In this chapter we will study nonlinear systems of differential equations

\[ \dot{x} = f(x), \]  

where \( f : E \rightarrow \mathbb{R}^n \) and \( E \) is an open subset of \( \mathbb{R}^n \). We will show that that under certain conditions on the function \( f \), the nonlinear system (1) has a unique solution through each point \( x_0 \in E \) defined on a maximal interval of existence \( (\alpha, \beta) \subseteq \mathbb{R} \). In general, it is not possible to solve the nonlinear system (1); however, a great deal of qualitative information about the local behavior of the solution is determined in this chapter. In particular, we establish the Hartman-Grobman Theorem and the Stable Manifold Theorem which show that topologically the local behavior of the nonlinear system (1) near an equilibrium point \( x_0 \) where \( f(x_0) = 0 \) is typically determined by the behavior of the linear system

\[ \dot{x} = Ax \]

near the origin when the matrix \( A = Df(x_0) \), the derivative of \( f \) at \( x_0 \).

### 2.1 Some Preliminary Concepts and Definitions

Before beginning our discussion of the fundamental theory of nonlinear systems of differential equations, we present some preliminary concepts and definitions. First of all, in this book we shall only consider autonomous systems of ordinary differential equations

\[ \dot{x} = f(x). \]  

#### Example 1. Consider the initial value problem

\[ \dot{x} = 3x^{2/3}, \quad x(0) = 0. \]

It has two different solutions

\[ x_1(t) = t^3 \]

and

\[ x_1(t) = 0 \]

\( \forall t \in \mathbb{R} \). Thus the solution is not unique.

#### Example 2. Consider \( \dot{x} = x^2 \), \( x(0) = 1 \). The solution is given by

\[ x(t) = \frac{1}{1 - t}. \]

Thus the solution is no longer defined for all \( t \in \mathbb{R} \). Indeed it is defined only for \( t \in (-\infty, 1) \).

#### Example 3. Consider \( \dot{x} = x \), \( x(0) = x_0 \). The solution \( x(t) = x_0 e^t \) is defined for all \( t \in \mathbb{R} \).

Note that \( x(t) = x_0 e^t \) is a solution for all \( x_0 \in \mathbb{R} \).
Definition. The function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be differentiable at $x_0 \in \mathbb{R}^n$ if there is a linear transformation $Df(x_0) \in L(\mathbb{R}^n)$ that satisfies

$$\lim_{{|h| \to 0}} \frac{|f(x_0 + h) - f(x_0) - Df(x_0)h|}{|h|} = 0.$$ 

The linear transformation $Df(x_0)$ is called the derivative of $f$ at $x_0$.

Theorem 1. If $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is differentiable at $x_0$, then the partial derivatives $\frac{\partial f_i}{\partial x_j}, i, j = 1, \cdots, n,$ all exist at $x_0$ and for all $x \in \mathbb{R}^n,$

$$Df(x_0)x = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x_0)x_j.$$ 

Thus, if $f$ is a differentiable function, the derivative $Df$ is given by the $n \times n$ Jacobian matrix $Df = \left[ \frac{\partial f_i}{\partial x_j} \right]$.

Example 1. Find $Df$ if

$$f(x) = \begin{bmatrix} x_1 - x_2^2 \\ -x_2 + x_1x_2 \end{bmatrix}$$

and evaluate it at the point $x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Solution:

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 & -2x_2 \\ x_2 & -1 + x_1 \end{bmatrix}.$$ 

Thus $Df(-1, 1) = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$. In most of the theorems in the remainder of this book, it is assumed that the function $f(x)$ is continuously differentiable; i.e., that the derivative $Df(x)$ considered as a mapping $Df : \mathbb{R}^n \rightarrow L(\mathbb{R}^n)$ is a continuous function of $x$ in some open set $E \subseteq \mathbb{R}^n$. Continuity is defined as usual:

Definition. Suppose that $V_1$ and $V_2$ are two normed linear spaces with respective norms $\| \|_1$ and $\| \|_2$. Then $F : V_1 \rightarrow V_2$ is continuous at $x_0 \in V_1$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $x \in V_1$ and $\|x - x_0\|_1 < \delta$ implies that $\|F(x) - F(x_0)\|_2 < \epsilon$.

And $F$ is said to be continuous on the set $E \subseteq V_1$ if it is continuous at each point $x \in E$. If $F$ is continuous on $E \subseteq V_1$, we write $F \in C(E)$.

Definition. Suppose that $f : E \rightarrow \mathbb{R}^n$ is differentiable on $E$. Then $f \in C^1(E)$ if the derivative $Df : E \rightarrow L(\mathbb{R}^n)$ is continuous on $E$.

Theorem 2. Suppose that $E$ is an open subset of $\mathbb{R}^n$ and that $f : E \rightarrow \mathbb{R}^n$. Then $f \in C^1(E)$ if and only if the partial derivatives $\frac{\partial f_i}{\partial x_j}, i, j = 1, \cdots, n,$ exist and are continuous on $E$. 


Remark. For $E$ an open subset of $\mathbb{R}^n$, the higher order derivatives $D^k f(x_0)$ of a function $f : E \to \mathbb{R}^n$ are defined in a similar way and it can be shown that $f \in C^k(E)$ if and only if the partial derivatives

$$\frac{\partial f_i}{\partial x_{j_1} \cdots \partial x_{j_k}},$$

with $i, j_1, \cdots, j_k = 1, \cdots, n$, exist and are continuous on $E$. Furthermore, $D^2 f(x_0) : E \times E \to \mathbb{R}^n$ and for $(x, y) \in E \times E$ we have

$$D^2 f(x_0)(x, y) = \sum_{j_1, j_2=1}^{n} \frac{\partial^2 f(x_0)}{\partial x_{j_1} \partial x_{j_2}} x_{j_1} y_{j_2}.$$