Chapter 6
Fourier Integrals and Fourier Transforms

In this chapter we discuss methods to solve partial differential equation in infinite domains.

6.1 Fourier Integrals

In earlier chapters, we have described Fourier series for periodic functions. However, functions which are not periodic cannot be represented by Fourier series. In many problems of physical interest, it is desirable to develop an integral representation for such a function that is analogous to a Fourier series.

Definition. (Fourier integrals ) Let \( f \) be a function and let

\[
A(\alpha) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos(\alpha x) dx, \\
B(\alpha) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin(\alpha x) dx.
\]

Then the integral

\[
\int_{0}^{\infty} [A(\alpha) \cos(\alpha x) + B(\alpha) \sin(\alpha x)] d\alpha
\]

is called the Fourier integral formula for \( f(x) \).

Theorem 1. Assume that \( f \) a piecewise smooth function on every finite interval \([a, b] \subseteq \mathbb{R}\) and assume that \( \int_{-\infty}^{\infty} |f(x)| dx \) converges. Then the Fourier integral of \( f \) converges to \( \frac{1}{2}[f(x+) + f(x-)] \) for all \( x \in \mathbb{R} \); that is

\[
\int_{0}^{\infty} [A(\alpha) \cos(\alpha x) + B(\alpha) \sin(\alpha x)] d\alpha = \frac{1}{2}[f(x+) + f(x-)], \quad \forall x \in \mathbb{R}.
\]

Example 1. Consider the function

\[
f(x) = \begin{cases} 
0, & x < 0, \\
x, & 0 < x < 1 \\
0, & x > 1.
\end{cases}
\]
(a) Graph \( f(x) \).

(b) Find the Fourier integral formula for \( f(x) \).

(c) Determine the convergence of the Fourier integral in part (b) at \( x = 1 \).

Solution:

Remarks.

(1) If \( f(x) \) is an even function, then \( B(\alpha) = 0 \) and

\[
A(\alpha) = \frac{2}{\pi} \int_{0}^{\infty} f(x) \cos(\alpha x) dx.
\]

(2) If \( f(x) \) is an odd function, then \( A(\alpha) = 0 \) and

\[
B(\alpha) = \frac{2}{\pi} \int_{0}^{\infty} f(x) \sin(\alpha x) dx.
\]

Example 1. Consider the function

\[
f(x) = \begin{cases} 
0, & x < -\pi, \\
-1, & -\pi < x < 0, \\
1, & 0 < x < \pi, \\
0, & x > \pi.
\end{cases}
\]

(a) Graph \( f(x) \).

(b) Find the Fourier integral representation for \( f(x) \).

(c) Determine the convergence of the Fourier integral in part (b) at \( x = -\pi \).

Solution:

Application of Fourier integrals to partial differential equations

Given a problem that is defined for \( x \) in an infinite interval, there are five basic steps in solving the problem by the Fourier integrals:

(1) Use separation of variables to convert the partial differential equation into two ordinary differential equations.

(2) Solve the boundary value problem for \( X(x) \) and find the eigenvalues and eigenfunctions.

(3) Solve the other ordinary differential equation.

(4) Use superposition to write \( u \) as a Fourier integral.

(5) Find the constants in the Fourier integral.
Example 1. Use Fourier integral to solve the initial-boundary value problem

\[ u_t = u_{xx}, \quad 0 < x < \infty, \quad t > 0, \]
\[ u(0, t) = 0, \quad t \geq 0, \]
\[ u(x, 0) = f(x), \]

where \( f(x) = \begin{cases} 
1, & 0 < x < 1, \\
0, & 1 < x < \infty.
\end{cases} \)

Solution:

Example 2. Use Fourier integral to solve the initial-boundary value problem

\[ u_{tt} = 4u_{xx}, \quad 0 < x < \infty, \quad t > 0, \]
\[ u_x(0, t) = 0, \quad t \geq 0, \]
\[ u(x, 0) = e^{-x}, \quad u_t(x, 0) = 0, \quad 0 < x < \infty. \]

Solution:

Exercises

(1) Find the Fourier integral formula for each of the following functions:

(a) \( f(x) = \begin{cases} 
0, & x < \pi, \\
x, & -\pi < x < 0 \\
1, & 0 < x < \pi \\
0, & x > \pi.
\end{cases} \)

(b) \( f(x) = \begin{cases} 
x + 1, & |x| < \pi, \\
0, & |x| > \pi.
\end{cases} \)

(c) \( f(x) = \begin{cases} 
sin x, & |x| < \pi, \\
0, & |x| > \pi.
\end{cases} \)

(2) Let \( f(x) = \begin{cases} 
-1, & -a < x < 0, \\
1, & 0 \leq x < a0, \quad \text{otherwise.}
\end{cases} \)

Show that \( f(x) \) has the Fourier sine integral representation

\[ f(x) \sim 2 \pi \int_{0}^{\infty} \frac{1}{\alpha} [1 - \cos(a\alpha)] \sin(x\alpha) d\alpha. \]

(3) Let \( f(x) = \begin{cases} 
0, & x < 0, \\
\cos x, & 0 \leq x < \pi0, \\
x > \pi.
\end{cases} \)

(a) Show that \( f(x) \) has the Fourier integral representation

\[ f(x) \sim \frac{1}{\pi} \int_{0}^{\infty} \frac{\alpha}{(1 - \alpha^2)} [\sin \alpha(\pi - x) - \sin(x\alpha)] d\alpha. \]

(b) When \( x = 0 \), show that

\[ \int_{0}^{\infty} \frac{\alpha \sin(x\alpha)}{(1 - \alpha^2)} d\alpha = \frac{\pi}{2}. \]
(4) Use Fourier integral to solve the initial-boundary value problem

\[ u_{tt} = u_{xx}, \quad 0 < x < \infty, \quad t > 0, \]
\[ u(0, t) = 0, \quad t \geq 0, \]
\[ u(x, 0) = \frac{1}{x^2 + 1}, \quad u_t(x, 0) = 0. \]

(5) Use Fourier integral to solve the initial-boundary value problem

\[ u_t = u_{xx}, \quad 0 < x < \infty, \quad t > 0, \]
\[ u(0, t) = 0, \quad t \geq 0, \]
\[ u(x, 0) = \begin{cases} 
  x, & 0 < x < 2, \\
  0, & 2 < x < \infty.
\end{cases} \]

6.2 Fourier Sine and Cosine Transforms

The Fourier sine and cosine are particularly appropriate in solving boundary-value problems for semi-infinite regions. We first give a formal definition of the Fourier sine and cosine transforms.

**Definition.** (Fourier sine and cosine Transforms)

Let \( f(x) \) be a continuous and piecewise smooth function such that \( \int_0^\infty |f(x)|dx \) converges.

1. The Fourier cosine transform of \( f \) is

\[ \mathcal{F}_c\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(x\alpha)dx = F_c(\alpha), \quad \alpha > 0 \]

and its inverse is

\[ f(x) = \mathcal{F}_c^{-1}\{F_c(\alpha)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\alpha) \cos(x\alpha)d\alpha, \quad x > 0. \]

2. The Fourier sine transform of \( f \) is

\[ \mathcal{F}_s\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(x\alpha)dx = F_s(\alpha), \quad \alpha > 0 \]

and its inverse is

\[ f(x) = \mathcal{F}_s^{-1}\{F_s(\alpha)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(\alpha) \sin(x\alpha)d\alpha, \quad x > 0. \]

**Example 1.** Find the Fourier sine and cosine transforms of \( f(x) = e^{-kx}, \quad k > 0 \).

**Solution:**
Theorem 2. (Fourier sine and cosine transforms for derivatives)
Assume that $f$, $f'$ and $f''$ are continuous and
\[ \int_0^\infty |f(x)|dx, \quad \int_0^\infty |f'(x)|dx, \quad \int_0^\infty |f''(x)|dx \]
converge. Moreover assume that
\[ \lim_{x \to \infty} f(x) = \lim_{x \to \infty} f'(x) = 0. \]

Then
\begin{align*}
(1) \quad & F_c\{f''(x)\} = -\sqrt{\frac{2}{\pi}} f'(0) - \alpha^2 F_c\{f(x)\}, \\
(2) \quad & F_s\{f''(x)\} = \alpha \sqrt{\frac{2}{\pi}} f(0) - \alpha^2 F_s\{f(x)\}. 
\end{align*}

Proof. Integration by parts. \hfill \square 

Example 1. Use $F_c\{f''(x)\} = -\sqrt{\frac{2}{\pi}} f'(0) - \alpha^2 F_c\{f(x)\}$ to find $F_c\{e^{-x}\}$.

Solution:

Example 2. Solve the integral equation
\[ \int_0^\infty f(x) \cos(x\alpha)dx = \begin{cases} 1, & 0 < \alpha < \pi, \\ 0, & \pi < \alpha < \infty. \end{cases} \]

Solution:

Example 3. Solve the integral equation
\[ \int_0^\infty f(x) \sin(x\alpha)dx = \begin{cases} \alpha, & 0 < \alpha < \pi, \\ 0, & \pi < \alpha < \infty. \end{cases} \]

Solution:

Properties of Fourier Sine and Cosine Transforms

Theorem 3. (The Fourier sine and cosine transforms are linear)
Let $f$ and $g$ be functions and let $a$ and $b$ be real numbers. Then
\begin{align*}
(a) \quad & F_s\{af + bg\} = aF_s\{f\} + bF_s\{g\}, \\
(b) \quad & F_c\{af + bg\} = aF_c\{f\} + bF_c\{g\}.
\end{align*}

Proof. \hfill \square
Application to initial-boundary value problems

Given a problem that is defined for \( x \in (0, \infty) \) there are three basic steps in solving the problem by the Fourier sine or cosine transform:

1. Apply the transform to the equation and to the given conditions to transform the problem.
2. Solve the transformed problem to find the transform of the solution \( u \).
3. Find the inverse of the transform obtained in step (2).

Example 1. Use Fourier sine transform to solve the initial-boundary value problem

\[
\begin{align*}
    u_t &= u_{xx}, \quad 0 < x < \infty, \quad t > 0, \\
    u(0, t) &= 0, \quad t \geq 0, \\
    u(x, 0) &= f(x),
\end{align*}
\]

where \( f(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & 1 < x < \infty. \end{cases} \)

Solution:

Example 2. Use Fourier cosine transform to solve the boundary value problem

\[
\begin{align*}
    u_{xx} + u_{yy} &= 0, \quad 0 < x < \infty, \quad 0 < y < 2, \\
    u_x(0, y) &= 0, \quad 0 < y < 2, \\
    u(x, 0) &= 0, \quad 0 < x < \infty, \\
    u(x, 2) &= f(x), \quad 0 < x < \infty,
\end{align*}
\]

where \( f(x) = \begin{cases} x, & 0 < x < 1, \\ 0, & 1 < x < \infty. \end{cases} \)

Solution:

Exercises

(1) Find the Fourier sine and cosine transforms of the following functions:
   (a) \( f(x) = xe^{-x} \).
   (b) \( f(x) = e^{-x} \cos x \).
   (c) \( f(x) = e^{-x} \sin x \).

(2) Solve the following integral equations
   (a) \( \int_0^\infty f(x) \sin(x\alpha)dx = e^{-\alpha} \).
   (b) \( \int_0^\infty f(x) \sin(x\alpha)dx = \begin{cases} 1 - \alpha, & 0 < \alpha < 1, \\ 0, & 1 < \alpha < \infty. \end{cases} \)
(3) Use Fourier cosine transform to solve the boundary value problems:

(a) \[ u_{tt} = u_{xx}, \quad 0 < x < \infty, \quad t > 0, \]
\[ u(0, t) = 0, \quad t \geq 0, \]
\[ u(x, 0) = xe^{-x}, \quad u_t(x, 0) = 0 \quad 0 < x < \infty. \]

(b) \[ u_t = u_{xx}, \quad 0 < x < \infty, \quad t > 0, \]
\[ u_x(0, t) = 1, \quad t \geq 0, \]
\[ u(x, 0) = 0, \quad 0 < x < \infty. \]

(3) Use Fourier sine transform to solve the boundary value problems:

(a) \[ u_t = u_{xx}, \quad 0 < x < \infty, \quad t > 0, \]
\[ u(0, t) = 1, \quad t \geq 0, \]
\[ u(x, 0) = 0, \quad 0 < x < \infty. \]

(b) \[ u_{xx} + u_{yy} = 0, \quad 0 < x < \infty, \quad 0 < y < 1, \]
\[ u_x(0, y) = y(1 - y), \quad 0 < y < 2, \]
\[ u(x, 0) = u(x, 1) = 0, \quad 0 < x < \infty. \]

(c) \[ u_t - u_{xx} + tu = 0, \quad 0 < x < \infty, \quad 0 < y < 1, \]
\[ u_x(0, t) = 0, \quad t \geq 0, \]
\[ u(x, 0) = e^{-x}, \quad 0 < x < \infty. \]

6.3 Fourier Transforms

Fourier Transform is an extremely powerful mathematical tool for the analysis of non-periodic functions. The Fourier transform is of fundamental importance in a broad range of applications, including both ordinary and partial differential equations, quantum mechanics, signal and image processing, control theory, and probability, to name but a few. We first give a formal definition of the Fourier transform by using the complex Fourier integral formula.

**Definition.** (Fourier Transform)

Let \( f(x) \) be a function such that \( \int_{-\infty}^{\infty} |f(x)| dx \) exists.

(1) The Fourier transform of \( f \) is
\[
    \mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\alpha x} dx = F(\alpha), \quad \alpha \in \mathbb{R}.
\]

(2) The inverse Fourier transform of a Fourier transform \( F(\alpha) \) is
\[
    \mathcal{F}^{-1}\{F(\alpha)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha)e^{i\alpha x} d\alpha \sim f(x), \quad x \in \mathbb{R}.
\]
Example 1. Find the Fourier transform of \( f(x) = \begin{cases} 1, & |x| < k, \\ 0, & |x| > k. \end{cases} \)

Solution:

Example 2. Use \( \int_{-\infty}^{\infty} e^{-ky^2} \, dy = \sqrt{\frac{\pi}{k}} \) to find the Fourier transform of \( f(x) = e^{-kx^2}, \, k > 0. \)

Solution:

Properties of Fourier Transforms

Theorem 4. (The Fourier transform and its inverse are linear)

Let \( f \) and \( g \) be functions with Fourier transforms \( F(\alpha) \) and \( G(\alpha) \) respectively. Then for any real numbers \( a \) and \( b \) we have

(a) \( \mathcal{F}\{af(x) + bg(x)\} = a\mathcal{F}\{f(x)\} + b\mathcal{F}\{g(x)\}, \)

(b) \( \mathcal{F}^{-1}\{aF(\alpha) + bG(\alpha)\} = a\mathcal{F}^{-1}\{F(\alpha)\} + b\mathcal{F}^{-1}\{G(\alpha)\}. \)

Proof. By linearity of integration.

Theorem 5. (Scaling)

Let \( \mathcal{F}\{f(x)\} \) be the Fourier transform of a function \( f(x) \). Then

\( \mathcal{F}\{f(x - c)\} = e^{-i\alpha c} \mathcal{F}\{f(x)\}, \)

where \( c \) is a real constant.

Proof.

Theorem 6. (Differentiation)

Let \( f, f', \ldots, f^{(n-1)} \) be continuous and let \( f^{(n)} \) be piecewise continuous on \( \mathbb{R} \). Assume that \( f(x), f'(x), \ldots, f^{(n-1)}(x) \) approach zero as \( |x| \to \infty \). If \( f, f', \ldots, f^{(n)} \) are absolutely integrable, then

\( \mathcal{F}\{f^{(n)}(x)\} = (i\alpha)^n \mathcal{F}\{f(x)\}, \quad n \in \mathbb{N}. \)

Proof. Integration by parts and induction.

Solving problems by Fourier transforms

Given an problem that is defined for \( x \in \mathbb{R} \) there are three basic steps in solving the problem by the Fourier transform:

1. Apply the Fourier transform to the equation and to the given conditions to transform the problem.

2. Solve the transformed problem to find the Fourier transform of the solution \( u \).

3. Find the inverse of the Fourier transform obtained in step (2).
**Example 1.** Use Fourier transform to solve the initial-boundary value problem
\[
  u_t = u_{xx}, \quad -\infty < x < \infty, \quad t > 0,
\]
\[
  u(x, 0) = f(x),
\]
where \( f(x) = \begin{cases} 
  1, & |x| < 1, \\
  0, & |x| > 1.
\end{cases} \)

**Solution:**

**Example 2.** Use Fourier transform to solve the boundary value problem
\[
  u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad 0 < y < \infty,
\]
\[
  u(x, 0) = f(x), \quad -\infty < x < \infty,
\]
\[
  u \text{ is bounded as } y \to \infty.
\]

**Solution:**

**Exercises**

1. **Find the Fourier transforms of the following functions:**
   - (a) \( f(x) = e^{-|x|}. \)
   - (b) \( f(x) = e^{-ax^2}. \)
   - (c) \( f(x) = \frac{1}{|x|}. \)
   - (d) \( f(x) = \cos x^2. \)
   - (e) \( f(x) = \frac{1}{x^2 + a^2}. \)
   - (f) \( f(x) = \begin{cases} 
  1 - |x|, & |x| < 1, \\
  0, & |x| > 1.
\end{cases} \)

2. **Use Fourier transform to solve the initial-boundary value problem**
   \[
   u_{tt} - u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0,
   \]
   \[
   u(x, 0) = f(x), \quad -\infty < x < \infty,
   \]
   \[
   u_t(x, 0) = g(x) \quad -\infty < x < \infty.
   \]

3. **Use Fourier transform to solve the initial-boundary value problem**
   \[
   u_t - u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0,
   \]
   \[
   u(x, 0) = e^{-x^2}, \quad -\infty < x < \infty.
   \]

4. **Use Fourier transform to solve the initial-boundary value problem**
   \[
   u_t - u_{xx} - u = 0, \quad -\infty < x < \infty, \quad t > 0,
   \]
   \[
   u(x, 0) = f(x), \quad -\infty < x < \infty.
   \]
(5) Use Fourier transform to solve the boundary value problem

\[ u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad 0 < y < 1, \]
\[ u(x, 0) = e^{-2|x|}, \quad u(x, 1) = 0, \quad -\infty < x < \infty. \]

(6) Use Fourier transform to solve the boundary value problem

\[ u_{xx} + u_{yy} = u, \quad -\infty < x < \infty, \quad 0 < y < 1, \]
\[ u_y(x, 0) = 0, \quad u(x, 1) = e^{-x^2}, \quad -\infty < x < \infty. \]

(7) Use Fourier transform to solve the boundary value problem

\[ u_{xx} + u_{yy} = e^{-x^2}, \quad -\infty < x < \infty, \quad 0 < y < 1, \]
\[ u(x, 0) = u(x, 1) = 0, \quad -\infty < x < \infty. \]