ ft}}{EI}\right) = 0
\]

\[
M_B = -1604 \text{ lb} \cdot \text{ft}
\]
Example 5
Example 6

Determine the support reactions on the frame shown EI is Constant.

_COMPATIBILITY EQUATION._ Reference to point B in Fig. 10–13b requires

\[ 0 = \Delta_B + B_x f_{BB} \]  \hfill (1)
Example 6

\[ \Delta_B = \int_0^L \frac{Mm}{EI} \, dx = \int_0^5 \frac{(20x_1 - 4x_1^2)(0.8x_1)}{EI} \, dx_1 + \int_0^4 \frac{(1x_2)}{EI} \, dx_2 \]

\[ = \frac{166.7}{EI} + 0 = \frac{166.7}{EI} \]

For \( f_{BB} \) we require application of a real unit load and a virtual unit load acting at \( B \), Fig. 10–13d. Thus,

\[ f_{BB} = \int_0^L \frac{mm}{EI} \, dx = \int_0^5 \frac{(0.8x_1)^2}{EI} \, dx_1 + \int_0^4 \frac{(1x_2)^2}{EI} \, dx_2 \]

\[ = \frac{26.7}{EI} + \frac{21.3}{EI} = \frac{48.0}{EI} \]

Substituting the data into Eq. (1) and solving yields

\[ 0 = \frac{166.7}{EI} + B_x \left( \frac{48.0}{EI} \right) \quad B_x = -3.47 \text{ kN} \quad \text{Ans.} \]
Example 6

\[ \sum F_x = 0; \quad A_x - 3.47 = 0 \quad A_x = 3.47 \text{ kN} \quad \text{Ans.} \]

\[ \sum M_A = 0; \quad -40(2.5) + B_y(5) - 3.47(4) = 0 \quad B_y = 22.8 \text{ kN} \quad \text{Ans.} \]

\[ \sum F_y = 0; \quad A_y - 40 + 22.8 = 0 \quad A_y = 17.2 \text{ kN} \quad \text{Ans.} \]
Example (Additional)

The frame, shown in the photo is used to support the bridge deck. Assuming EI is constant, a drawing of it along with the dimensions and loading is shown. Determine the support reactions.


**Compatibility Equation.** Reference to point A in Fig. 10–12b requires

\[
0 = \Delta_A + A_x f_{AA} \quad (1)
\]
\[ m_2 = 200x_2 \]
\[ m_3 = 200(5 + x_3) - 40x_3 \left( \frac{x_3}{2} \right) \]
\[ = 1000 + 200x_3 - 20x_3^2 \]

40 kN/m

5 m

\[ x_2 \]

\[ x_1 \]

\[ x_3 \]

200 kN

200 kN
\[ \Delta_A = \int_0^L \frac{Mm}{EI} \, dx = 2 \int_0^5 \frac{(0)(1x_1) \, dx_1}{EI} + 2 \int_0^5 \frac{(200x_2)(-5) \, dx_2}{EI} \\
+ 2 \int_0^5 \frac{(1000 + 200x_3 - 20x_3^2)(-5) \, dx_3}{EI} \]

\[ = 0 - \frac{25000}{EI} - \frac{66666.7}{EI} = -\frac{91666.7}{EI} \]
\[ f_{AA} = \int_0^L \frac{mm}{EI} \, dx = 2 \int_0^5 \frac{(1x_1)^2 \, dx_1}{EI} + 2 \int_0^5 (5)^2 \, dx_2 + 2 \int_0^5 (5)^2 \, dx_3 \]

\[ = \frac{583.33}{EI} \]

Substituting the results into Eq. (1) and solving yields

\[ 0 = -\frac{91.666.7}{EI} + A_x \left(\frac{583.33}{EI}\right) \]

\[ A_x = 157 \text{ kN} \]
Example 7

Determine the moment at the fixed support A for the frame shown EI is constant.
Example 7

**Compatibility Equation.** Reference to point $A$ in Fig. 10–14b requires

$$(	heta^+ + ) 0 = \theta_A + M_A \alpha_{AA}$$

(1)

As in the preceding example, $\theta_A$ and $\alpha_{AA}$ will be computed using the method of virtual work. The frame’s $x$ coordinates and internal moments are shown in Fig. 10–14c and 10–14d.
Example 7

For $\theta_A$ we require application of the real loads, Fig. 10–14c, and a virtual unit couple moment at $A$, Fig. 10–14d. Thus,

$$\theta_A = \sum \int_0^L \frac{Mm_\theta}{EI} \, dx$$

$$= \int_0^8 \frac{(29.17x_1)(1 - 0.0833x_1)}{EI} \, dx_1$$

$$+ \int_0^5 \frac{(296.7x_2 - 50x_2^2)(0.0667x_2)}{EI} \, dx_2$$

$$= \frac{518.5}{EI} + \frac{303.2}{EI} = \frac{821.8}{EI}$$
\[ \alpha_{AA} = \sum \int_{0}^{L} \frac{m_{0}m_{0}}{EI} \, dx \]

\[ = \int_{0}^{8} \frac{(1 - 0.0833x_{1})^2}{EI} \, dx_{1} + \int_{0}^{5} \frac{(0.0667x_{2})^2}{EI} \, dx_{2} \]

\[ = \frac{3.85}{EI} + \frac{0.185}{EI} = \frac{4.04}{EI} \]

Substituting these results into Eq. (1) and solving yields

\[ 0 = \frac{821.8}{EI} + M_{A}\left(\frac{4.04}{EI}\right) \quad M_{A} = -204 \text{ lb} \cdot \text{ft} \quad \text{Ans.} \]
Force Method of Analysis: Truss

Example 8

Determine the force in member AC.
Assume EA is the same for all the members.
Example 8

\[ 0 = \Delta_{AC} + F_{AC} f_{AC AC} \]

actual truss

primary structure

redundant $F_{AC}$ applied
Force Method of Analysis: Truss

\[ \Delta_{AC} = \sum \frac{nNL}{AE} \]

\[ = 2\left[ \frac{(-0.8)(400)(8)}{AE} \right] + \frac{(-0.6)(0)(6)}{AE} + \frac{(-0.6)(300)(6)}{AE} \]

\[ + \frac{(1)(-500)(10)}{AE} + \frac{(1)(0)(10)}{AE} \]

\[ = \frac{-11200}{AE} \]

\[ f_{ACAC} = \sum \frac{n^2L}{AE} \]

\[ = 2\left[ \frac{(-0.8)^2(8)}{AE} \right] + 2\left[ \frac{(-0.6)^2(6)}{AE} \right] + 2\left[ \frac{(1)^210}{AE} \right] \]

\[ = \frac{34.56}{AE} \]

Substituting the data into Eq. (1) and solving yields

\[ 0 = \frac{-11200}{AE} + \frac{34.56}{AE} F_{AC} \]

\[ F_{AC} = 324 \text{ lb} \ (T) \]
Example 9

- Determine the force in each member if the turnbuckle on member AC is used to shorten the member by 0.5 in. Each member has a cross-section area of 0.2 in\(^2\) & E=29(10^6)\text{psi}
Example 9

[Diagram of a truss structure with labels and notation]

- Primary structure: \( \Delta_{AC} = 0 \)
- Redundant force \( F_{AC} \) applied
Example 9

\[ f_{AC \ AC} = \frac{34.56}{AE} \]

Assuming the amount by which the bar is shortened is positive, the compatibility equation for the bar is therefore

\[ 0.5 \text{ in.} = 0 + \frac{34.56}{AE} F_{AC} \]

Realizing that \( f_{AC \ AC} \) is a measure of displacement per unit force, we have

\[ 0.5 \text{ in.} = 0 + \frac{34.56 \text{ ft}(12 \text{ in./ft})}{(0.2 \text{ in}^2)[29(10^6) \text{ lb/in}^2]} F_{AC} \]

Thus,

\[ F_{AC} = 6993 \text{ lb} = 6.99 \text{ k (T)} \quad \text{Ans.} \]

Since no external forces act on the truss, the external reactions are zero. Therefore, using \( F_{AC} \) and analyzing the truss by the method of joints yields the results shown in Fig. 10–16c.
Example 10

The beam shown is supported by a pin at A and two pin-connected bar at B. Determine the force in member BD. Take $E=29\times10^3$, $I=800$ in$^4$ for the beam and $A=3$ in$^2$ for each bar.

$$0 = \Delta_{BD} + F_{BD}f_{BD BD}$$
Example 10

- (c) Diagram with forces and equations:
  - $M = 6.67x - 0.0667x^3$

- (d) Diagram with forces and angles:
  - $m = 0$
Example 10

\[ \Delta_{BD} = \int_0^L \frac{Mm}{EI} \, dx + \sum \frac{nNL}{AE} = \int_0^{10} \frac{(6.67x - 0.0667x^3)(0) \, dx}{EI} + \frac{(-15.40)(-0.816)(6/\cos 30^\circ)(12)}{AE} + \frac{(0)(1)(6/\cos 45^\circ)(12)}{AE} \]

\[ = 0 + \frac{1045.1}{3[29(10^3)]} + 0 = 0.0120 \text{ in.} \]

For \( f_{BD \, BD} \) we require application of a real unit load and a virtual unit load at the cut ends of member \( BD \), Fig. 10–17d. Thus,

\[ f_{BD \, BD} = \int_0^L \frac{m^2 \, dx}{EI} + \sum \frac{n^2L}{AE} = \int_0^{10} \frac{(0)^2 \, dx}{EI} + \frac{(-0.816)^2(6/\cos 30^\circ)(12)}{AE} + \frac{(1)^2(6/\cos 45^\circ)(12)}{AE} = \frac{157.2}{3[29(10^3)]} = 0.001807 \]

Substituting the data into Eq. (1) yields

\[ 0 = \Delta_{BD} + F_{BD} f_{BD \, BD} \]

\[ 0 = 0.0120 + F_{BD}(0.001807) \]

\[ F_{BD} = -6.65 \text{ k} = 6.65 \text{ k (C)} \]
Example 11

The simply supported beam shown in the photo is to be designed to support a uniform load of 2 kN/m. Determine the force developed in member CE. Neglect the thickness of the beam and assume the truss members are pin connected to the beam. Also, neglect the effect of axial compression and shear in the beam. The cross-sectional area of each strut is 400 mm$^2$, and for the beam $I=20(10^6)$mm$^4$. Take $E=200$ GPa
Example 11

\[ 0 = \Delta_{CE} + F_{CE} f_{CECE} \]
Example 11

\[ 0 = \Delta_{CE} + F_{CEF} f_{CECE} \]

\[ M_1 = 6x_1 - x_2^2 \]

\[ M_2 = 6x_2 - x_2^2 \]
Example 11

\[
0 = \Delta_{CE} + F_{CEF}f_{CE}f_{CE}
\]

\[
m_1 = -0.5x_1
\]

\[
m_2 = -0.5x_2 + 0.5(x_2 - 2) = -1
\]
Example 11

\[ 0 = \Delta_{CE} + F_{CEF}f_{CECE} \]

\[ \Delta_{CE} = \int_0^L \frac{Mm}{EI} \, dx + \sum_{} \frac{nNL}{AE} = 2 \int_0^2 \frac{(6x_1 - x_1^2)(-0.5x_1) \, dx_1}{EI} \]

\[ + 2 \int_2^3 \frac{(6x_2 - x_2^2)(-1) \, dx_2}{EI} + 2 \left( \frac{(1.118)(0)(\sqrt{5})}{AE} \right) \]

\[ + 2 \left( \frac{(-0.5)(0)(1)}{AE} \right) + \left( \frac{1(0)2}{AE} \right) \]

\[ = -\frac{12}{EI} - \frac{17.33}{EI} + 0 + 0 + 0 \]

\[ = \frac{-29.33(10^3)}{200(10^9)(20)(10^{-6})} = -7.333(10^{-3}) \, m \]
Example 11

\[ f_{CECE} = \int_0^L \frac{m^2 dx}{EI} + \sum \frac{n^2 L}{AE} = 2 \int_0^2 \frac{(-0.5x_1)^2 dx_1}{EI} + 2 \int_2^3 \frac{(-1)^2 dx_2}{EI} + 2\left( \frac{(1.118)^2(\sqrt{5})}{AE} \right) + 2\left( \frac{(-0.5)^2(1)}{AE} \right) + \left( \frac{(1)^2(2)}{AE} \right) \]

\[ = \frac{1.3333}{EI} + \frac{2}{EI} + \frac{5.590}{AE} + \frac{0.5}{AE} + \frac{2}{AE} \]

\[ = \frac{3.333(10^3)}{200(10^9)(20)(10^{-6})} + \frac{8.090(10^3)}{400(10^{-6})(200(10^9))} \]

\[ = 0.9345(10^{-3}) \text{ m/kN} \]

\[ 0 = \Delta_{CE} + F_{CE} f_{CECE} \]

\[ 0 = -7.333(10^{-3}) \text{ m} + F_{CE}(0.9345(10^{-3}) \text{ m/kN}) \]

\[ F_{CE} = 7.85 \text{ kN} \]
Symmetric Structures
Antisymmetric Structures
Transformation of Loading

The diagram illustrates the transformation of loading from a single concentrated load to a combination of symmetric and antisymmetric loading. The initial configuration shows a single load of 8 kN and a distributed load of 2 kN/m. This is transformed into symmetric loading with 4 kN and 1 kN/m on the left and right, respectively, and antisymmetric loading with 4 kN and 1 kN/m on the left and right, respectively.
Influence Lines for Statically Indeterminate Beams

Reaction at A

- Actual beam
- Primary structure
- Redundant $A_y$ applied
Influence Lines for Statically Indeterminate Beams

Reaction at A

\[ A_y = \left( \frac{1}{f_{AA}} \right) f_{DA} \]

Scale Factor \( 1/f_{AA} \)
Influence Lines for Statically Indeterminate Beams

Shear at E

\[ V_E = \left( \frac{1}{f_{EE}} \right) f_{DE} \]

Scale Factor \( 1/f_{EE} \)
Influence Lines for Statically Indeterminate Beams

Moment at E

\[ M_E = \left( \frac{1}{\alpha_{EE}} \right) f_{DE} \]

Scale Factor \( 1/\alpha_{EE} \)
Example 11

Draw the influence line for the vertical reaction at $A$ for the beam in Fig. 10–25a. $EI$ is constant. Plot numerical values every 6 ft.
Example 11

real beam
(b)

qualitative influence line
for reaction at A
(c)

conjugate beam
(d)
Example 11

For $B'$, since no moment exists on the conjugate beam at $B'$, Fig. 10–25d, then

$$\Delta_B = M_{B'} = 0$$

For $D'$, Fig. 10–25e:

$$\Sigma M'_{D'} = 0; \quad \Delta_D = M_{D'} = \frac{162}{EI} (6) - \frac{1}{2} \left( \frac{6}{EI} \right) (6)(2) = \frac{936}{EI}$$

For $C'$, Fig. 10–25f:

$$\Sigma M'_{C'} = 0; \quad \Delta_C = M_{C'} = \frac{162}{EI} (12) - \frac{1}{2} \left( \frac{12}{EI} \right) (12)(4) = \frac{1656}{EI}$$

For $A'$, Fig. 10–27d:

$$\Delta_A = M_{A'} = \frac{1944}{EI}$$
Example 11

Since a vertical 1-k load acting at A on the beam in Fig. 10–25a will cause a vertical reaction at A of 1 k, the displacement at A, $\Delta_A = 1944/EI$, should correspond to a numerical value of 1 for the influence-line ordinate at A. Thus, dividing the other computed displacements by this factor, we obtain

<table>
<thead>
<tr>
<th>$x$</th>
<th>$A_y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>1</td>
</tr>
<tr>
<td>$C$</td>
<td>0.852</td>
</tr>
<tr>
<td>$D$</td>
<td>0.481</td>
</tr>
<tr>
<td>$B$</td>
<td>0</td>
</tr>
</tbody>
</table>

A plot of these values yields the influence line shown in Fig. 10–25g.
Example 12

Draw the influence line for the shear at $D$ for the beam in Fig. 10–26a. $EI$ is constant. Plot numerical values every 9 ft.
Example 12

Since there is a discontinuity of moment at \( D' \), the internal moment just to the left and right of \( D' \) will be computed. Just to the left of \( D' \), Fig. 10–26f, we have

\[
\Sigma M_{D'_L} = 0; \quad \Delta_{D_L} = M_{D'_L} = \frac{40.5}{EI} (3) - \frac{270}{EI} (9) = -\frac{2308.5}{EI}
\]

Just to the right of \( D' \), Fig. 10–26g, we have

\[
\Sigma M_{D'_R} = 0; \quad \Delta_{D_R} = M_{D'_R} = \frac{40.5}{EI} (3) - \frac{270}{EI} (9) + \frac{3888}{EI} = \frac{1579.5}{EI}
\]

From Fig. 10–26e,

\[
\Delta_A = M_{A'} = 0 \quad \Delta_B = M_{B'} = 0 \quad \Delta_C = M_{C'} = 0
\]

For point \( E \), Fig. 10–26b, using the method of sections at the corresponding point \( E' \) on the conjugate beam, Fig. 10–26h, we have

\[
\Sigma M_{E'} = 0; \quad \Delta_E = M_{E'} = \frac{40.5}{EI} (3) - \frac{54}{EI} (9) = -\frac{364.5}{EI}
\]
Example 12

The ordinates of the influence line are obtained by dividing each of the above values by the scale factor $M_D = 3888/EI$. We have

<table>
<thead>
<tr>
<th>$x$</th>
<th>$V_D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>0</td>
</tr>
<tr>
<td>$D_L$</td>
<td>-0.594</td>
</tr>
<tr>
<td>$D_R$</td>
<td>0.406</td>
</tr>
<tr>
<td>$B$</td>
<td>0</td>
</tr>
<tr>
<td>$E$</td>
<td>-0.0938</td>
</tr>
<tr>
<td>$C$</td>
<td>0</td>
</tr>
</tbody>
</table>

A plot of these values yields the influence line shown in Fig. 10–26i.
Example 13

Draw the influence line for the moment at $D$ for the beam in Fig. 10–27a. $EI$ is constant. Plot numerical values every 9 ft.
Example 13

Real beam (b)

Qualitative influence line for moment at D (c)

(d)
Example 13

For point $D'$, Fig. 10–27e:

$$\Sigma M_{D'} = 0; \quad \Delta_D = M_{D'} = \frac{4.5}{EI} (3) + \frac{18}{EI} (9) = \frac{175.5}{EI}$$

For point $E'$, Fig. 10–27f:

$$\Sigma M_{E'} = 0; \quad \Delta_E = M_{E'} = \frac{4.5}{EI} (3) - \frac{6}{EI} (9) = -\frac{40.5}{EI}$$
Example 13

The angular displacement $\alpha_{DD}$ at $D$ of the "real beam" in Fig. 10–27c is defined by the reaction at $D'$ on the conjugate beam. This factor, $D'_y = 48/EI$, is divided into the above values to give the ordinates of the influence line, that is,

<table>
<thead>
<tr>
<th>$x$</th>
<th>$M_D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>0</td>
</tr>
<tr>
<td>$D$</td>
<td>3.656</td>
</tr>
<tr>
<td>$B$</td>
<td>0</td>
</tr>
<tr>
<td>$E$</td>
<td>$-0.844$</td>
</tr>
<tr>
<td>$C$</td>
<td>0</td>
</tr>
</tbody>
</table>

A plot of these values yields the influence line shown in Fig. 10–27g.
Live Load Pattern in Continuous Beams
Support Reactions

Influence Line for positive reaction at support 1

Load pattern for maximum positive reaction at support 1
Support Negative Moment

Influence line for negative moment at support 2

Load pattern for maximum negative moment at support 2
Span Positive Moment

Influence line for positive moment at 7

Load pattern for maximum positive moment at 7
Internal Shear

Influence line for positive shear at 7

Load pattern for maximum positive shear at 7
Live Load Pattern for Three Span Beam