Chapter 2
First Order Equations

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A first-order PDE for an known function \( u(x, y) \) has the general form

\[
F(x, y, u, u_x, u_y) = 0.
\]

First-order equations have many applications such as the transport of material in a fluid flow and propagation of wavefronts in optics.

The main result for first-order PDEs is the fact that the general solution of such equations can be obtained by solving systems of ordinary differential equations. This is not true for higher order equations or for systems of first-order equations.

2.1 Linear First-order Equations

A first-order PDE is said to be linear if it has the form

\[
A(x, y)u_x + B(x, y)u_y + C(x, y)u = G(x, y).
\]

It is assumed that \( A, B, C, \) and \( G \) have continuous first-order derivatives with respect to \( x \) and \( y \) in some region \( \Omega \) of the \( xy \)-plane. Moreover it is assumed that at least one of the coefficient functions \( A \) and \( B \) is not identically zero on \( \Omega \).

Example 1. \( u_x + u_y = 1 \) is a first-order PDE.

Remarks.

(1) The linear equation

\[
A(x, y)u_x + B(x, y)u_y + C(x, y)u = G(x, y)
\]

can be written as

\[
L(u) = G(x, y),
\]

where \( L \) is the differential operator

\[
L = A \frac{\partial}{\partial x} + B \frac{\partial}{\partial y} + C.
\]
The operator $L$ is linear; that is, for each pair of functions $u_1, u_2$, and each pair of constants $c_1, c_2$, we have

$$L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2).$$

**Definition.** (Solution of linear equation)

1. A solution of $L(u) = G$ on a region $\Omega$ is a function $u = \phi(x,y)$ with continuous first-order partial derivatives in $\Omega$ such that $L(\phi) = G$.

2. If $u = \phi(x,y)$ is a solution of $L(u) = G$, then the corresponding surface in the $xyu$ space is called an integral surface.

3. The general solution of $L(u) = 0$ is a relation involving an arbitrary function such that each choice for the arbitrary function yields a solution of $L(u) = 0$.

4. The general solution of $L(u) = G$ is a solution which includes all solutions of $L(u) = G$.

**Example 1.** Show that $u(x,y) = e^{-y}f(x-y)$ is the general solution of $u_x + u_y + u = 0$.

**Solution:** $u(x,y) = e^{-y}f(x-y)$ involves an arbitrary function $f$ and satisfies the equation $u_x + u_y + u = 0$.

**Theorem 1.** (General solution of $L(u) = G$)

Let $u_h$ be the general solution of $L(u) = 0$ and let $u_p$ be any particular solution of $L(u) = G$. Then $u = u_h + u_p$ is the general solution of $L(u) = G$.

**Proof.** First of all $L(u) = L(u_h + u_p) = L(u_h) + L(u_p) = 0 + G = G$. Thus we have shown that $u = u_h + u_p$ is a solution of $L(u) = G$.

Next we will show that $u = u_h + u_p$ is the general solution. Let $w$ be any solution of $L(u) = G$. Then

$$L(w - u_p) = L(w) - L(u_p) = G - G = 0.$$

Since $u_h$ is the general solution of $L(u) = 0$ we have $w - u_p = u_h$. This implies that $w = u_h + u_p$. □

**Example 1.** Find the general solution of the linear equation $u_x + u = x$.

**Solution:**

**Example 2.** Find the general solution of $u_x + u_y + u = 0$.

**Solution:**

**How to obtain the general solution of $L(u) = 0$**

In the above examples we have seen how we can solve special cases of homogeneous linear partial differential equations. In this section we will study a general method to solve homogeneous linear partial differential equations.
Canonical form of first-order linear equation

Consider a homogeneous linear partial differential equations

\[ L(u) = A(x, y)u_x + B(x, y)u_y + C(x, y)u = 0. \]

We look for a transformation

\[ \xi = \xi(x, y), \quad \eta = \eta(x, y) \]

that transforms \( L(u) = 0 \) to a simple equation. We assume that the Jacobian

\[ \frac{\partial (\xi, \eta)}{\partial (x, y)} = \left| \begin{array}{cc} \xi_x & \eta_x \\ \xi_y & \eta_y \end{array} \right| \]

is not zero. This guarantee the existence of the inverse transformation

\[ x = x(\xi, \eta), \quad y = y(\xi, \eta). \]

**Definition.** (Nonsingular transformation)

A transformation

\[ \xi = \xi(x, y), \quad \eta = \eta(x, y) \]

is called non singular if the Jacobian \( J = \frac{\partial (\xi, \eta)}{\partial (x, y)} = \xi_x \eta_y - \xi_y \eta_x \neq 0 \) of at any point \((x, y)\).

Now consider \( u(x, y) = u(\xi, \eta) \). Using the chain rule we obtain

\[ u_x = u_\xi \xi_x + u_\eta \eta_x, \]
\[ u_y = u_\xi \xi_y + u_\eta \eta_y. \]

Substituting into \( L(u) = 0 \) we obtain

\[ (A \xi_x + B \xi_y)u_\xi + (A \eta_x + B \eta_y)u_\eta + Cu = 0. \]

To simplify the last equation we may choose \( \eta \) such that

\[ A \eta_x + B \eta_y = 0. \]

A solution \( \eta(x, y) \) of the last equation can be accomplished as follows. Assume \( A(x, y) \neq 0 \) and consider the ordinary differential equation

\[ \frac{dy}{dx} = \frac{B(x, y)}{A(x, y)}. \]

Let \( \eta(x, y) = k \) be the general solution of this ODE with \( \eta_y \neq 0 \). Then

\[ \eta_x + \eta_y \frac{dy}{dx} = 0. \]

Using \( \frac{dy}{dx} = \frac{B(x, y)}{A(x, y)} \), we obtain

\[ A \eta_x + B \eta_y = 0. \]
Therefore we can choose \( \eta(x, y) \) any solution of the ordinary differential equation

\[
\frac{d\eta}{dx} = \frac{B(x, y)}{A(x, y)}.
\]

Let us take \( \xi = x \). Then

\[
\frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} 1 & \eta_x \\ 0 & \eta_y \end{vmatrix} = \eta_y \neq 0.
\]

Thus the transformation

\[
\xi = x, \quad \eta = \eta(x, y),
\]

where \( \eta(x, y) = k \) is a solution of the ODE \( \frac{dy}{dx} = \frac{B(x, y)}{A(x, y)} \) with \( \eta_y \neq 0 \), is invertible.

**Definition.** (Canonical form of first-order linear equation)

The equation

\[
A(\xi, \eta)u_\xi + C(\xi, \eta)u = G(\xi, \eta)
\]

is the canonical form of first-order linear equation

\[
A(x, y)u_x + B(x, y)u_y + C(x, y)u = G(x, y).
\]

Now we can solve the simple equation

\[
A(\xi, \eta)u_\xi + C(\xi, \eta)u = 0.
\]

It is separable; that is it can be written as

\[
\frac{u_\xi}{u} = -\frac{C}{A}.
\]

Integrating both sides with respect to \( x \), we obtain

\[
u(\xi, \eta) = f(\eta) \exp \left( -\int \frac{C(\xi, \eta)}{A(\xi, \eta)} d\xi \right),
\]

where \( f \) is an arbitrary function.

**Theorem 2.** (Solution of homogeneous linear PDE)

Let

\[
L(u) = A(x, y)u_x + B(x, y)u_y + C(x, y)u = 0
\]

be a first-order linear partial differential equation with \( A(x, y) \neq 0 \). Assume that \( \xi = x \) and \( \eta(x, y) = k \) is the general solution of the ordinary differential equation \( \frac{dy}{dx} = \frac{B(x, y)}{A(x, y)} \) with \( \eta_y \neq 0 \).

Then the general solution of \( L(u) = 0 \) is given by

\[
u(x, y) = f(\eta) \exp \left( -\int \frac{C(\xi, \eta)}{A(\xi, \eta)} d\xi \right),
\]

where \( f \) is an arbitrary function.
Example 1. Solve the partial differential equation
\[ x^2u_x - x y u_y + y u = 0. \]
Solution:

Example 2. Find the general solution of the partial differential equation
\[ x u_x + y u_y = 2u. \]
Solution:

Example 3. Find the general solution of the partial differential equation
\[ 2u_x + 3u_y = x^2. \]
Solution:

Example 4. Solve the partial differential equation
\[ 3u_x + 6u_y + u = x + 2e^y. \]
Solution:

Exercises

(1) Hold one independent variable constant and integrate with respect to the remaining variable to obtain the general solution for each of the following equations. Verify that your answer is correct by substituting into the equation.

(a) \( u_y = x^2 + y^2 \)
(b) \( u_x = \sin \left( \frac{x}{y} \right) \)
(c) \( u_x + xu = x^3 + 3xy \)

(2) Consider the first-order equation \( Au_x + Bu_y = 0 \), where \( A \) and \( B \) are constants. Assume a solution of the form \( u = f(\alpha x + \beta y) \), where \( f \) is an arbitrary function and \( \alpha \) and \( \beta \) are constants. Substitute into the differential equation to determine suitable values of \( \alpha \) and \( \beta \).

(3) Show that if \( A, B, \) and \( C \) are constants, then the general solution of the homogeneous first-order linear equation \( Au_x + Bu_y + Cu = 0 \) is given by \( u = f(Bx - Ay)e^{-Cx/A} \), where \( f \) is an arbitrary function.

(4) Use exercise (3) to obtain the general solution of the following equations:

(a) \( 3u_x - 4u_y = x^2 \)
(b) \( u_x - 3u_y = \sin x + \cos y \)
(c) \( 5u_x + 4u_y + u = x^2 + 1 + 2e^y \)
(d) \( u_x + 2u_y - 5u = \cos x + y^2 + 1 \)
(e) \( u_x - au_y = e^{mx} \cos(by) \), where \( a, b, m \) are constants.
(5) Make the change of independent variables \( \xi = \ln x, \ \eta = \ln y \) and reduce the given differential equation to one with constant coefficients. Then obtain the general solution.

(a) \( 4xu_x - 2yu_y = 0 \)
(b) \( 2xu_x + 3yu_y = \ln x \)
(c) \( xu_x - 7yu_y = x^2y \)
(d) \( 8xu_x - 5yu_y + 4u = x^2 \cos x \)
(e) \( axu_x + byu_y + cu = x^2 + y^2, \) where \( a, \ b, \ c \) are constants.

(6) Find the general solution of the following equations:

(a) \( xyu_x - x^2yu_y + yu = 0 \)
(b) \( yu_x - xu_y = 0 \)
(c) \( (x + a)u_x + (y + b)u_y + cu = 0, \) where \( a, \ b, \ c \) are constants.
(d) \( x^2u_x + y^2u_y = 2xy \)
(e) \( x^2u_x + y^2u_y = (x + y)u \)

2.2 Quasilinear First-order Equations

A first-order PDE \( F(x, y, u, u_x, u_y) = 0 \) is called quasilinear if it is linear in \( u_x \) and \( u_y \). The general form of quasilinear first-order PDE is

\[
P(x, y, u)u_x + Q(x, y, u)u_y = R(x, y, u).
\]

Remark. The linear first-order PDEs are special cases of quasilinear first-order PDEs.

The Method of Lagrange

Consider the system of first-order ordinary differential equations

\[
\frac{dx}{P(x, y, u)} = \frac{dy}{Q(x, y, u)} = \frac{du}{R(x, y, u)}.
\]

(1)

This system is called subsidiary equations. If \( P \neq 0 \), then the subsidiary equations is equivalent to the system

\[
\frac{dy}{dx} = \frac{Q(x, y, u)}{P(x, y, u)}, \quad \frac{du}{dx} = \frac{R(x, y, u)}{P(x, y, u)}.
\]

(2)

Similarly, if \( Q \neq 0 \) or \( R \neq 0 \), then the subsidiary equations may be written on the equivalent form

\[
\frac{dx}{dy} = \frac{P(x, y, u)}{Q(x, y, u)}, \quad \frac{du}{dy} = \frac{R(x, y, u)}{Q(x, y, u)}
\]

(3)

or

\[
\frac{dx}{du} = \frac{P(x, y, u)}{R(x, y, u)}, \quad \frac{dy}{du} = \frac{Q(x, y, u)}{R(x, y, u)}.
\]

(4)
respectively. The advantage of the subsidiary system is it avoids the distinguishing between dependent and independent variable.

The general solution of system (2) has the form

\[ y = y(x, c_1, c_2), \quad u = u(x, c_1, c_2), \]

where \( c_1, c_2 \) are arbitrary constants. If these equations are solved for \( c_1 \) and \( c_2 \), then the general solution of the subsidiary equations (1) can be written in the form

\[ v(x, y, u) = c_1, \quad w(x, y, u) = c_2. \]

Each relation \( v = c_1, w = c_2 \) is called an integral of the subsidiary equations.

If \( v \) and \( w \) are functionally independent is some region \( G \) in the \( xyu \) space; that is, the Jacobians

\[ \frac{\partial(v, w)}{\partial(x, y)}, \quad \frac{\partial(v, w)}{\partial(x, u)}, \quad \frac{\partial(v, w)}{\partial(y, u)} \]

are not all zero at any point of \( G \), then the general solution of the quasilinear equation is given by

\[ F(v, w) = 0, \]

where \( F \) is an arbitrary function. The solution \( F(v, w) = 0 \) can be written in the alternative forms

\[ w = f(v) \quad \text{or} \quad v = g(w), \]

where \( f \) and \( g \) are arbitrary functions.

**Theorem 3. (Solution of quasilinear equation)**

Let \( v \) and \( w \) be two functionally independent solutions of the subsidiary equation (1) in a domain \( \Omega \subseteq \mathbb{R}^3 \). Let \( F(v, w) \) be an arbitrary function with continuous first-order derivatives. Then the equation

\[ F(v(x, y, u), w(x, y, u)) = 0 \]

defines \( u \) implicitly as a function of \( x \) and \( y \) and this function is a solution of the equation

\[ P(x, y, u)u_x + Q(x, y, u)u_y = R(x, y, u). \]

**Example 1.** Find the general solution of the equation

\[ xuu_x + yuu_y + x^2 + y^2 = 0. \]

**Solution:** The general solution is

\[ F\left(\frac{y}{x}, u^2 + x^2 + y^2\right) = 0, \]

where \( F \) is an arbitrary function.

Alternatively, the solution can be written as

\[ y = xf(u^2 + x^2 + y^2) \quad \text{or} \quad u^2 + x^2 + y^2 = g(y/x), \]

where \( f \) and \( g \) are arbitrary functions.

**Exercise:** Show that \( F\left(\frac{y}{x}, u^2 + x^2 + y^2\right) = 0 \) is a solution of \( xuu_x + yuu_y + x^2 + y^2 = 0. \)
Example 2. Solve the equation

\[ xu_x + yu_y + u = 0. \]

Solution:
The general solution is

\[ F \left( \frac{y}{x}, xu \right) = 0, \]

where \( F \) is an arbitrary function.

Alternatively, the solution can be written as

\[ u = \frac{1}{x} f(y/x) \quad \text{or} \quad y = xg(xu), \]

where \( f \) and \( g \) are arbitrary functions.

The Method of Multipliers

A useful technique for integrating a system of first-order equations is the method of multipliers.

Proposition 1. If \( \frac{a}{b} = \frac{c}{d} \), then

\[ \frac{\lambda a + \mu c}{\lambda b + \mu d} = \frac{a}{b} = \frac{c}{d}, \]

for arbitrary values of multipliers \( \lambda, \mu \).

Proof. Assume that \( \frac{a}{b} = \frac{c}{d} \). Then we have \( ad = bc \). Multiplying both sides by \( \lambda \) we get

\[ \lambda ad = \lambda bc. \] (5)

Similarly multiplying both sides by \( \mu \) we get

\[ \mu ad = \mu bc. \] (6)

Adding equations (5) and (6) we obtain

\[ a(\lambda b + \mu d) = b(\lambda a + \mu c) \]

and hence we obtain the result. \( \square \)

Corollary 1. If \( \frac{a}{b} = \frac{c}{d} = \frac{e}{f} \), then

\[ \frac{\lambda a + \mu c + \nu e}{\lambda b + \mu d + \nu f} = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}, \]

for arbitrary values of multipliers \( \lambda, \mu, \nu \).

Proof. Exercise \( \square \)
Remark. Applying Corollary 1 to the subsidiary system

\[
\frac{dx}{P(x, y, u)} = \frac{dy}{Q(x, y, u)} = \frac{du}{R(x, y, u)}
\]

we obtain

\[
\frac{\lambda dx + \mu dy + \nu du}{\lambda P + \mu Q + \nu R} = \frac{dx}{P(x, y, u)} = \frac{dy}{Q(x, y, u)} = \frac{du}{R(x, y, u)}.
\]

In this way related differential equations can be formed, some of which may be easy to integrate. In particular if \( \lambda, \mu, \nu \) are chosen such that

\[
\lambda P + \mu Q + \nu R = 0,
\]

then

\[
\lambda dx + \mu dy + \nu du = 0.
\]

Now if there exist a function \( v \) such that

\[
dv = \lambda dx + \mu dy + \nu du,
\]

then \( v(x, y, u) = c_1 \) is an integral of the subsidiary equations.

**Example 1.** Find the general solution of the equation

\[
u u_x + y u_y = x.
\]

**Solution:** The general solution is given by

\[
F\left(x^2 - u^2, \frac{x + u}{y}\right) = 0.
\]

**Example 2.** Find the general solution of the equation

\[
(y - x)u_x + (y + x)u_y = \frac{x^2 + y^2}{u}.
\]

**Solution:** The general solution is given by

\[
F\left(x^2 + 2xy - y^2, x^2 - y^2 + u^2\right) = 0.
\]

**Exercises**

Find the general solution of each of the following PDEs:

(a) \( u_x + xu_y = u \)

(b) \( xu_x + yu_y = nu, \) where \( n \) is a constant.

(c) \( (x + u)u_x + (y + u)u_y = 0 \)

(d) \( xu_x + yu_y = y \)

(e) \( (x + y)(u_x - u_y) = u \)
2.3 Cauchy Problem for Quasilinear First-order Equations

A fundamental problem in the study of differential equations is to solve an initial value problem; that is, a differential equation subject to an initial condition. In the case of partial differential equation the basic initial value problem is to determine an integral surface that passes through a given curve in the $xyu$–space. The problem will also be called Cauchy problem in honor of the French mathematician Augustin Louis Cauchy (1789 – 1857).

Definition. (Cauchy problem ) Let $\Gamma$ be a curve in $\mathbb{R}^3$ described parametrically by the equations

$$\Gamma: \quad x = x_0(s), \quad y = y_0(s), \quad u_0(s), \quad s \in [a, b],$$

where $x_0(s), y_0(s),$ and $u_0(s)$ are functions with continuous first-order derivatives on $[a, b]$. The initial value problem (Cauchy problem) for a quasilinear equation

$$P(x, y, u)u_x + Q(x, y, u)u_y = R(x, y, u)$$

is to find a function $u = u(x, y)$ defined in a domain $\Omega$ of $\mathbb{R}^2$ such that:

(a) $u = u(x, y)$ is a solution of

$$P(x, y, u)u_x + Q(x, y, u)u_y = R(x, y, u)$$

in $\Omega$.

(b) On the curve $\Gamma$, $u$ is equal to the given function $u_0(s)$; that is

$$u(x_0(s), y_0(s)) = u_0(s), \quad s \in [a, b].$$

The curve $\Gamma$ will be called the initial curve and the condition

$$u(x_0(s), y_0(s)) = u_0(s), \quad s \in [a, b],$$

will be called the initial condition.

Example 1. Solve the Cauchy problem

$$yu_x - xu_y = x^3y + xy^3$$

(a) $u = u(x, y)$ is a solution of

$$yu_x - xu_y = x^3y + xy^3$$

in $\Omega$.

(b) On the curve $\Gamma$, $u$ is equal to the given function $u_0(s)$; that is

$$u(x_0(s), y_0(s)) = u_0(s), \quad s \in [a, b].$$

The curve $\Gamma$ will be called the initial curve and the condition

$$u(x_0(s), y_0(s)) = u_0(s), \quad s \in [a, b],$$

will be called the initial condition.

Example 1. Solve the Cauchy problem

$$yu_x - xu_y = x^3y + xy^3$$

(a) $u = u(x, y)$ is a solution of

$$yu_x - xu_y = x^3y + xy^3$$

in $\Omega$.
Solution: The solution of the initial value problem is
\[ u(x, y) = (x^2 + y^2)^2. \]

Example 2. Solve the Cauchy problem
\[ yu_x + u_y = u, \quad u = \sin x, \ y = 0. \]

Solution:
\[ u(x, y) = e^y \sin(x - y). \]

Existence and uniqueness of solution

Theorem 4. (Existence and uniqueness Theorem)
Consider the quasilinear equation
\[ P(x, y, u)u_x + Q(x, y, u)u_y = R(x, y, u), \]
where \( P, Q, R \) are continuous functions with continuous first-order partial derivatives in a domain \( \Omega \subseteq \mathbb{R}^3 \). Let
\[ \Gamma : \ x = x_0(s), \ y = y_0(t), \ u = u_0(s), \quad s \in [a, b], \]
be an initial smooth curve in \( \Omega \). If
\[ J = P \frac{dy_0}{ds} - B \frac{dx_0}{ds} \neq 0, \quad s \in [a, b], \]
then there exists a unique solution \( u = u(x, y) \) defined in a neighborhood of the initial curve \( \Gamma \), which satisfies the quasilinear equation
\[ P(x, y, u)u_x + Q(x, y, u)u_y = R(x, y, u) \]
and the initial condition
\[ u(x_0(s), y_0(s)) = u_0(s), \quad s \in [a, b]. \]

If
\[ J = P(x_0, y_0, u_0) \frac{dy_0}{ds} - Q(x_0, y_0, u_0) \frac{dx_0}{ds} = 0, \]
then the Cauchy problem has either no solution at all, or it has infinitely many solutions.

Example 1. Solve the equation \( u_x = 1 \) subject to the initial condition \( u(0, y) = y \).

Solution: The unique solution is
\[ u(x, y) = x + y. \]

Example 2. Determine whether the Cauchy problem
\[ yu_x - xu_y = 0, \quad \Gamma : \ x = \cos s, \ y = \sin s, \ u = \sin s, \]
has a unique solution, infinitely many solutions, or no solution.
Solution: There is no solution of this Cauchy problem.

Example 3. Show that the Cauchy problem

\[ u_x + u_y = 1, \quad u(x, x) = x \]

has infinitely many solutions.

Solution: There is infinitely many solutions of the Cauchy problem:

\[ u(x, y) = x + f(y - x), \]

where \( f \) is any function with \( f(0) = 0 \).

Alternative method for solving Cauchy problem

Let \( \Gamma \) be a given smooth curve defined parameterically by

\[ x = x(t), \quad y = y(t), \quad u = u(t), \quad t \in [a, b]. \]

Let \( v = c_1 \) and \( w = c_2 \) be two functionally independent integral of the subsidiary equations. Write down the equations

\[ v(x(t), y(t), u(t)) = c_1, \quad w(x(t), y(t), u(t)) = c_2. \]

Eliminating \( t \) from these equations gives a functional relation

\[ F(c_1, c_2) = 0 \]

between \( c_1 \) and \( c_2 \). Then the solution of the Cauchy problem is given by

\[ F(v(x, y, u), w(x, y, u)) = 0. \]

Example 1. Find the integral surface of the equation

\[ (y + xu)u_x + (x + yu)u_y = u^2 - 1 \]

which passes through the parabola

\[ x = t, \quad y = 1, \quad u = t^2. \]

Solution: The solution of the Cauchy problem is

\[ (u - 1)^2 + \frac{x + y}{y - x}(u^2 - 1) + 2(x + y)(u - 1) + 2(x + y)^2 = 0. \]

Example 2. Solve the Cauchy problem

\[ uu_x + uu_y = y + x, \quad u = y^2, \quad x = 1. \]

Solution: The solution of the Cauchy problem is

\[ 2(x - y) + (x + y)^2 - u^2 = 3. \]
Exercises

(1) Find the solution of each of the following Cauchy problem:

(a) \( u_x + u_y = u; \quad u = \cos x, y = 0. \)
(b) \( u(u_x - u_y) = y - x; \quad x = 1, u = y^2. \)
(c) \( (x + u)u_x + (y + u)u_y = 0; \quad x = 1 - t, y = 1 + t, u = t. \)
(d) \( xu_x - yu_y = 0; \quad x = y = u = t. \)
(e) \( x^2u_x + y^2u_y = y^2; \quad x = t, y = 2t, u = 1. \)
(f) \( xuu_x + yuu_y + xy = 0; \quad xy = 1, u = 4. \)
(g) \( 2xuu_x + 2yuu_y = u^2 - x^2 - y^2; \quad x + y + u = 0, x^2 + y^2 + u^2 = 1. \)
(h) \( (u0y)u_x + (u - x)u_y = x - y; \quad x = t, y = 2t, u = 0. \)
(i) \( x(x^2 + y^2)u_x + 2y^2(xu_x + yu_y - u) = 0; \quad x^2 + y^2 = a^2, u = b, \) where \( a, b \) are constants.
(j) \( (x^2 + y^2)u_x + 2xyu_y = xu; \quad x = a, y^2 + u^2 = a^2, \) where \( a \) is a constant.
(k) \( x(y - u)u_x + y(u - x)u_y = u(x - y); \quad x = y = u = t. \)
(l) \( u(x + u)u_x - y(y + u)u_y = 0; \quad x = 1, y = t, u = \sqrt{t}. \)
(m) \( \sec(x)u_x + u_y = \cot y; \quad u(0, y) = \sin y. \)

(2) Consider the equation \( yu_x - xu_y = 0, \quad (y > 0). \) Check for each of the following initial conditions whether the problem is solvable. If it is solvable, find a solution. If it is not, explain why.

(a) \( u(x, 0) = x^2. \)
(b) \( u(x, 0) = x. \)
(c) \( u(x, 0) = x, \quad x > 0. \)

2.4 The method of characteristics

The method of characteristics is a method to solve Cauchy problems for first-order partial differential equations. This method was developed in the middle of the nineteenth century by Hamilton.

The characteristics method is based on generating the solution surface from a one-parameter family of curves that intersect a given curve in space.

Consider the general first-order linear equation

\[ A(x, y)u_x + B(x, y)u_y + C(x, y)u = G(x, y) \]  \hspace{1cm} (7)

and write the initial condition parameterically:

\[ \Gamma : \quad x = x_0(s), \quad y = y_0(s), \quad u = u_0(s), \quad s \in [a, b]. \]

The curve \( \Gamma \) will be called the initial curve.

A solution of equation (7) defines an integral surface \( u = u(x, y) \) in the \( xyu \)-space. The normal to this surface at a point \( P \) is the vector \( n = (u_x, u_y, -1) \). Let \( v = (A, B, -C + G) \) be a vector. Then
the equation (7) can be interpreted as the condition that at each point \( P \) of the integral surface, the vector \( v \) is tangent to the integral surface. Thus the linear equation (7) can be written as

\[
(A, B, -Cu + G) \cdot (u_x, u_y, -1) = 0.
\]

In other words, the vector direction \( v = (A, B, -Cu + G) \) is tangential to the integral surface at each point. The direction \( v = (A, B, -Cu + G) \) at any point on the surface is called the characteristic direction. A space curve

\[
(x(t), y(t), u(t))
\]

whose tangent at every point coincides with the characteristic direction is called a characteristic curve and is given by the equations

\[
\frac{dx}{dt} = A(x(t), y(t)), \\
\frac{dy}{dt} = B(x(t), y(t)), \\
\frac{du}{dt} = -C(x(t), y(t))u(t) + G(x(t), y(t)),
\]

defines curves lying on the solution surface. This system is called the system of characteristic equations or, for short, the characteristic equations.

In order to determine a characteristic curve we need an initial condition. We shall require the initial point to lie on the initial curve \( \Gamma \). Since each curve \((x(t), y(t), u(t))\) emanates from a different point \( \Gamma(s) \), we shall explicitly write the curves in the form \((x(t, s), y(t, s), u(t, s))\). The initial conditions are written as:

\[
x(0, s) = x_0(s), \quad y(0, s) = y_0(s), \quad u(0, s) = u_0(s).
\]

Notice that we selected the parameter \( t \) such that the characteristic curve is located on \( \Gamma \) when \( t = 0 \). One may, of course, select any other parameterization. We also notice that, in general, the parameterization \((x(t, s), y(t, s), u(t, s))\) represents a surface in \( \mathbb{R}^3 \).

One can readily verify that the method of characteristics applies to the quasilinear equation

\[
P(x, y, u)u_x + Q(x, y, u)u_y = R(x, y, u)
\]

as well. Namely, each point on the initial curve \( \Gamma \) is a starting point for a characteristic curve. The characteristic equations are now

\[
x_t(t) = P(x, y, u), \quad y_t(t) = Q(x, y, u), \quad u_t(t) = R(x, y, u),
\]

supplemented by the initial condition (9).

**Example 1.** Solve the equation

\[u_x + u_y = 2\]

subject to the initial condition \( u(x, 0) = x^2 \).

**Solution:** A parametric representation of the integral surface

\[x(t, s) = t + s, \quad y(t, s) = t, \quad u(t, s) = 2t + s^2.\]

The explicit representation of the integral surface is given by

\[u(x, y) = 2y + (x - y)^2.\]
Example 2. Solve the equation $u_x = 1$ subject to the initial condition $u(0, y) = g(y)$.

Solution: A parametric representation of the integral surface

$$x(t, s) = t, \quad y(t, s) = s, \quad u(t, s) = t + g(s).$$

The explicit representation of the integral surface is given by

$$u(x, y) = x + g(y).$$

Example 3. Determine whether the Cauchy problem

$$u_x + yu_y = 2u,$$

$\Gamma : x = 1, \quad y = s, \quad u = s,$

has a unique solution, infinitely many solutions, or no solution.

Solution:

A parametric representation of the integral surface

$$x(t, s) = t + 1, \quad y(t, s) = se^t, \quad u(t, s) = se^{2t}.$$

The explicit representation of the integral surface is given by

$$u(x, y) = ye^{x-1}.$$

Exercises

(1) Solve the initial value problem

$$u_x + u_y = u^2, \quad u(s, 0) = s^2.$$

(2) Solve the Cauchy problem

$$xu_x + (y + x^2)u_y = u, \quad u(2, s) = s - 4.$$

(3) Find the integral surface of the equation

$$u_x + u_y = u^2$$

passing through the curve

$$\Gamma : x = s, \quad y = -s, \quad u = s, \quad s \in \mathbb{R}.$$

(4) Find the solution of the equation

$$uu_x + u_y = 1$$

for the following initial conditions:

(a) $u(x, 0) = x.$
(b) \( u(x, 0) = x^2. \)

(5) Consider the equation

\[ uu_x + u_y = \frac{1}{2} u. \]

(a) Show that there is a unique integral surface in a neighborhood of the curve

\[ \Gamma : (x, y, u) = (s, 0, \sin s), \quad s \in \mathbb{R}. \]

(b) Find the parametric representation

\[ x = x(t, s), \quad y = y(t, s), \quad u = u(t, s) \]

of the integral surface \( S \) for initial condition of part (a).

(c) Find an integral surface \( S_1 \) of the same PDE passing through the initial curve

\[ \Gamma_1 : (x, y, u) = (s, s, 0), \quad s \in \mathbb{R}. \]