Noetherian and Artinian Modules and Rings

Artinian and Noetherian rings have some measure of finiteness associated with them. In fact, the conditions for Artinian and Noetherian rings, called respectively the descending and ascending chain conditions, are often termed the minimum and maximum conditions. These properties make Artinian and Noetherian rings of interest to an algebraist. Furthermore, these two types of rings are related.

**Definition 1.** Let $M$ be an $R$-module, and suppose that we have an increasing sequence of submodules $M_1 \subseteq M_2 \subseteq M_3 \subseteq \ldots$, or a decreasing sequence $M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots$. We say that the sequence stabilizes if for some $t$, $M_t = M_{t+1} = M_{t+2} = \ldots$.

The question of stabilization of sequences of submodules appears in a fundamental way in many areas of abstract algebra and its applications.

**Definition 2.** The module $M$ is said to satisfy the ascending chain condition (acc) if every increasing sequence of submodules stabilizes; $M$ satisfies the descending chain condition (dcc) if every decreasing sequence of submodules stabilizes.

**Definition 3.** An $R$-module $M$ is called Artinian if descending chain conditions (dcc) hold for submodules of $M$. 
Definition 4. An $R$-module $M$ is called Noetherian if ascending chain condition (acc) holds for submodules $M$.

Proposition 1. The following conditions on $M$ are equivalent:
(i) $M$ is Artinian;
(ii) Every nonempty collection of submodules of $M$ has a minimal element.
Proof. (i) $\rightarrow$ (ii) Let $F = \{ M_i \mid i \in I \}$ be a nonempty family of submodules of $M$. Pick any $i_1 \in I$. If $M_{i_1}$ is a minimal element of $F$, we are done. If not, there exists $i_2 \in I$ such that $M_{i_2} \subset M_{i_1}$. If $M_{i_2}$ is minimal, we are done. Else $\exists i_3 \in I$ such that $M_{i_3} \subset M_{i_2}$. If $F$ does not contain a minimal element, we obtain an infinite strictly descending chain $M_{i_1} \supset M_{i_2} \supset M_{i_3} \supset \ldots$ which is a contradiction as $M$ is Artinian.
(ii) $\rightarrow$ (i) Consider a descending chain of submodules of $M$, $M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots$. Let $F = \{ M_i \mid i \in I \}$, then $F$ has a minimal element say, $M_t$. Now $M_t = M_{t+1} = M_{t+2} = \ldots$. Hence $M$ is Artinian.

Corollary 1. Every non-zero Artinian module contains a minimal submodule.
Proof. Let $M$ be a non-zero Artinian $R$-module. Let $F$ be a family of all proper submodules of $M$. Then $(0) \in F$ implies $F \neq \emptyset$. Then $F$ has a minimal
element say, N. Clearly N is a minimal submodule of M.

Proposition 2.
The following conditions on an R-module M are equivalent:
(1) M is a a Noetherian module;
(2) Every nonempty collection of submodules of M has a maximal element (with respect to inclusion).

Proof. Assume (1), and let S be a nonempty collection of submodules. Choose \( M_1 \in S \).
If \( M_1 \) is maximal, we are finished; otherwise we have \( M_1 \subseteq M_2 \) for some \( M_2 \in S \). If we continue inductively, the process must terminate at a maximal element; otherwise the acc would be violated.
Conversely, assume (2), and let \( M_1 \subseteq M_2 \subseteq \ldots \).
The sequence must stabilize; otherwise \( \{M_1, M_2, \ldots \} \) would be a nonempty collection of submodules with no maximal element.

There is another equivalent condition in the Noetherian case.

Proposition 3.
M is Noetherian if and only if every submodule of M is finitely generated.

Proof. \( \Leftarrow \) If the sequence \( M_1 \subseteq M_2 \subseteq \ldots \) does not stabilize, let \( N = \bigcup_{r=1}^{\infty} M_r \). Then N is a submodule of M, and it cannot be finitely
generated. For if \( x_1, \ldots, x_s \) generate \( N \), then for sufficiently large \( t \), all the \( x_i \) belong to \( M_t \). But then \( N \subseteq M_t \subseteq M_{t+1} \subseteq \cdots \subseteq N \), so \( M_t = M_{t+1} = \cdots \).

Conversely, assume that the acc holds, and let \( N \) be a submodule of \( M \). If \( N \neq 0 \), choose \( x_1 \in N \). If \( Rx_1 = N \), then \( N \) is finitely generated. Otherwise, there exists \( x_2 \in N - Rx_1 \). If \( x_1 \) and \( x_2 \) generate \( N \), we are finished. Otherwise, there exists \( x_3 \in N - (Rx_1 + Rx_2) \). Since \( Rx_1 \subseteq Rx_1 + Rx_2 \subseteq Rx_1 + Rx_2 + Rx_3 \subseteq \cdots \), the acc forces the process to terminate at some stage \( t \), in which case \( x_1, \ldots, x_t \) generate \( N \).

**Definition 5**
A ring \( R \) is Noetherian [resp. Artinian] if it is Noetherian [resp. Artinian] as a module over itself. If we need to distinguish between \( R \) as a left, as opposed to right, \( R \)-module, we will refer to a *left Noetherian* and a *right Noetherian* ring, and similarly for Artinian rings.

**Examples**
1. Every PID is Noetherian. This follows since every ideal is generated by a single element.

2. \( \mathbb{Z} \) is Noetherian (a special case of Example 1) but not Artinian. There are many descending chains of ideals that do not stabilize, e.g., \( \mathbb{Z} \supseteq (2) \supseteq (4) \supseteq (8) \supseteq \ldots \).
3. If $F$ is a field, then the polynomial ring $F[X]$ is Noetherian (another special case of Example 1) but not Artinian. A descending chain of ideals that does not stabilize is $(X) \supset (X^2) \supset (X^3) \supset \ldots$.

4. The ring $F[X_1, X_2, \ldots]$ of polynomials over $F$ in infinitely many variables is neither Artinian nor Noetherian. A descending chain of ideals that does not stabilize is constructed as in Example 3, and an ascending chain of ideals that does not stabilize is $(X_1) \subset (X_1, X_2) \subset (X_1, X_2, X_3) \subset \ldots$.

5. A module with only finitely many submodules is Artinian and Noetherian. In particular, finite abelian groups are both Artinian and Noetherian over $\mathbb{Z}$.

6. Let $V$ be a finite dimensional vector space over a field $F$, say, $\dim V = n$. Then $V$ is both Artinian as well as Noetherian $F$-module. For, if $W$ is a proper subspace of $V$, then $\dim W < n$. Thus any proper ascending or descending chain of subspace of $V$ can have at most $n + 1$ terms. However, infinite dimensional vector spaces are not Artinian. For, suppose $V$ is infinite dimensional vector space over the field $K$ with basis $B$. Let $u_i \in B$, $\forall i \in \mathbb{N}$ be distinct elements in $B$. Define $U_n = \langle u_n, u_{n+1}, \ldots \rangle$ as a subspace of $V$. Then $U_1 \supset U_2 \supset \cdots$ is an infinite descending chain of subspaces.
7- Q has only two ideals, (0) and Q. Clearly, Q ⊃ (0). This finite chain of ideals clearly bottoms out. This is the only possible chain of proper ideal inclusions in Q since Q only has two ideals. Thus, Q is Artinian. The field of rational numbers provides us with an example of an Artinian ring. It should be noted that by the same logic as in the preceding paragraph, Q is also Noetherian.

**Remark** The following observations will be useful in deriving properties of Noetherian and Artinian modules. If N ⊆ M, then a submodule L of M that contains N can always be written in the form K+N for some submodule K. (K = L is one possibility.) By the correspondence theorem, (K1 + N)/N = (K2 + N)/N implies K1 + N = K2 + N and (K1 + N)/N ⊆ (K2 + N)/N implies K1 + N ⊆ K2 + N.

**Proposition 4.** Submodules Noetherian [resp. Artinian] modules are Noetherian [resp. Artinian].

**Proof.** Assume M is Noetherian and let N be its submodule. Then Any submodule of N is also a submodule of M. Hence acc holds for submodules of N (similarly any descending chain of submodules of N is a chain of submodules of M). Hence the result follows.

**Proposition 5.** If N is a submodule of M, then M is Noetherian [resp. Artinian] if and only if N and M/N are Noetherian [resp. Artinian].
**Proof.** Assume $M$ is Noetherian. Then $N$ is Noetherian. An ascending chain of submodules of $M/N$ looks like $(M_1+N)/N \subseteq (M_2+N)/N \subseteq \cdots$. But then the $M_i+N$ form an ascending sequence of submodules of $M$, which must stabilize. Consequently, the sequence $(M_i+N)/N, i = 1, 2, \ldots$ must stabilize.

Conversely, assume that $N$ and $M/N$ are Noetherian, and let $M_1 \subseteq M_2 \subseteq \cdots$ be an increasing sequence of submodules of $M$. Take $i$ large enough so that both sequences $\{M_i \cap N\}$ and $\{M_i+N\}$ have stabilized. If $x \in M_{i+1}$, then $x+N \in M_{i+1}+N = M_i+N$, so $x = y + z$ where $y \in M_i$ and $z \in N$. Thus $x - y \in M_{i+1} \cap N = M_i \cap N$, and since $y \in M_i$ we have $x \in M_i$ as well. Consequently, $M_i = M_{i+1}$ and the sequence of $M_i$’s has stabilized. The Artinian case is handled by reversing inequalities (and interchanging indices $i$ and $i + 1$ in the second half of the proof).

**Corollary**

If $M_1, \ldots, M_n$ are Noetherian [resp. Artinian] $R$-modules, then so is $M_1 \oplus M_2 \oplus \cdots \oplus M_n$.

**Proof.** It suffices to consider $n = 2$ (induction will take care of higher values of $n$). The submodule $N = M_1$ of $M = M_1 \oplus M_2$ is Noetherian by hypothesis, and $M/N \cong M_2$ (apply the first isomorphism theorem to the natural projection of $M$ onto $M_2$, i.e. $f : M \to M_2$, where $f(a,b) = b$) is also Noetherian.
By **Proposition 5**, \( M \) is Noetherian. The Artinian case is done the same way.

**Proposition.** *In an artinian ring all the prime ideals are maximal.*

**Proof.** Let \( p \) be a prime ideal. We must show that for each element \( f \in A \setminus p \) we have that \( Af + p = A \). Since \( A \) is artinian the chain \( Af + p \supseteq Af^2 + p \supseteq \cdots \) must stabilize. Hence there is a positive integer \( n \) such that \( f^n = g f^{n+1} + h \) for some \( g \in A \) and \( h \in p \). Hence \( f^n(1 - gf) \in p \). Since \( p \) is a prime ideal and \( f \notin p \) we have that \( 1 - gf \in p \). Hence there is an \( e \in p \) such that \( 1 - gf = e \). The ideal \( Af + p \) consequently contains the element \( gf - e = 1 \) and thus is equal to \( A \) is we wanted to prove.
Proposition. Let \( 0 \rightarrow L \overset{\alpha}{\rightarrow} M \overset{\beta}{\rightarrow} N \rightarrow 0 \) be a short exact sequence of \( A \)-modules. Then,

\[ M \text{ is Noetherian } \iff L \text{ and } N \text{ are Noetherian} \]

**Proof.** \((\Rightarrow)\) Given an ascending chain of submodules \( \{L_i\}_{i=1}^{\infty} \) in \( L \), we get ascending chain of submodules \( \{\alpha(L_i)\}_{i=1}^{\infty} \) in \( M \). Since \( M \) is Noetherian, there exists a positive integer \( n \) such that \( \alpha(L_n) = \alpha(L_{n+1}) = \cdots \). Applying \( \alpha^{-1} \) to both sides

\[ \alpha^{-1}(\alpha(L_n)) = \alpha^{-1}(\alpha(L_{n+1})) = \alpha^{-1}(\alpha(L_{n+2})) = \cdots \]

Since \( \alpha \) is injective, \( \alpha^{-1}(\alpha(L_i)) = L_i \) for each \( i \). So we obtain

\[ L_n = L_{n+1} = L_{n+2} = \cdots \]

showing that \( L \) is Noetherian.

Similarly, given an ascending chain of submodules \( \{N_i\}_{i=1}^{\infty} \) in \( N \), we get ascending chain of submodules \( \{\beta^{-1}(L_i)\}_{i=1}^{\infty} \) in \( M \). Since \( M \) is Noetherian, there exists a positive integer \( p \) such that \( \beta^{-1}(N_p) = \beta^{-1}(N_{p+1}) = \cdots \). Applying \( \beta \) to both sides,

\[ \beta(\beta^{-1}(N_p)) = \beta(\beta^{-1}(N_{p+1})) = \beta(\beta^{-1}(N_{p+2})) = \cdots \]

Since \( \beta \) is surjective, \( \beta(\beta^{-1}(N_i)) = N_i \) for each \( i \). So we obtain

\[ N_p = N_{p+1} = N_{p+2} = \cdots \]

showing that \( N \) is Noetherian.
(⇐) Suppose \( \{M_i\}_{i=1}^{\infty} \) is an ascending chain of submodules of \( M \), then identifying \( \alpha(L) \) with \( L \) (which can be done, since \( \alpha \) is injective), and taking intersections, we get a chain of the form

\[
L \cap M_1 \subseteq L \cap M_2 \subseteq \cdots
\]

of submodules in \( L \). Similarly, applying \( \beta \) gives a chain

\[
\beta(M_1) \subseteq \beta(M_2) \subseteq \cdots
\]

of submodules in \( N \). Since \( L \) and \( N \) are Noetherian, each of these chains terminate. To prove that \( M \) is Noetherian, it suffices to prove the following lemma:

**Lemma.** For submodules, \( M_1 \subseteq M_2 \subseteq M \),

\[
\alpha(L) \cap M_1 = \alpha(L) \cap M_2 \quad \text{and} \quad \beta(M_1) = \beta(M_2) \quad \Rightarrow \quad M_1 = M_2
\]

**Proof.** Suppose \( m \in M_1 \). Then, \( \beta(m) \in \beta(M_1) = \beta(M_2) \), so that there is \( n \in M_2 \) such that \( \beta(m) = \beta(n) \). Then, \( \beta(m-n) = 0 \), so that \( m-n \in M_1 \cap \ker(\beta) = M_1 \cap \alpha(L) = M_2 \cap \alpha(L) \). It follows that \( m-n \in M_2 \), so that \( m \in M_2 \). This shows that \( M_1 \subseteq M_2 \). Similarly, we can prove \( M_2 \subseteq M_1 \). Thus, \( M_1 = M_2 \), as desired.
Example 5. Let $M = \left\{ \frac{a}{2^n} | a \in \mathbb{Z}, n \geq 0 \right\}$, and consider $M$ as a $\mathbb{Z}$–module. Note here that $\mathbb{Z} \subseteq M \subseteq \mathbb{Q}$. Also, let $T = M/\mathbb{Z}$. Then $T$ is Artinian but not Noetherian. We’ll show this, and for notational purposes, let $\frac{a}{2^n}$ denote $\frac{a}{2^n} + \mathbb{Z}$.

For $n \geq 0$ let $N_n = \mathbb{Z} \cdot \frac{1}{2^n}$ and notice that $N_n \subseteq T$.

First, we show that $N_n \subseteq N_{n+1}$ which shows that $T$ is not Noetherian as it provides an ascending chain that doesn’t stabilize. Since $\frac{1}{2^n} = 2 \cdot \frac{1}{2^{n+1}}$, then for any $a \in \mathbb{Z}$, we have $a \cdot \frac{1}{2^n} = 2a \cdot \frac{1}{2^{n+1}}$ which gives the containment. To show the containment is strict, suppose that $\frac{1}{2^{n+1}} = a \cdot \frac{1}{2^n}$ for some $a \in \mathbb{Z}$. Then we get $\frac{1}{2^{n+1}} - \frac{a}{2^n} \in \mathbb{Z}$ but that $\frac{1}{2^{n+1}} - \frac{a}{2^n} = \frac{1 - 2a}{2^{n+1}} \notin \mathbb{Z}$ so we have a contradiction, and hence the containment is strict.

Now, we claim that every proper $\mathbb{Z}$–submodule of $T$ is $N_n$ for some $n$. If we can show this, then any descending chain of submodules of $T$ must start with $N_n$ for some $n$, and the only submodules of $N_n$ are $N_i$ where $0 \leq i < n$ and so the chain will stabilize, and we’ll get that $T$ is Artinian. So let $A \subseteq T$ be a proper submodule. Choose $n$ to be the largest integer so that $\frac{1}{2^n} \in A$. Such an element exists as $A$ is a proper submodule, and $\frac{1}{2^0} \in A$. So we claim that $A = N_n$. We clearly have that $N_n \subseteq A$, so let $\frac{a}{2^m} \in A$ and suppose that $\frac{a}{2^m} \notin N_n$. Without loss of generality, we may assume that $a$ is odd, and we also have that $m \geq n + 1$. Since $\gcd(a, 2^m) = 1$, then there exist integers $x, y$ so that $ax + 2^m y = 1$. Hence, $x \cdot \frac{a}{2^m} + y = \frac{1}{2^m}$. Modding out by $Z$ gives $x \cdot \frac{a}{2^m} = \frac{1}{2^m}$ and since $\frac{a}{2^m} \in A$ this gives that $\frac{1}{2^m} \in A$. This is a contradiction as $m \geq n + 1$ and $n$ was chosen to be maximal. Hence, $A = N_n$.\footnote{Note: The specific example given here is a classic example of a module that is Artinian but not Noetherian, illustrating the distinction between these two module properties.}
**Proposition**. A quotient ring of an Artinian (Noetherian) ring is Artinian (Noetherian).

**Proof.** We prove for Artinian rings. The Noetherian case follows on similar lines.

Let $R$ be an Artinian ring and $I$ be an ideal of $R$. Then $R/I$ is an Artinian $R$-module (as seen in the module case).

Now any left ideal of $R/I$ is of the form $J/I$, where $J$ is a left ideal of $R$. Hence a descending chain of left ideals of $R/I$ gives descending chain of left ideals of $R$. Hence $R/I$ is an Artinian ring.

**Remark**. Subring of Artinian or Noetherian may not be Artinian or Noetherian. For example, the subring $\mathbb{Z}$ of the Artinian ring $\mathbb{Q}$ is not Artinian.

**Proposition**. A finitely generated module over an Artinian (Noetherian) ring is Artinian (Noetherian).

**Proof.** Let $R$ be a ring and $M = \langle x_1, x_2, \ldots, x_n \rangle$, be a finitely generated $R$-module. Let $\psi : R^n \to M$ be defined by $\psi(r_1, r_2, \ldots, r_n) = \sum_{i=1}^{n} r_i x_i$. Then it is easy to see that $\psi$ is an onto $R$-module homomorphism.

Thus $R^n / \text{ker}\psi \cong M$. Now if $R$ is Artinian or Noetherian, so is $R^n / \text{ker}\psi$. Hence so is $M$. 
**Corollary.** Matrix rings over Artinian (Noetherian) rings are Artinian (Noetherian).

Proof. Let $R$ be a ring and $S = M_n(R)$. Then $S$ is finitely generated as an $R$-module, generated by $n^2$ elements. If $R$ is Artinian or Noetherian, then by previous result, so is $S$ as an $R$-module. Now any left ideal of $S$ is also a left $R$-module. Hence every chain of left ideals is a chain of left $R$-submodules of $S$. Thus if $R$ is Artinian (Noetherian) then $S$ is also Artinian (Noetherian) ring.

**Theorem.** Let $R$ be an Artinian ring. Then
(i) Every non-zero divisor in $R$ is a unit.
(ii) If $R$ is commutative, then every prime ideal is maximal. ( another prof )

Proof. (i) Let $r \in R$ be a non-zero divisor. Then clearly $r^n$ is also a non-zero divisor for any positive integer $n$. Let $(r^n)$ denote the left ideal of $R$ generated by $r^n$ for some positive integer $n$. Then the descending chain $(r) \supseteq (r^2) \supseteq ... \supseteq (r^n) \supseteq ...$ must terminate, as $R$ is Artinian.

Let $(r^m) = (r^{m+1}) = ...$ for some positive integer $m$.
Now, $r^m \in (r^{m+1})$ implies $r^m = sr^{m+1}$ for some $s \in R$. Thus $(1 - sr) r^m = 0$ implies $1 - sr = 0$ implies $sr = 1$ as $r^m$ is a non-zero divisor. Hence $r$ has a left inverse. Now $r = rsr = (rs)r$ implies $(1 - rs)r = 0$ implies $rs = 1$. Thus $r$ is a unit with $r^{-1} = s$.

(ii) Suppose $R$ is a commutative Artinian ring. Let $P$ be a prime ideal of $R$. Then $R/P$ is an Artinian
Integral domain. By (i) every non-zero element of $R/P$ is invertible. Hence $R/P$ is a field and thus $P$ is maximal.

In particular, we obtain that every Artinian integral domain is a field.

In commutative algebra, polynomial rings form an important class of examples of commutative rings, both because they are interesting in their own right and because of their connections to algebraic geometry.

If we have a general commutative ring $R$, we can form a ring of polynomials with coefficients in $R$ using point-wise addition and multiplication. We denote it by $R[X]$.

A natural question is of course: “Which properties of $R$ are preserved in $R[X]$?” We can show that if $R$ is an integral domain, then so is $R[X]$. On the other hand, even if $R$ is a field, $R[X]$ is not a field, since almost none of its elements are invertible. It is an Euclidean domain, however, which is a weaker but still a useful result.

These two examples demonstrate that the question we formulated is indeed interesting; the properties are not inherited in general, but sometimes they might be. And sometimes they are not. And sometimes we can infer something weaker, but still relevant.
Hilbert’s basis theorem is one particular example of such statement. It says that if $R$ is Noetherian, then so is $R[X]$. Given that most of the rings we deal with (at least in the initial stages of commutative algebra) are Noetherian, this is a powerful result. An easy consequence of the theorem is that the polynomials over a PID are not necessarily a PID (that is not every ideal is guaranteed be generated by one element), but they are Noetherian (ideals generated by finitely many elements).

**Theorem** (Hilbert basis theorem).
Let $R$ be a Noetherian ring. Then $R[x]$ is also Noetherian.

**Proof:** Let $J$ be a non-trivial ideal of $R[x]$ and $m$ the least degree of a non-zero polynomial in $J$. Then for $n \geq m$ define:

$I_n = \{a \in R \mid a$ is the leading coefficient of an $n$-th degree polynomial in $J\} \cup \{0\}$. It is a routine to check that the $I_n$’s are ideals of $R$ and that $I_n \subseteq I_{n+1}$, because if $a \in I_n$, then there exists a polynomial $f(x)$ of degree $n$ and leading coefficient $a$. So $xf(x) \in J$ is of degree $n+1$ and leading coefficient $a$. Thus $a \in I_{n+1}$. Since $R$ is a Noetherian ring, each of the $I_n$ is finitely generated and there exists $k \in \mathbb{N}$ such that $I_n = I_k$ for $n \geq k$. For each $n$ with $m \leq n \leq k$, let $A_n$ be a finite set of polynomials of degree $n$ such that their leading coefficients generate $I_n$. Let $A = \bigcup A_n$. Then $A$ is a finite set
and we will show that it generates $J$. We will use induction on the degree of a polynomial in $J$.

If $\deg p(x) = m$ (nothing smaller is possible for a non-zero polynomial, since $m$ is the least degree of a non-zero polynomial in $J$), then there are $q_i$’s in $A_m$ and $a_i$’s in $R$ such that the leading coefficient of $p(x)$ coincides with the leading coefficient of $\sum a_i q_i(x)$. This means that $p(x) - \sum a_i q_i(x)$ has degree strictly less than $m$, which implies that it is the zero polynomial and our induction is complete for $m$.

Now, assuming the claim for all naturals between $m$ and $n$ we are going to check it for $n + 1$. If $n + 1 \leq k$ and $\deg p(x) = n + 1$, then there exist $q_i(x)$ in $A_{n+1}$ and $a_i \in R$ such that $p(x) - \sum a_i q_i(x)$ is of degree less than $n + 1$. This polynomial can now be written in terms of the elements of $A$ by induction hypothesis. On the other hand, if $n + 1 > k$, then there are polynomials of degree $n$, $q_i(x)$ in $J$ and $a_i \in R$ so that the leading coefficient of $p(x)$ coincides with that of $x \sum a_i q_i(x)$. Thus the difference $p(x) - x \sum a_i q_i(x)$ is in $J$ and has degree less than $n + 1$. The inductive hypothesis applied both on the $q_i(x)$ and $p(x) - x \sum a_i q_i(x)$ concludes the proof.

Next result proves that the converse of Hilbert basis theorem is satisfied.
Remark: Let $R[x]$ be a Noetherian ring. Then is also Noetherian.

Proof. Suppose $R[X]$ is Noetherian. Then $h : R[X] \to R$ given by $h(f(x)) = f(0)$ is an onto ring homomorphism. Also $\ker h = \{ f(X) \in R[X] \mid f(0) = 0 \} =$ all polynomials with constant term $0 = (x)$. By first isomorphism theorem $\frac{R[x]}{(x)} \cong R$. Thus $R$ is Noetherian.

Definition. An ideal $I$ of a ring $R$ is called nil if for every $a \in I$, there exists a natural number $n$ such that $a^n = 0$. This $n$ is dependent on the element $a$.

Definition. An ideal $I$ of a ring $R$ is called nilpotent if there exists a natural number $n$ such that $I^n = (0)$. Here $I^n = \{ \sum_{finite} a_{i1} a_{i2} \ldots a_{in} \mid a_{it} \in I, t = 1, 2, \ldots, n \}$.

Definition. An element $x$ of a ring $R$ is said to be nilpotent if there exists a natural number $n$ such that $x^n = 0$. Thus an ideal $I$ of the ring $R$ is nil if every element in $I$ is nilpotent.

Remark. Clearly, every nilpotent ideal is nil. However, the converse need not hold as shown in the following example.
Example. Let \( R = \bigoplus_{n=1}^{\infty} \frac{\mathbb{Z}}{(p^n)} \), where \( p \) is any prime.

Then \( R \) is a commutative ring. Let \( I \) be the set of all nilpotent elements of \( R \). Clearly, \( I \) is an ideal of \( R \), as \( R \) is commutative. Also \( I \) is a nil ideal.

Suppose \( I \) is nilpotent. Then there exists a natural number \( m \) such that \( I^m = (0) \). Let \( r = (0 + (p), 0 + (p^2), \ldots, 0 + (p^m), p + (p^{m+1}), 0 + (p^{m+2}), \ldots) \)

Then \( r^{m+1} = 0 \), but \( r^m \neq 0 \). This is contradiction as \( r \in I \) and \( r^m \neq 0 \). Hence \( I \) is nil but not nilpotent.

The following result gives the condition that makes a nil ideal nilpotent.

**Theorem.** A nil left ideal of an Artinian ring is nilpotent.

**Proof.** Let \( J \) be a nil left ideal of an Artinian ring \( R \). Suppose \( J \) is not nilpotent. Consider \( F = \{ J, J^2, J^3, \ldots \} \). Then \( (0) \) is not an element of \( F \). Since \( R \) is Artinian, \( F \) has a minimal element, say, \( I = J^n \).

Now \( I^2 = J^{2n} \subseteq J^n = I \) and \( I^2 \in F \). Hence by minimality of \( I \), \( I^2 = I \).

Now consider the family \( S = \{ L \mid L \) is a left ideal of \( R, L \subseteq I \) and \( IL \neq (0) \} \)

Now \( I^2 = J^{2n} \neq (0) \), we have \( I \in S \). Let \( A \) be a minimal element of \( S \). Then \( IA \neq (0) \). This implies there exists \( a \in A \) such that \( IA \neq (0) \).

Since \( I(IA) = I^2a = Ia \neq (0) \), then \( Ia \in S \). By minimality of \( A \) we get \( Ia = A \). This implies that there exists \( x \in I \) such that \( xa \neq a \). Thus \( x^\prime a = a \) for
each \( i \geq 0 \). As \( J \) is nil, and \( I = J^n \), we get \( I \) is nil. Thus \( x \) is nilpotent, which gives \( a = 0 \), a contradiction. Therefore, \( J \) is nilpotent.