1.9 Stability Theory

In this section we define the stable, unstable and center subspace, \( E^s \), \( E^u \) and \( E^c \) respectively, of a linear system

\[
\dot{x} = Ax.
\]

(1)

Recall that \( E^s \) and \( E^u \) were defined in Section 1.2 in the case when \( A \) had distinct eigenvalues. We also establish some important properties of these subspaces in this section.

**Definition.** (stable, unstable and center subspaces)

Let \( w_j = u_j + iv_j \) be a generalized eigenvector of the (real) matrix \( A \) corresponding to an eigenvalue \( \lambda_j = a_j + ib_j \). And let \( B = \{ u_1, \ldots, u_k, u_{k+1}, \ldots, u_m, v_m \} \) be a basis of \( \mathbb{R}^n \) (with \( n = 2m - k \)). as established by Theorems 1 and 2 and the Remark in Section 1.7. Then

\[
E^s = \text{Span}\{u_j, v_j \mid a_j < 0\}
\]

\[
E^c = \text{Span}\{u_j, v_j \mid a_j = 0\}
\]

and

\[
E^u = \text{Span}\{u_j, v_j \mid a_j > 0\};
\]

i.e., \( E^s \), \( E^c \) and \( E^u \) are the subspaces of \( \mathbb{R}^n \) spanned by the real and imaginary parts of the generalized eigenvectors \( w_j \) corresponding to eigenvalues \( \lambda_j \) with negative, zero and positive real parts respectively.

**Example 1.** Find the stable, unstable, and center subspaces of the linear system \( \dot{x} = Ax \) if

\[
A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.
\]

**Solution:** The eigenvalues of \( A \) are \( \lambda_1 = \lambda_2 = 0 \) with corresponding generalized eigenvectors \( u_1 = (0, 1)^T \) and \( u_2 = (1, 0)^T \). Thus \( E^c = \mathbb{R}^2 \). The solution of the system is given by

\[
x_1(t) = c_1, \quad x_2(t) = c_1t + c_2.
\]

The solutions with \( c_1 = 0 \) are bounded (points on the \( x_2 \)-axis) while the others solutions are not bounded.

**Example 2.** Find the stable, unstable, and center subspaces of the linear system \( \dot{x} = Ax \) if

\[
A = \begin{bmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.
\]

**Solution:** The eigenvalues of \( A \) are \( \lambda_1 = -2 + i \), \( \lambda_2 = -2 - i \) and \( \lambda_3 = 3 \) with corresponding eigenvectors

\[
w_1 = u_1 + iv_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\]

and

\[
u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\]

The stable subspace is \( E^s = \text{Span}\{u_1, v_1\} \); i.e., the \( x_1x_2 \) plane.
The unstable subspace is $E^u = \text{Span}\{u_2\}$; i.e., the $x_3$-axis. The solution is given by

$$x(t) = \begin{bmatrix} e^{-2t} \cos t & -e^{-2t} \sin t & 0 \\ e^{-2t} \sin t & e^{-2t} \cos t & 0 \\ 0 & 0 & e^{3t} \end{bmatrix} x_0.$$ 

Note that all solutions in $E^s$ (i.e. with $x_3 = 0$) approach the equilibrium point $x = 0$ as $t \to \infty$ and all solutions in $E^u$ approach the equilibrium point $x = 0$ as $t \to -\infty$.

**Example 3.** Find the stable, unstable, and center subspaces of the linear system $\dot{x} = Ax$ if

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$ 

**Solution:** The eigenvalues of $A$ are $\lambda_1 = i$, $\lambda_2 = -i$ and $\lambda_3 = 2$ with corresponding eigenvectors

$$w_1 = u_1 + iv_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and

$$u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$ 

The center subspace is $E^c = \text{Span}\{u_1, v_1\}$; i.e., the $x_1x_2$ plane. The unstable subspace is $E^u = \text{Span}\{u_2\}$; i.e., the $x_3$-axis.

The solution is given by

$$x(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} x_0.$$ 

All solutions lie on the cylinder $x_1^2 + x_2^2 = c_1^2 + c_2^2.$

Note that all solutions in $E^u$ approach the equilibrium point $x = 0$ as $t \to -\infty$ and all solutions in $E^c$ are bounded if and if $x(0) \neq 0$, then they are bounded away from $x = 0$ for all $t$.

### Flow of a system of differential equations

Consider the initial value problem

$$\dot{x} = Ax, \quad x(0) = x_0.$$ 

Its solution is given by $x(t) = e^{tA}x_0$. The mapping $e^{tA} : \mathbb{R}^n \to \mathbb{R}^n$ may be regarded as describing the motion of points $x_0 \in \mathbb{R}^n$ along the trajectories of (1).

**Definition.** (Flow)

The set of mappings $e^{tA} : \mathbb{R}^n \to \mathbb{R}^n$ is called the flow of the linear system $\dot{x} = Ax$.

**Definition.** (Hyperbolic flow)

If all eigenvalues of the matrix $A$ have nonzero real parts, then the flow $e^{tA} : \mathbb{R}^n \to \mathbb{R}^n$ is called a hyperbolic flow of the linear system $\dot{x} = Ax$ and the system is called a hyperbolic linear system.
Definition. (Invariant subspace)
A subspace $E \subset \mathbb{R}^n$ is said to be invariant with respect to the flow $e^{tA} : \mathbb{R}^n \to \mathbb{R}^n$ if

$$e^{tA}E \subset E$$

for all $t \in \mathbb{R}$.

We next show that the stable, unstable and center subspaces, $E^s$, $E^u$ and $E^c$ of the linear system $\dot{x} = Ax$ are invariant under the flow $e^{At}$ of the linear system; i.e., any solution starting in $E^s$, $E^u$ or $E^c$ at time $t = 0$ remains in $E^s$, $E^u$ or $E^c$ respectively for all $t \in \mathbb{R}$.

Lemma. Let $E$ be the generalized eigenspace of a matrix $A$ corresponding to an eigenvalue $\lambda$. Then

$$AE \subset E.$$

Proof. $AE = \{Av : v \in E\}$. Let $\{v_1, ..., v_k\}$ be a basis of generalized eigenvectors for $E$. Then given $v \in E$,

$$v = \sum_{j=1}^{k} c_j v_j$$

and hence

$$Av = \sum_{j=1}^{k} c_j Av_j.$$

Now since each $v_j$ is a generalized eigenvector, it satisfies

$$(A - \lambda I)^{k_j} v_j = 0$$

for some minimal $k_j$. Thus we have

$$(A - \lambda I)^{k_j} Av_j = (A - \lambda I)^{k_j} Av_j - \lambda(A - \lambda I)^{k_j} v_j$$

$$= (A - \lambda I)^{k_j} (A - \lambda I)v_j = 0.$$ 

So, it follows by induction that $Av_j \in E$ and since $E$ is a subspace of $\mathbb{R}^n$, it follows that

$$Av = \sum_{j=1}^{k} c_j Av_j \in E.$$

Theorem 1. ($E^s$, $E^u$, and $E^c$ are invariant)
Let $A$ be a real $n \times n$ matrix. Then

$$\mathbb{R}^n = E^s \oplus E^u \oplus E^c,$$

where $E^s$, $E^u$, and $E^c$ are the stable, unstable and center subspaces of the system $\dot{x} = Ax$ respectively; furthermore, $E^s$, $E^u$, and $E^c$ are invariant with respect to the flow $e^{At}$ of the system.

Proof. Let $u_j$, $j = 1, ..., k$, $w_j = u_j + iv_j$, $j = k + 1, \cdots m$ be generalized eigenvectors of $A$ and let $B = \{u_1, ..., u_k, u_{k+1}, v_{k+1}, \cdots, u_m, v_m\}$ be a basis for $\mathbb{R}^n$. Then it follows from the definition of $E^s$, $E^u$, and $E^c$ that $\mathbb{R}^n = E^s \oplus E^u \oplus E^c$. 

3
If \( x_0 \in E^s \), then
\[
x_0 = \sum_{j=1}^{n_s} c_j V_j,
\]
where \( V_j = v_j \) or \( u_j \) and \( \{V_1, \ldots, V_{n_s}\} \subset B \) is a basis for \( E^s \). Then by the linearity of \( e^{At} \), it follows that
\[
e^{tA}x_0 = \sum_{j=1}^{n_s} c_j e^{tA}V_j.
\]
But
\[
e^{tA}V_j = \lim_{k \to \infty} \left[ I + At + \cdots + \frac{A^k t^k}{k!} \right] V_j \in E^s
\]
since for \( j = 1, ..., n \), by the above lemma \( A^k V_j \in E^s \) and since \( E^s \) is complete. Thus, for all \( t \in \mathbb{R}^n \), \( e^{tA}x_0 \in E^s \) and therefore \( e^{tA}E^s \subset E^s \).

It can similarly be shown that \( E^u \) and \( E^c \) are invariant under the flow \( e^{tA} \).

**Example 1.** Consider the linear system \( \dot{x} = Ax \) where
\[
A = \begin{bmatrix}
-2 & -1 & 0 \\
1 & -2 & 0 \\
0 & 0 & 3
\end{bmatrix}.
\]

**Solution:** The eigenvalues of \( A \) are \( \lambda_1 = -2 + i \), \( \lambda_2 = -2 - i \) and \( \lambda_3 = 3 \) with corresponding eigenvectors
\[
w_1 = u_1 + iv_1 = \begin{bmatrix} 0 \\ 1 + i \\ 0 \end{bmatrix}
\]
and
\[
u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\]
The stable subspace is \( E^s = \text{Span}\{u_1, v_1\} \); i.e., the \( x_1x_2 \) plane.
The unstable subspace is \( E^u = \text{Span}\{u_2\} \); i.e., the \( x_3 \)-axis.
The solution is given by
\[
x(t) = \begin{bmatrix}
e^{-2t} \cos t & -e^{-2t} \sin t & 0 \\
e^{-2t} \sin t & e^{-2t} \cos t & 0 \\
0 & 0 & e^{3t}
\end{bmatrix} x_0.
\]

If \( x_0 \in E^s \), then \( x_0 = \begin{bmatrix} c_1 \\ c_2 \\ 0 \end{bmatrix} \) and hence the solution
\[
x(t) = e^{tA}x_0 = \begin{bmatrix} c_1 e^{-2t} \cos t - c_2 e^{-2t} \sin t \\
c_1 e^{-2t} \sin t + c_2 e^{-2t} \cos t \\
0 \end{bmatrix} \in E^s.
\]
Similarly, if \( x_0 = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} \in E^u \), then
\[
x(t) = e^{tA}x_0 = \begin{bmatrix} 0 \\ 0 \\ ce^{3t} \end{bmatrix} \in E^u.
**Definition.** (Sinks and sources) If all of the eigenvalues of $A$ have negative (positive) real parts, the origin is called a sink (source) for the linear system $\dot{x} = Ax$.

**Example 1.** Consider the linear system $\dot{x} = Ax$ if

$$A = \begin{bmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$  

**Solution:** The eigenvalues of $A$ are $\lambda_1 = -2 + i$, $\lambda_2 = -2 - i$ and $\lambda_3 = -3$ with corresponding eigenvectors

$$w_1 = u_1 + iv_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and

$$u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$  

$E^s = \mathbb{R}^3$ and the origin is a sink for this example.

**Theorem 2.** The following statements are equivalent:

(a) For all $x_0 \in \mathbb{R}^n$, $\lim_{t \to \infty} e^{tA}x_0 = 0$ and for $x_0 \neq 0$, $\lim_{t \to -\infty} |e^{tA}x_0| = \infty$.

(b) All eigenvalues of $A$ have negative real parts.

(c) There are positive constants $a$, $c$, $m$ and $M$ such that for all $x_0 \in \mathbb{R}^n$

$$|e^{tA}x_0| \leq Me^{-ct}|x_0| \quad \text{for} \quad t \geq 0$$

and

$$|e^{tA}x_0| \geq me^{-at}|x_0| \quad \text{for} \quad t \leq 0.$$  

**Proof.** Each coordinate in the solution $x(t)$ of the initial value problem

$$\dot{x} = Ax, \quad x(0) = x_0$$

is a linear combination of functions of the form

$$t^ke^{at} \cos(bt) \quad \text{or} \quad t^ke^{at} \sin(bt),$$

where $\lambda = a + ib$ is an eigenvalue of the matrix $A$ and $0 < k < n - 1$.

**Theorem 3.** The following statements are equivalent:

(a) For all $x_0 \in \mathbb{R}^n$, $\lim_{t \to -\infty} e^{tA}x_0 = 0$ and for $x_0 \neq 0$, $\lim_{t \to \infty} |e^{tA}x_0| = \infty$.

(b) All eigenvalues of $A$ have positive real parts.

(c) There are positive constants $a$, $c$, $m$ and $M$ such that for all $x_0 \in \mathbb{R}^n$

$$|e^{tA}x_0| \leq Me^{ct}|x_0| \quad \text{for} \quad t \leq 0$$

and

$$|e^{tA}x_0| \geq me^{at}|x_0| \quad \text{for} \quad t \geq 0.$$
Corollary.

(a) If \( x_0 \in E^s \), then \( e^{tA} x_0 \in E^s \) for all \( t \in \mathbb{R} \) and
\[
\lim_{t \to \infty} e^{tA} x_0 = 0.
\]

(a) If \( x_0 \in E^u \), then \( e^{tA} x_0 \in E^u \) for all \( t \in \mathbb{R} \) and
\[
\lim_{t \to -\infty} e^{tA} x_0 = 0.
\]

Thus, we see that all solutions of \( \dot{x} = Ax \) which start in the stable space \( E^s \) remain in \( E^s \) for all \( t \) and approach the origin exponentially fast as \( t \to \infty \); and all solutions of ( which start in the unstable manifold \( E^u \) remain in \( E^u \) for all \( t \) and approach the origin exponentially fast as \( t \to -\infty \).