Tensor Products of Modules

The construction of the concept of a tensor product of two modules yields an additive abelian group that is unique up to isomorphism. The assumption that the expression “R-module” means right R-module. Let M be an R-module, N a left R-module and G an additive abelian group. Then

a mapping \( \rho : M \times N \to G \) is said to be \( R\)-balanced if

\[
\rho(x_1 + x_2, y) = \rho(x_1, y) + \rho(x_2, y),
\]

\[
\rho(x, y_1 + y_2) = \rho(x, y_1) + \rho(x, y_2)
\]

and

\[
\rho(ax, y) = \rho(x, ay)
\]

for all \( x, x_1, x_2 \in M, y, y_1, y_2 \in N \) and \( a \in R \). (\( R\)-balanced mappings are also called bilinear mappings.)

R-balanced mappings play a central role in the development of the tensor product of modules.

**Definition** If M is an R-module and N is a left R-module, then an additive abelian group T together with an \( R\)-balanced mapping

\[
\rho : M \times N \to T
\]
is said to be a tensor product of M and N, if whenever G is an additive abelian group and

\[ \rho' : M \times N \to G \]

is an R-balanced mapping, there is a unique group homomorphism

\[ f : T \to G \]

that completes the diagram

\[
\begin{array}{ccc}
M \times N & \xrightarrow{\rho'} & G \\
\downarrow{\rho} & & \downarrow{f} \\
T & & \\
\end{array}
\]

commutatively. The map \( \rho \) is called the canonical R-balanced map from \( M \times N \) to T. A tensor product of \( M_R \) and \( R_N \) will be denoted by \( (T, \rho) \).

**Proposition 2.3.2.** If a tensor product \( (T, \rho) \) of \( M_R \) and \( R_N \) exists, then T is unique up to group isomorphism.

**Proof.** Let \( (T, \rho) \) and \( (T', \rho') \) be tensor products of M and N. Then there are group homomorphisms
and \( f' : T' \to T \) such that the diagram

\[
\begin{array}{ccc}
M \times N & \xrightarrow{\rho'} & T' \\
\downarrow{\rho} & & \downarrow{f'} \\
T & \xrightarrow{f} & T'
\end{array}
\]

is commutative. But \( \text{id}_T : T \to T \) makes the outer triangle commute, so the uniqueness of \( \text{id}_T \) gives \( f'f = \text{id}_T \). Similarly, \( ff' = \text{id}_{T'} \) and so \( f \) is a group isomorphism.

Our next task is to show that tensor products always exist.

**Proposition** Every pair of modules \( M_R \) and \( N_R \) has a tensor product.

**Proof.** Consider the free \( \mathbb{Z} \)-module \( F = \mathbb{Z}^{(M \times N)} \) on \( M \times N \).

Since \( F \) is free, we can write

\[
F = \left\{ \sum n_{(x,y)}(x,y) \mid n_{(x,y)} \in \mathbb{Z}, (x,y) \in M \times N \right\}.
\]

and almost all \( n_{(x,y)} = 0 \)
Now let $H$ be the subgroup of $F$ generated by elements of the form

$$(1) \quad (x_1 + x_2, y) - (x_1, y) - (x_2, y),$$

$$(2) \quad (x, y_1 + y_2) - (x, y_1) - (x, y_2) \quad \text{and}$$

$$(3) \quad (xa, y) - (x, ay),$$

where $x, x_1, x_2 \in M$, $y, y_1, y_2 \in N$ and $a \in R$. If $\rho : M \times N \to F/H$ is such that $\rho((x, y)) = (x, y) + H$, then $\rho$ is $R$-balanced and we claim that $(F/H, \rho)$ is a tensor product of $M$ and $N$. Suppose that $\rho' : M \times N \to G$ is an $R$-balanced mapping. Since

$M \times N$ is a basis for $F$, $\rho'$ can be extended uniquely to a group homomorphism $g : F \to G$. But $\rho'$ is $R$-balanced, so $g(H) = 0$. Hence, there is an induced group homomorphism $f : F \to G$ such that the diagram
is commutative. Since \( g \) is unique, \( f \) is unique, so \((F/H, \rho)\) is a tensor product of \( M \) and \( N \), as asserted. \( \square \)

The additive abelian group \( F/H \), constructed in the preceding proof, will now be denoted by \( M \otimes_R N \) and we will call \( M \otimes_R N \) the tensor product of \( M \) and \( N \).

If the cosets \((x, y) + H\) in \( M \otimes_R N \) are denoted by \( x \otimes y \), then

\[(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y,\]
\[x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2 \quad \text{and} \]
\[xa \otimes y = x \otimes ay\]

for all \( x, x_1, x_2 \in M, y, y_1, y_2 \in N \) and \( a \in R \). Under this notation the canonical \( R \)-balanced mapping \( \rho : M \times N \to M \otimes_R N \) is now given by \( \rho((x, y)) = x \otimes y \). 

\[\]
In general, the additive abelian group $M \otimes_R N$ is neither a left nor a right $R$-module. It is simply a $\mathbb{Z}$-module. Recall that if $R$ and $S$ are rings, then an additive abelian group $M$ that is a left $R$-module and an $S$-module is said to be an $(R, S)$-bimodule, denoted by $R M S$ if $(ax)b = a(xb)$ for all $a \in R$, $x \in M$ and $b \in S$. If we are given an $(R, S)$-bimodule $R M S$ and a left $S$-module $S N$, then $M \otimes_R N$ is a left $R$-module under the operation $a(x \otimes y) = ax \otimes y$ for all $a \in R$ and $x \otimes y \in M \otimes_R N$. Likewise, given an $(R, S)$-bimodule $R N S$ and an $R$-module $M$, then $M \otimes_R N$ is an $S$-module via $(x \otimes y)b = x \otimes yb$. We also point out that not every element of $M \otimes_R N$ can be written as $x \otimes y$. The set $\{x \otimes y\}_{(x, y) \in M \times N}$ is a set of generators of $M \otimes_R N$, so a general element of $M \otimes_R N$ is written as $\sum_{i=1}^{m} n_i (x_i \otimes y_i)$, where $n_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, m$. If $0_M, 0_N$ and $0_R$ denote the additive identities of $M, N$ and $R$ respectively, then for $x \in M$ and $y \in N$

$$(x \otimes y) + (0_M \otimes 0_N) = (x \otimes y) + (0_M \otimes 0_R y) = (x \otimes y) + (0_M 0_R \otimes y)$$

$$= (x \otimes y) + (0_M \otimes y) = (x + 0_M) \otimes y = x \otimes y.$$

Simplifying notation, it follows that
\[(0 \otimes 0) + (x \otimes y) = (x \otimes y) + (0 \otimes 0) = x \otimes y,\]

so \(0 \otimes 0\) is the additive identity of \(M \otimes_R N\). Similarly,

\[x \otimes 0 = 0 \otimes y = 0 \otimes 0\]

for any \(x \in M\) and \(y \in N\).

**Remark.** Care must be taken when attempting to define a \(\mathbb{Z}\)-linear mapping with domain \(M \otimes_R N\) by specifying the image of each element of \(M \otimes_R N\). For example, if \(f : M \to M'\) is an \(R\)-linear mapping, then one may be tempted to define a group homomorphism \(g : M \otimes_R N \to M' \otimes_R N\) by setting \(g(\sum_{i=1}^{m} n_i (x_i \otimes y_i)) = \sum_{i=1}^{m} n_i (f(x_i) \otimes y_i)\). As we will see, this map actually works, but when \(g\) is specified
in this manner it is difficult to show that it is well defined. This difficulty can be avoided by working with an $R$-balanced mapping and going through an intermediate step. To see this, consider the commutative diagram

\[
\begin{array}{ccc}
M \times N & \xrightarrow{\rho} & M \otimes_R N \\
\downarrow f \times \text{id}_N & h & \downarrow g \\
M' \times N & \xrightarrow{\rho'} & M' \otimes_R N
\end{array}
\]

where $\rho$ and $\rho'$ are the canonical $R$-balanced mappings. Since the map $h = \rho'(f \times \text{id}_N)$ is an $R$-balanced map, the existence of the group homomorphism $g$ displayed in the diagram is guaranteed by the definition of the tensor product $M \otimes_R N$. Thus,

\[
g(x \otimes y) = \rho'(f \times \text{id}_N)((x, y)) = \rho'(f(x), y) = f(x) \otimes y,
\]

so

\[
g\left(\sum_{i=1}^{m} n_i (x_i \otimes y_i)\right) = \sum_{i=1}^{m} n_i (f(x_i) \otimes y_i) \quad \text{as expected.}
\]
**Proposition** If $M$ is an $R$-module, then $M \otimes_R R \cong M$ as $R$-modules.

**Proof.** If $\rho' : M \times R \to M$ is defined by $\rho'(x, a) = xa$, then $\rho'$ is an $R$-balanced mapping. Thus, there is a unique group homomorphism $f : M \otimes_R R \to M$ such that $f \rho = \rho'$, where $\rho : M \times R \to M \otimes_R R$ is the canonical $R$-balanced map. Hence, we see that $f(x \otimes a) = xa$ for every generator $x \otimes a$ of $M \otimes_R R$. Note that if $b \in R$, then

$$f((x \otimes a)b) = f(x \otimes ab) = x(ab) = (xa)b = f(x \otimes a)b,$$

so $f$ is $R$-linear. Now define $f' : M \to M \otimes_R R$ by $f'(x) = x \otimes 1$. Then $f'$ is clearly well defined and additive. Furthermore,

$$f'(xa) = xa \otimes 1 = x \otimes a = (x \otimes 1)a = f'(x)a,$$

so $f'$ is $R$-linear. Since $f'f(x \otimes a) = f'(xa) = xa \otimes 1 = x \otimes a$ for each generator $x \otimes a$ of $M \otimes_R R$, we see that $f'f = \text{id}_{M \otimes_R R}$. Similarly, $ff'(x) = f(x \otimes 1) = x1 = x$, so $ff' = \text{id}_M$. Hence, $f$ is an isomorphism and we have $M \otimes_R R \cong M$. $\square$

The proof of the previous Proposition is clearly symmetrical, so we have $R \otimes_R N \cong N$ for every left $R$-module $N$. 
Proposition

If \( f : M_R \to M'_R \) and \( g : N_R \to N'_R \) are R-linear mappings, then there is a unique group homomorphism

\[
f \otimes g : M \otimes_R N \to M' \otimes_R N'
\]

such that \((f \otimes g)(x \otimes y) = f(x) \otimes g(y)\) for all \(x \otimes y \in M \otimes_R N\).

Proof. Consider the commutative diagram,

\[
\begin{array}{ccc}
M \times N & \xrightarrow{\rho} & M \otimes_R N \\
\downarrow{f \times g} & & \downarrow{h} \\
M' \times N' & \xrightarrow{\rho'} & M' \otimes_R N'
\end{array}
\]

where \(\rho\) and \(\rho'\) are the canonical R-balanced maps. Moreover, \(h = \rho'(f \times g)\) is \(R\)-balanced and the unique group homomorphism \(f \otimes g\) is given by the tensor product \(M \otimes_R N\). From this we see that \((f \otimes g)\rho = \rho'(f \times g)\), so if \((x, y) \in M \times N\), then

\[
(f \otimes g)(x \otimes y) = (f \otimes g)((x, y)) \\
= \rho'(f \times g)((x, y)) = \rho'((f(x), g(x)) \\
= f(x) \otimes g(x).
\]

\[\square\]

The following proposition shows that tensor products and direct sums enjoy a special relationship, that is, they commute.
**Proposition** If \( M \) is an \( R \)-module and \( \{N_\alpha\}_\Delta \) is a family of left \( R \)-modules, then

\[
M \otimes_R \left( \bigoplus_{\Delta} N_\alpha \right) \cong \bigoplus_{\Delta} (M \otimes_R N_\alpha).
\]

Furthermore, the group isomorphism is unique and given by \( x \otimes (y_\alpha) \mapsto (x \otimes y_\alpha) \).

**Proof.** The mapping \( \rho' : M \times (\bigoplus_{\Delta} N_\alpha) \to \bigoplus_{\Delta} (M \otimes_R N_\alpha) \) defined by \( \rho'(x, (y_\alpha)) = (x \otimes y_\alpha) \) is \( R \)-balanced, so the definition of a tensor product produces a unique group homomorphism

\[
f : M \otimes_R \left( \bigoplus_{\Delta} N_\alpha \right) \to \bigoplus_{\Delta} (M \otimes_R N_\alpha)
\]

such that \( f \rho = \rho' \), where \( \rho : M \times (\bigoplus_{\Delta} N_\alpha) \to M \otimes_R \left( \bigoplus_{\Delta} N_\alpha \right) \) is the canonical \( R \)-balanced map given by \( \rho(x, (y_\alpha)) = x \otimes (y_\alpha) \). It follows that \( f(x \otimes (y_\alpha)) = (x \otimes y_\alpha) \) for each generator \( x \otimes (y_\alpha) \in M \otimes_R \left( \bigoplus_{\Delta} N_\alpha \right) \).

The proof will be complete if we can find a \( \mathbb{Z} \)-linear mapping

\[
f' : \bigoplus_{\Delta} (M \otimes_R N_\alpha) \to M \otimes_R \left( \bigoplus_{\Delta} N_\alpha \right)
\]

that serves as an inverse for \( f \). For this, let \( i_\beta : M_\beta \to \bigoplus_{\Delta} M_\alpha \) be the canonical injection for each \( \beta \in \Delta \). Then for each \( \beta \in \Delta \) we have a \( \mathbb{Z} \)-linear mapping \( f_\beta = \text{id}_M \otimes i_\beta : M \otimes_R M_\beta \to M \otimes_R \left( \bigoplus_{\alpha} M_\alpha \right) \) defined by \( x \otimes x_\beta \mapsto x \otimes (y_\alpha) \), where
\[ y_\alpha = x_\beta \text{ when } \alpha = \beta \text{ and } y_\alpha = 0 \text{ if } \alpha \neq \beta. \] If \( i_\beta : M \otimes_R N_\beta \to \bigoplus_{\Delta}(M \otimes_R N_\alpha) \) is the canonical injection for each \( \beta \in \Delta \), then the definition of a direct sum produces a unique \( \mathbb{Z} \)-linear map \( f' : \bigoplus_{\Delta}(M \otimes_R N_\alpha) \to M \otimes_R (\bigoplus_{\Delta} N_\alpha) \) such that \( f'i_\beta = f_\beta \) for each \( \beta \in \Delta \). It is not difficult to verify that

\[
ff'((x \otimes y_\alpha)) = f(x \otimes (y_\alpha)) = (x \otimes y_\alpha) \quad \text{and}
\]

\[
f'f(x \otimes (y_\alpha)) = f'(x \otimes y_\alpha) = x \otimes (y_\alpha),
\]

so \( ff' = \text{id}_{\bigoplus_{\Delta}(M \otimes_R N_\alpha)} \) and \( f'f = \text{id}_{M \otimes_R (\bigoplus_{\Delta} N_\alpha)}. \) Hence, \( f \) is an isomorphism.

By symmetry, we see that if \( \{M_\alpha\}_\Delta \) is a family of \( R \)-modules, then \( (\bigoplus_{\Delta} M_\alpha) \otimes_R N \cong \bigoplus_{\Delta}(M_\alpha \otimes_R N) \) for any left \( R \)-module \( N \).

**Corollary** If \( F \) is a free \( R \)-module and \( M \) is a left \( R \)-module, then there is a set \( \Delta \) such that \( F \otimes_R M \cong M^{(\Delta)}. \)

**Proof.** Since \( F \) is a free \( R \)-module, there is a set \( \Delta \) such that \( F \cong R^{(\Delta)}. \) Hence, we have \( F \otimes_R M \cong R^{(\Delta)} \otimes_R M \cong (R \otimes_R M)^{(\Delta)} \cong M^{(\Delta)}. \) \( \Box \)
Corollary. If $\{M_\alpha\}_\Delta$ is a family of $R$-modules and $\{N_\beta\}_\Gamma$ is a family of left $R$-modules, then $\left(\bigoplus_\Delta M_\alpha \otimes_R \bigoplus_\Gamma N_\beta\right) \cong (M_\alpha \otimes_R N_\beta)^{(\Delta \times \Gamma)}$.

Proof. \[
\left(\bigoplus_\Delta M_\alpha\right) \otimes_R N \cong \bigoplus_\Delta (M_\alpha \otimes_R N) \cong \bigoplus_\Delta \left(M_\alpha \otimes_R \left(\bigoplus_\Gamma N_\beta\right)\right)
\cong \bigoplus_\Delta \bigoplus_\Gamma (M_\alpha \otimes_R N_\beta) \cong (M_\alpha \otimes_R N_\beta)^{(\Delta \times \Gamma)}.
\]

Examples

1. Monomorphisms under Tensor Products. If $i : \mathbb{Z} \to \mathbb{Q}$ is the canonical injection and $\text{id}_{\mathbb{Z}_6} : \mathbb{Z}_6 \to \mathbb{Z}_6$ is the identity map, then $i \otimes \text{id}_{\mathbb{Z}_6} : \mathbb{Z} \otimes \mathbb{Z} \mathbb{Z}_6 \to \mathbb{Q} \otimes \mathbb{Z} \mathbb{Z}_6$ is not an injection. Thus, injective maps are not, in general, preserved under tensor products.

2. Epimorphisms under Tensor Products. If $f : M_R \to M'_R$ and $g : R N \to R N'$ are epimorphisms, then

$$f \otimes g : M \otimes_R N \to M' \otimes_R N'$$

is a group epimorphism.

3. Composition of Maps under Tensor Products. If

$$f : M_R \to M'_R, \quad f' : M'_R \to M'''_R, \quad g : R N \to R N' \quad \text{and} \quad g' : R N' \to R N''$$

are $R$-linear mappings, then $(f' \otimes g')(f \otimes g) = (f' \otimes g') \otimes (g' \otimes g)$.

4. Tensor Products Preserve Isomorphisms. If $M_R \cong M'_R$ and $R N \cong R N'$, then $M \otimes_R N \cong M' \otimes_R N'$. This follows easily from Example 3, since if $f : M_R \to M'_R$ and $g : R N \to R N'$ are $R$-isomorphisms, then $f \otimes g : M \otimes_R N \to M' \otimes_R N'$ is a group homomorphism with inverse $f^{-1} \otimes g^{-1} : M' \otimes_R N' \to M \otimes_R N$. 
Examples:

1) Let \( f : 2\mathbb{Z} \rightarrow \mathbb{Z} \), defined by \( f(2n) = 2n \). Then \( f \) is monomorphism. Let \( I : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \) be the identity. Then \( I \) is monomorphism. Note that

\[
\begin{align*}
f \otimes I : 2\mathbb{Z} \otimes \mathbb{Z}_2 &\rightarrow \mathbb{Z} \otimes \mathbb{Z}_2 \text{ is not monomorphism, because} \\
(f \otimes I)(2 \otimes 1) &= f(2) \otimes 1 = 2 \otimes 1 = 1 \otimes 2(1) = 1 \otimes 0 = 0,
\end{align*}
\]

but \( 2 \otimes 1 \neq 0 \) in \( 2\mathbb{Z} \otimes \mathbb{Z}_2 \). Thus \( \ker(f \otimes I) \neq 0 \).

Now, consider the sequence

\[
\begin{array}{c}
0 \rightarrow 2\mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_2 \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \rightarrow 0
\end{array}
\]

Where \( \pi(n) = n + 2\mathbb{Z} \). Since \( \ker(\pi) = \text{Im}(f) = 2\mathbb{Z} \), then the sequence is short exact, however

\[
\begin{array}{c}
2\mathbb{Z} \otimes \mathbb{Z}_2 \xrightarrow{f \otimes I} \mathbb{Z} \otimes \mathbb{Z}_2 \xrightarrow{\pi \otimes I} \mathbb{Z}_2 \otimes \mathbb{Z}_2 \rightarrow 0
\end{array}
\]

Is not short exact sequence since as noted before, \( f \otimes I \) is not monomorphism.
2) The sequence \( 0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Q} \xrightarrow{\pi} \mathbb{Q} / \mathbb{Z} \to 0 \), where \( i \) is the injection map and \( \pi \) is the canonical projection is exact, while the sequence

\[
\mathbb{Z} \otimes \mathbb{Z}_6 \xrightarrow{i \otimes I} \mathbb{Q} \otimes \mathbb{Z}_6 \xrightarrow{\pi \otimes I} \mathbb{Q} / \mathbb{Z} \otimes \mathbb{Z}_6 \to 0
\]

is not exact since \( i \otimes I \) is not injective (note that the map \( I \) is the identity map on \( \mathbb{Z}_6 \)), this is because \( \mathbb{Q} \otimes \mathbb{Z}_6 = 0 \). One can see that because if \( \frac{p}{q} \otimes n \in \mathbb{Q} \otimes \mathbb{Z}_6 \), then \( \frac{p}{q} \otimes \frac{6}{6}n = \frac{p}{6q} \otimes 6n = \frac{p}{q} \otimes 0 = 0 \), but \( \mathbb{Z} \otimes \mathbb{Z}_6 \cong \mathbb{Z}_6 \neq 0 \). Hence \( i \otimes I \) is not injective.

Now, we study cases in which the sequence preserve exactness when taking its tensor.

**Theorem:** Let \( 0 \to A \xrightarrow{f} B \xrightarrow{\pi} C \to 0 \) be a short exact sequence which splits. Let \( M \) be an \( R \)-module, then

\[
0 \to A \otimes M \xrightarrow{f \otimes I} B \otimes M \xrightarrow{g \otimes I} C \otimes M \to 0
\]

is a short exact sequence which splits.
Proof: Since \( 0 \to A \xrightarrow{f} B \xrightarrow{\pi} C \to 0 \) is a short exact sequence which splits, then there exists a left inverse of \( f \), \( k : B \to A \), where \( k \circ f = I_A \).

Now, \( (k \otimes I) \circ (f \otimes I) = (k \circ f \otimes I) = I \otimes I = I \)

Thus \( f \otimes I \) is monomorphism and \( k \otimes I \) is a left inverse of \( f \otimes I \). Hence the proof is complete.

**Theorem:** Let \( 0 \to A \xrightarrow{f} B \xrightarrow{\pi} C \to 0 \) be a short exact sequence. Let \( F \) be a free \( R \)-module, then \( 0 \to F \otimes A \xrightarrow{I \otimes f} F \otimes B \xrightarrow{I \otimes g} F \otimes C \to 0 \) is a short exact sequence.

Proof: We only must prove that \( I \otimes f \) is injective.

Let \( w \in \ker(I \otimes f) \). Then \( w = \sum_{x \in S} x \otimes y_x \), where \( S \) is a basis for \( F \) and \( y_x = 0 \) almost everywhere. Let

\[
w = x_1 \otimes y_1 + x_2 \otimes y_2 + \ldots + x_n \otimes y_n.
\]

Then

\[
(I \otimes f)(w) = x_1 \otimes f(y_1) + \ldots + x_n \otimes f(y_n) = 0,
\]

but

\[
x_1 \otimes 0 + x_2 \otimes 0 + \ldots + x_n \otimes 0 = 0 \in F \otimes B.
\]

Thus it must be written uniquely. Hence

\[
f(y_1) = \ldots = f(y_n) = 0.
\]

Since \( f \) is injective, then \( y_1 = y_2 = \ldots = y_n = 0 \). Hence \( w = 0 \).
Remark: Let $0 \to A \xrightarrow{f} B \xrightarrow{\pi} C \to 0$ be a short exact sequence. Let $F$ be a free $R$-module, then $0 \to A \otimes F \xrightarrow{f \otimes I} B \otimes F \xrightarrow{g \otimes I} C \otimes F \to 0$ is a short exact sequence. In examples 1 and 2, $\mathbb{Z}_2$ and $\mathbb{Z}_6$ are not free $\mathbb{Z}$-modules.
Problems:

1. Show that each of the following hold.

   (a) If \(d\) is the greatest common divisor of two positive integers \(m\) and \(n\), then \(\mathbb{Z}_m \otimes \mathbb{Z} \mathbb{Z}_n = \mathbb{Z}_d\). [Hint: Show that the mapping \(\rho': \mathbb{Z}_m \times \mathbb{Z}_n \to \mathbb{Z}_d\) defined by \(\rho'(([a], [b])) = [ab]\) is a well defined \(\mathbb{Z}\)-balanced mapping and then consider the mapping \(f: \mathbb{Z}_m \otimes \mathbb{Z} \mathbb{Z}_n \to \mathbb{Z}_d\) given by the tensor product \(\mathbb{Z}_m \otimes \mathbb{Z} \mathbb{Z}_n\).] Conclude that if \(m\) and \(n\) are relatively prime, then \(\mathbb{Z}_m \otimes \mathbb{Z} \mathbb{Z}_n = 0\).

   (b) \(\mathbb{Q}/\mathbb{Z} \otimes \mathbb{Z} \mathbb{Q}/\mathbb{Z} = 0\)

   (c) \(\mathbb{Q} \cong \mathbb{Q} \otimes \mathbb{Z} \mathbb{Q}\)

   (d) If \(G\) is a torsion \(\mathbb{Z}\)-module, show that \(G \otimes \mathbb{Z} \mathbb{Q} = 0\).

2. If \(I\) is a left ideal of \(R\) and \(M\) is an \(R\)-module, prove that \(M \otimes_R (R/I) \cong M/M I\). [Hint: Show that the mapping \(h: M \to M \otimes_R (R/I)\) defined by \(h(x) = x \otimes (1 + I)\) is an epimorphism. Next, show that \(MI \subseteq \text{Ker} h\) so that we have an induced epimorphism \(g: M/M I \to M \otimes_R (R/I)\). Now the map \(\rho': M \times R/I \to M/M I\) given by \(\rho'((x, a + I)) = xa + MI\) is \(R\)-balanced, so if \(f: M \otimes_R (R/I) \to M/M I\) is the map given by the tensor product \(M \otimes_R (R/I)\), show that \(fg = \text{id}_{M/M I}\) and \(gf = \text{id}_{M \otimes_R (R/I)}\).]
3. Let $R$ and $S$ be rings and consider the modules $L_R, R_M, S_N$. Prove that $(L \otimes_R M) \otimes_S N \cong L \otimes_R (M \otimes_S N)$.

4. If $R$ is a commutative ring and $M$ and $N$ are $R$-modules, then are $M \otimes_R N$ and $N \otimes_R M$ isomorphic?

5. Verify the assertions of Examples 1 through 5.

6. If $I_1$ and $I_2$ are ideals of $R$, prove that $R/I_1 \otimes_R R/I_2 \cong R/(I_1 + I_2)$. [Hint: The balanced map $\rho': R/I_1 \times R/I_2 \to R/(I_1 + I_2)$ defined by $\rho'((a + I_1, b + I_2)) = ab + I_1 + I_2$ gives a group homomorphism $f: R/I_1 \otimes_R R/I_2 \to R/(I_1 + I_2)$ such that $f(a + I_1 \otimes b + I_2) = ab + I_1 + I_2$. So show that $g: R/(I_1 + I_1) \to R/I_1 \otimes_R R/I_2$ given by $f(a + I_1 + I_2) = 1 + I_1 \otimes a + I_2$ is a well defined group homomorphism such that $gf = \text{id}_{R/(I_1+I_2)}$ and $fg = \text{id}_{R/(I_1+I_2)}$.]