Chapter I
Vector Analysis
1.1 Vector Algebra

It is well-known that any vector can be written as

\[ \vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \]

Vectors obey the following algebraic rules:

(i) \[ \vec{A} \pm \vec{B} = (A_x \pm B_x) \hat{i} + (A_y \pm B_y) \hat{j} + (A_z \pm B_z) \hat{k} \]

(ii) \[ \vec{A} + \vec{B} = \vec{B} + \vec{A} \quad \text{Commutative} \]

(iii) \[ (\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C}) \quad \text{Associative} \]

(iv) \[ \lambda \vec{A} = \lambda A_x \hat{i} + \lambda A_y \hat{j} + \lambda A_z \hat{k} \quad \lambda \text{ is a scalar} \]

(v) \[ \vec{A} \cdot \vec{B} = AB \cos \theta = A_x B_x + A_y B_y + A_z B_z \]

(vi) \[ \vec{A} \cdot \vec{A} = A^2 \]
The law of cosines

Let \( \vec{C} = \vec{A} - \vec{B} \) \implies

\[
\vec{C} \cdot \vec{C} = (\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B}) = \vec{A} \cdot \vec{A} - 2\vec{A} \cdot \vec{B} + \vec{B} \cdot \vec{B}
\]

\[
C^2 = A^2 + B^2 - 2AB \cos \theta
\]

(iii) \( \vec{A} \times \vec{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \)

(ix) \( \vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \) not commutative

Triple Product

(i) \( \vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{C} \times \vec{A}) \)

volume of parallelepiped

(ii) \( \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B} (\vec{A} \cdot \vec{C}) - \vec{C} (\vec{A} \cdot \vec{B}) \) bac – cab rule
The position vector of a point in 3-D is expressed in Cartesian coordinates as

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

The infinitesimal displacement vector is

$$d\vec{l} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

A source point $r'$ is the point where an e. charge is located.

A field point $r$ is the point at which you are calculating electric or magnetic filed.

The separation vector from the source point to the field point is defined as

$$d\vec{r} = \vec{r} - \vec{r}'$$
Differential Calculus

Ordinary Derivatives

It is known that \[ df = \left( \frac{df}{dx} \right) dx \]
that is if \( x \) is changed by \( dx \), the function changes by \( df \) with \( \frac{df}{dx} \) is the proportionality factor.

In other words we say that the derivative is small if the function varies slowly with \( x \) and large if the function varies rapidly with \( x \).

Geometrically we say that \( \frac{df}{dx} \) is the slope of the graph \( f(x) \) vs. \( x \).

Gradient (Directional Derivative)

Let \( T \) be a scalar function of 3-variables, i.e., \( T = T(x, y, z) \) \( \Rightarrow \)

\[
dT = \left( \frac{\partial T}{\partial x} \right) dx + \left( \frac{\partial T}{\partial y} \right) dy + \left( \frac{\partial T}{\partial z} \right) dz \quad (1)
\]

This tells us how \( T \) varies as we go a small distance \((dx, dy, dz)\) away from the point \((x, y, z)\). Let us rewrite Eq. (1) as
\[
d T = \left\{ \left( \frac{\partial T}{\partial x} \right) \hat{i} + \left( \frac{\partial T}{\partial y} \right) \hat{j} + \left( \frac{\partial T}{\partial z} \right) \hat{k} \right\} \cdot \{dx \hat{i} + dy \hat{j} + dz \hat{k} \}
\]

or \(dT = \nabla T \cdot d\vec{l}\)

with \(\nabla T = \frac{\partial T}{\partial x} \hat{i} + \frac{\partial T}{\partial y} \hat{j} + \frac{\partial T}{\partial z} \hat{k}\)

is called the gradient of the function \(T\).

The symbol \(\nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}\) is called the del operator.

Like any vector the gradient has a magnitude and a direction. Now Eq. (3) can be written as

\[
d T = |\nabla T| |d\vec{l}| \cos \theta
\]

For fixed \(|d\vec{l}|\), \(dT\) is maximum when \(\theta = 0\), that is \(dT\) is maximum when we move in the same direction as \(|\nabla T|\).
The gradient $\nabla T$ points in the direction of maximum increase of the function $T$.

The magnitude $|\nabla T|$ gives the slope along this maximal direction.

$\nabla T$ is directed normal to the level surface of $T$ through the point being considered, i.e., $\nabla T$ is perpendicular to the surface $T=\text{constant}$.

**Example** Find the gradient of the position vector.

**Solution** In Cartesian coordinates the magnitude of the position vector is

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\nabla r = \frac{\partial r}{\partial x} \hat{i} + \frac{\partial r}{\partial y} \hat{j} + \frac{\partial r}{\partial z} \hat{k} = \frac{x}{r} \hat{i} + \frac{y}{r} \hat{j} + \frac{z}{r} \hat{k} = \vec{r} = \hat{r}$$

This means that the distance from the origin increases most rapidly in the radial direction.
Divergence

The divergence of a vector $\vec{A}$ is defined as

$$\nabla \cdot \vec{A} = \left( \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

The divergence is a measure of how much the vector spreads out (diverge) from the point in question.

It is also defined as the net rate of flow per unit volume, i.e.,

$$\nabla \cdot \vec{A} = \text{the source density}$$

Example Find the divergence of the vectors $\vec{r}$, $\hat{k}$, and $z\hat{k}$

Solution: Since $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\nabla \cdot \vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

$$\nabla \cdot \hat{k} = \frac{\partial 0}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial 1}{\partial z} = 0$$

$$\nabla \cdot (z\hat{k}) = \frac{\partial 0}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial z}{\partial z} = 1$$
The Curl

The curl of a vector \( \vec{A} \) is defined as

\[
\vec{\nabla} \times \vec{A} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
A_x & A_y & A_z \\
\end{vmatrix}
\]

The curl of a vector is a measure of how much the vector curl around the point in question.

Example Find the curl of the vectors \( \vec{A} = -y\hat{i} + x\hat{j} \) and \( \vec{B} = x\hat{j} \)

Solution:

\[
\vec{\nabla} \times \vec{A} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
-y & x & 0 \\
\end{vmatrix} = \left( \frac{\partial 0}{\partial y} - \frac{\partial x}{\partial z} \right)\hat{i} + \left( -\frac{\partial y}{\partial z} - \frac{\partial 0}{\partial x} \right)\hat{j} + \left( \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} \right)\hat{k} = 2\hat{k}
\]
The Del Operator

\[ \vec{\nabla} \times \vec{B} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \partial x & \partial y & \partial z \\ 0 & x & 0 \end{bmatrix} \left( \frac{\partial 0}{\partial y} - \frac{\partial x}{\partial z} \right) \hat{i} + \left( \frac{\partial 0}{\partial z} - \frac{\partial 0}{\partial y} \right) \hat{j} + \left( \frac{\partial x}{\partial z} + \frac{\partial 0}{\partial y} \right) \hat{k} = \hat{k} \]

The Del Operator Operations

(i) \[ \vec{\nabla} \cdot \vec{\nabla} T = \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \]

(ii) \[ \vec{\nabla} \times \vec{\nabla} T = 0 \]

(iii) \[ \vec{\nabla} \cdot \vec{\nabla} \times \vec{A} = 0 \]

(iv) \[ \vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} \]
Integral Calculus

Line Integral

The line integral is expressed as

\[ \int_{a}^{b} \mathbf{A} \cdot d\mathbf{l} \]

where \( \mathbf{A} \) is a vector function and \( d\mathbf{l} \) is an infinitesimal displacement vector along a path from point \( a \) to point \( b \).

If the path forms a closed loop, a circle is put on the integral sign, i.e.,

\[ \oint \mathbf{A} \cdot d\mathbf{l} \]

If the line integral is independent on the path followed, the vector \( \mathbf{A} \) is called conservative.
Example Let \( \vec{A} = y^2 \hat{i} + 2x(y+1) \hat{j} \) find \( \int_a^b \vec{A} \cdot d\vec{l} \) from point \( a=(1,1,0) \) to point \( b=(2,2,0) \) along the solid path (path 1) and along the dashed path (path 2).

Solution:

\[
d\vec{l} = dx \hat{i} + dy \hat{j} + dz \hat{k} \quad \Rightarrow
\]

we have for the first path

\[
\int_a^b \vec{A} \cdot d\vec{l} = \int_a^b A_x dx + A_y dy + A_z dz
\]

\[
= \int_a^b y^2 dx + 2x(y+1) dy
\]

\[
= \int_a^c y^2 dx + \int_c^b 2x(y+1) dy
\]

Now along the path \( ac \ y=1 \), and along the path \( cb \ x=2 \). This leads to

\[
\int_a^b \vec{A} \cdot d\vec{l} = \int_1^2 dx + 4\int_1^2 (y+1) dy = 1 + 10 = 11
\]
For the second path we have \( x = y \) and this gives

\[
\int_{a}^{b} \mathbf{A} \cdot d\mathbf{l} = \int_{1}^{2} y^2 \, dx + \int_{1}^{2} 2x(y + 1) \, dy = \int_{1}^{2} x^2 \, dx + 2 \int_{1}^{2} (y^2 + y) \, dy = 10
\]

\[
= \int_{1}^{2} x^2 \, dx + 2 \int_{1}^{2} (y^2 + y) \, dy = 10
\]

Note that the two results are different, i.e., the vector is not conservative.

**Surface Integral**

The surface integral is expressed as

\[
\int_{S} \mathbf{A} \cdot d\mathbf{S}
\]

where \( \mathbf{A} \) is a vector function and \( d\mathbf{S} \) is an infinitesimal element of area.

Again if the surface is closed we put a circle on the integral sign, that is

\[
\oint \mathbf{A} \cdot d\mathbf{S}
\]

The direction of \( d\mathbf{S} \) is perpendicular to the surface and directed outward for closed surfaces and arbitrary for open surfaces.
Example: Let \( \mathbf{\vec{A}} = 2xz\mathbf{\hat{i}} + (x+2)\mathbf{\hat{j}} + y(z^2-3)\mathbf{\hat{k}} \)

Find \( \int \mathbf{\vec{A}} \cdot d\mathbf{\vec{S}} \) over the 5-sides of a cube of side 2, as shown in the figure (excluding the bottom).

Solution: For the top side \( d\mathbf{\vec{S}}_{\text{top}} = dydx\mathbf{\hat{k}} \) \( \Rightarrow \)

\[
\int_{\text{top}} A \cdot d\mathbf{\vec{S}} = \iint \mathbf{A} \cdot d\mathbf{\vec{S}} = \iint y(z^2-3)dx\,dy
\]

But on the top side \( z=2 \) \( \Rightarrow \) \( \int_{\text{top}} A \cdot d\mathbf{\vec{S}} = \int_0^2 dx \int_0^2 y\,dy = 4 \)

For the right side \( d\mathbf{\vec{S}}_{\text{right}} = dxdz\mathbf{\hat{j}} \) \( \Rightarrow \)

\[
\int_{\text{right}} A \cdot d\mathbf{\vec{S}} = \iint (x+2)dxdz = \int_0^2 (x+2)dx \int_0^2 dz = 12
\]

For the left side \( d\mathbf{\vec{S}}_{\text{left}} = -dxdz\mathbf{\hat{j}} \) \( \Rightarrow \)

\[
\int_{\text{left}} A \cdot d\mathbf{\vec{S}} = -\iint (x+2)dxdz = -\int_0^2 (x+2)dx \int_0^2 dz = -12
\]

For the front side \( d\mathbf{\vec{S}} = dzdy\mathbf{\hat{i}} \) \( \Rightarrow \)

\[
\int_{\text{front}} A \cdot d\mathbf{\vec{S}} = \iint 2xz\,dy\,dz
\]

But on the top side \( x=2 \) \( \Rightarrow \)

\[
\int_{\text{front}} A \cdot d\mathbf{\vec{S}} = 4\int_0^2 dy \int_0^2 z\,dz = 16
\]

For the back side \( d\mathbf{\vec{S}} = -dzdy\mathbf{\hat{i}} \) \( \Rightarrow \)

\[
\int_{\text{back}} A \cdot d\mathbf{\vec{S}} = -\iint 2xz\,dy\,dz
\]

But on the top side \( x=0 \) \( \Rightarrow \)

\[
\int_{\text{back}} A \cdot d\mathbf{\vec{S}} = 0
\]
Volume Integral
The volume integral is expressed as
\[ \int_V T \, d\tau \]
where \( T \) is a vector function and \( d\tau \) is an infinitesimal element of volume.

The Fundamental Theorem of Calculus

\[ \int_a^b \frac{df}{dx} \, dx = f(b) - f(a) \]

The Fundamental Theorem of Gradient

where \( T = T(x,y,z) \) be a scalar function of three variables, then

\[ \int_a^b (\nabla T) \cdot d\vec{l} = T(b) - T(a) \]

Since the right side of the last equation depends only on the end points and not on the path followed we conclude that
Corollary 1: \( \int_a^b (\nabla T) \cdot d\vec{l} \) is independent on the path followed from \( a \) to \( b \).

Corollary 2 \[ \int \nabla T \cdot d\vec{l} = 0 \]

Example: Let \( T = xy^2 \)

Check the fundamental theorem of gradient by taking two paths from point \( a (0,0,0) \) to point \( b (2,1,0) \).

Solution: The first path is 2-steps: step (i) along the \( x \)-axis and then up step (ii).

Now \( d\vec{l} = dx\hat{i} + dy\hat{j} + dz\hat{k} \) and \( \nabla T = y^2\hat{i} + 2xy\hat{j} \)

For the 1\(^{st} \) path
\[ \int_a^b (\nabla T) \cdot d\vec{l} = \int_{(i)} \nabla T \cdot dx\hat{i} + \int_{(ii)} \nabla T \cdot dy\hat{j} \]
\[ = \int_0^2 y^2 \, dx + \int_0^1 2xy \, dy \]

But for step (i) \( y=0 \), and for step (ii) \( x=2 \) \( \Rightarrow \int_a^b (\nabla T) \cdot d\vec{l} = 0 + 2 = 2 \)

For the 2\(^{nd} \) path \( y = \frac{1}{2}x \) \( \Rightarrow dy = \frac{1}{2} \, dx \)

\[ \int_a^b (\nabla T) \cdot d\vec{l} = \int_{(iii)} y^2 \, dx + 2xy \, dy = \int_0^1 \frac{1}{4} x^2 \, dx + \frac{1}{2} x^2 \, dx = \int_0^2 \frac{3}{4} x^2 \, dx = 2 \]
The Fundamental Theorem of Divergence

\[ \int_v \nabla \cdot \mathbf{A} \, d\tau = \int_S \mathbf{A} \cdot d\mathbf{S} \]

It states that the integral of a divergence over a volume is equal to the value of the function at the boundary.

In another world, the divergence theorem states that the outward flux of a vector field through a surface is equal to the triple integral of the divergence on the region inside the surface.

**Example:** Check the divergence theorem using the vector

\[ \mathbf{A} = y^2 \hat{i} + (2xy + z^2) \hat{j} + 2yz \hat{k} \]

Over the unit cube situated at the origin.

**Solution:**

\[ \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 2x + 2y \quad \Rightarrow \]

\[ \int_v \nabla \cdot \mathbf{A} \, d\tau = \iiint_{000} (2x + 2y) \, dxdydz = 2 \]
To find $\int_S \vec{A} \cdot d\vec{S}$ we have to calculate the integral over all the faces:

$$\int_{top} \vec{A} \cdot d\vec{S} = \iint S 2yz \, dx \, dy$$

But for the top side $z=1$, so we have

$$\int_{top} \vec{A} \cdot d\vec{S} = \int_0^1 2 \, dx \int_0^1 y \, dy = 1$$

$$\int_{bottom} \vec{A} \cdot d\vec{S} = -\iint S 2yz \, dx \, dy$$

But for the top side $z=0$, so we have $\int_{bottom} \vec{A} \cdot d\vec{S} = 0$

$$\int_{right} \vec{A} \cdot d\vec{S} = \iint (2xy + z^2) \, dx \, dz$$

But for the right side $y=1$, so we have $\int_{right} \vec{A} \cdot d\vec{S} = \int_0^1 (2x) \, dx + \int_0^1 z^2 \, dz = \frac{4}{3}$
\[ \int_{left} \vec{A} \cdot d\vec{S} = - \iint (2xy + z^2) \, dx\, dz \]

But for the left side \( y=0 \), so we have
\[ \int_{left} \vec{A} \cdot d\vec{S} = - \int_{0}^{1} z^2 \, dz = -\frac{1}{3} \]

\[ \int_{front} \vec{A} \cdot d\vec{S} = \iint y^2 \, dy\, dz = \int_{0}^{1} y^2 \, dy \int_{0}^{1} dz = \frac{1}{3} \]

\[ \int_{back} \vec{A} \cdot d\vec{S} = - \iint y^2 \, dy\, dz = \int_{0}^{1} y^2 \, dy \int_{0}^{1} dz = -\frac{1}{3} \]

\[ \oint \vec{A} \cdot d\vec{S} = 1 + 0 + \frac{4}{3} - \frac{1}{3} + \frac{1}{3} - \frac{1}{3} = 2 \]

**Stokes' Theorem**

\[ \oint_{S} (\nabla \times \vec{A}) \cdot d\vec{S} = \oint_{C} \vec{A} \cdot d\vec{l} \]

Since the boundary line for any closed surface shrink down to a point, then
\[ \oint_{S} (\nabla \times \vec{A}) \cdot d\vec{S} = 0 \]
Example: Check the Stokes' theorem using the vector

\[ A = \left(2x z + 3y^2 \right) \hat{j} + 4yz^2 \hat{k} \]

Over the square surface shown.

Solution: \[ \nabla \times A = \left(4z^2 - 2x \right) \hat{i} + 2z \hat{k} \]

\[ d\vec{S} = dydz \hat{i} \]

\[ \oint (\nabla \times A) \cdot d\vec{S} = \int_{0}^{1} \int_{0}^{2} \left(4z^2 - 2x \right) dydz. \]

But on the surface \( x=0 \), so we have

\[ \oint (\nabla \times A) \cdot d\vec{S} = \int_{0}^{1} \int_{0}^{2} 4z^2 dydz = \frac{4}{3} \]
\[ \int \vec{A} \cdot d\vec{l} = \int_{\text{bottom}} \vec{A} \cdot dy\hat{j} + \int_{\text{right}} \vec{A} \cdot dz\hat{k} + \int_{\text{top}} \vec{A} \cdot (-dy\hat{j}) + \int_{\text{left}} \vec{A} \cdot (-dz\hat{k}) \]

Along the bottom side \( x=z=0 \), so we have

\[ \int_{\text{bottom}} \vec{A} \cdot dy\hat{j} = \int_{0}^{1} (2xz + 3y^2) dy = 1 \]

Along the top side \( x=0, z=1 \) so we have

\[ -\int_{\text{top}} \vec{A} \cdot dy\hat{j} = -\int_{0}^{1} (2xz + 3y^2) dy = -1 \]

Along the right side \( x=0, y=1 \) so we have

\[ \int_{\text{right}} \vec{A} \cdot dz\hat{k} = \int_{0}^{1} (4yz^2) dz = \frac{4}{3} \]

Along the left side \( x=0, y=0 \) so we have

\[ \int_{\text{left}} \vec{A} \cdot dz\hat{k} = -\int_{0}^{1} (4yz^2) dz = 0 \]

\[ \int \vec{A} \cdot d\vec{l} = 1 + \frac{4}{3} - 1 + 0 = \frac{4}{3} \]
Integration by Parts

It is known that \( \frac{d}{dx}(fg) = f \frac{dg}{dx} + g \frac{df}{dx} \)

Integrating both sides we get \( \int_{a}^{b} \frac{d}{dx}(fg) \, dx = \int_{a}^{b} f \frac{dg}{dx} \, dx + \int_{a}^{b} g \frac{df}{dx} \, dx \)

Using the fundamental theorem of calculus we get

\( \int_{a}^{b} f \frac{dg}{dx} \, dx = fg \bigg|_{a}^{b} - \int_{a}^{b} g \frac{df}{dx} \, dx \)

Example: Evaluate the integral \( \int_{0}^{\infty} xe^{-x} \, dx \)

Solution: It is known that \( \int_{0}^{\infty} xe^{-x} \, dx = \int_{0}^{\infty} x \frac{d}{dx} \left(-e^{-x}\right) \, dx \quad \Rightarrow \quad \int_{0}^{\infty} xe^{-x} \, dx = \left[-e^{-x}\right]_{0}^{\infty} = 1 \)
Curvilinear Coordinates

**Spherical coordinates** \((r, \theta, \phi)\)

- **\(r\):** is the distance from the origin (from 0 to \(\infty\))
- **\(\theta\):** the polar angle, is the angle between \(r\) and the \(z\)-axis (from 0 to \(\pi\))
- **\(\phi\):** the azimuthal angle is the angle between the projection of \(r\) to the \(x-y\) plane and the \(x\)-axis (from 0 to \(2\pi\))

The relation between the Cartesian coordinates and the spherical coordinates can be written as

\[
\begin{align*}
x &= r \sin \theta \cos \phi \\
y &= r \sin \theta \sin \phi \\
z &= r \sin \cos \theta
\end{align*}
\]

The unit vectors associated with the spherical coordinates are related to the corresponding unit vectors in the Cartesian coordinates as

\[
\begin{align*}
\hat{r} &= \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k} \\
\hat{\theta} &= \cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k} \\
\hat{\phi} &= -\sin \phi \hat{i} + \cos \phi \hat{j}
\end{align*}
\]
The infinitesimal displacement vector in spherical coordinates is expressed as

$$d\mathbf{l} = dr \hat{r} + r d\theta \hat{\theta} + r \sin \theta d\phi \hat{\phi}$$

The volume element is expressed as

$$dV = dV_r d\theta d\phi = r^2 \sin \theta d\theta d\phi dr$$

For the surface elements we have

- $$d\mathbf{S}_1 = dl_r dl_\theta d\phi = r d\theta dr \hat{\phi} \quad \phi \text{ is constant}$$
- $$d\mathbf{S}_2 = dl_r dl_\phi d\theta = r \sin \theta d\phi dr \hat{\theta} \quad \theta \text{ is constant}$$
- $$d\mathbf{S}_3 = dl_\theta dl_\phi \hat{r} = r^2 \sin \theta d\theta d\phi \hat{r} \quad r \text{ is constant}$$

To find the volume of a sphere of radius $R$ we have

$$V = \int dV = \int_0^R \int_0^\pi \int_0^{2\pi} r^2 \sin \theta dr d\theta d\phi = \frac{4}{3} \pi R^3$$
To find the gradient in spherical coordinates let \( T=T(r, \theta, \phi) \) so

\[
\mathbf{d}T = \mathbf{\nabla} T \cdot \mathbf{d}\mathbf{l} = (\mathbf{\nabla} T)_r \mathbf{\hat{r}} \, dr + (\mathbf{\nabla} T)_\theta r \, d\theta + (\mathbf{\nabla} T)_\phi r \sin \theta \, d\phi
\]  

(1)

\[
dT = \left( \frac{\partial T}{\partial r} \right) dr + \left( \frac{\partial T}{\partial \theta} \right) d\theta + \left( \frac{\partial T}{\partial \phi} \right) d\phi
\]  

(2)

Equating the above two equations we get

\[
(\mathbf{\nabla} T)_r = \frac{\partial T}{\partial r} \mathbf{\hat{r}} \quad (\mathbf{\nabla} T)_\theta = \frac{1}{r} \frac{\partial T}{\partial \theta} \mathbf{\hat{r}} \quad (\mathbf{\nabla} T)_\phi = \frac{1}{r \sin \theta} \frac{\partial T}{\partial \phi} \mathbf{\hat{r}}
\]

or

\[
\mathbf{\nabla} T = \frac{\partial T}{\partial r} \mathbf{\hat{r}} + \frac{1}{r} \frac{\partial T}{\partial \theta} \mathbf{\hat{r}} + \frac{1}{r \sin \theta} \frac{\partial T}{\partial \phi} \mathbf{\hat{r}} \quad \Rightarrow
\]

\[
\mathbf{\nabla} = \frac{\partial}{\partial r} \mathbf{\hat{r}} + \frac{1}{r} \frac{\partial}{\partial \theta} \mathbf{\hat{r}} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \mathbf{\hat{r}}
\]

Similarly, one can find the divergence and the curl in spherical coordinates.
\[ \nabla \cdot \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 A_r \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta A_\theta \right) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi} \]

\[ \nabla \times \vec{A} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{r} & r \hat{\theta} & r \sin \theta \hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & r A_\theta & r \sin \theta A_\phi \end{vmatrix} \]

The Laplacian is defined as

\[ \nabla^2 T = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \phi^2} \]
Cylindrical coordinates \((\rho, \theta, \phi)\)

\(\rho\) : is the distance from the \(z\)-axis (from 0 to \(\infty\))

\(\phi\) : the azimuthal angle is the angle between \(\rho\) and the \(x\)-axis (from 0 to \(2\pi\))

\(z\) : the distance from the \(x-y\) plane (from \(-\infty\) to \(\infty\))

The relation between the Cartesian coordinates and the cylindrical coordinates can be written as

\[ x = \rho \cos \phi \quad y = \rho \sin \phi \quad z = z \]

The unit vectors associated with the cylindrical coordinates are related to the corresponding unit vectors in the Cartesian coordinates as

\[ \hat{\rho} = \cos \phi \hat{i} + \sin \phi \hat{j} \]
\[ \hat{\phi} = -\sin \phi \hat{i} + \cos \phi \hat{j} \]
\[ \hat{z} = \hat{k} \]
The infinitesimal displacement vector in cylindrical coordinates is expressed as

\[ d\vec{l} = d\rho \hat{\rho} + \rho d\phi \hat{\phi} + dz \hat{z} \]

The volume element is expressed as

\[ d\tau = dl_\rho dl_\phi dl_z = \rho d\rho d\phi dz \]

For the surface elements we have

\[ d\vec{S}_1 = dl_\rho dl_\phi \hat{z} = \rho d\rho d\phi \hat{z} \quad \text{z is constant} \]

\[ d\vec{S}_2 = dl_r dl_z \hat{\phi} = d\rho dz \hat{\phi} \quad \phi \text{ is constant} \]

\[ d\vec{S}_3 = dl_\phi dl_z \hat{\rho} = \rho d\phi dz \hat{\rho} \quad \rho \text{ is constant} \]
The Del

\[ \vec{\nabla} = \frac{\partial}{\partial \rho} \hat{\rho} + \frac{1}{\rho} \frac{\partial}{\partial \phi} \hat{\phi} + \frac{\partial}{\partial z} \hat{z} \]

The Divergence

\[ \vec{\nabla} \cdot \vec{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \]

The Curl

\[ \vec{\nabla} \times \vec{A} = \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho \hat{\phi} & \rho \sin \theta \hat{\phi} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} \]

The Laplacian

\[ \nabla^2 T = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial T}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} \]
The Dirac Delta Function

Consider the function \( \vec{A} = \frac{1}{r^2} \hat{r} \)

Now \( \vec{\nabla} \cdot \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{1}{r^2} \right) = 0 \)

and \( \int_S \vec{A} \cdot d\vec{S} = \int_0^\pi \int_0^{2\pi} \left( \frac{1}{r^2} \hat{r} \right) \cdot \left( r^2 \sin \theta \ d\theta \ d\phi \hat{r} \right) = \int_0^\pi \int_0^{2\pi} \left( \sin \theta \ d\theta \ d\phi \right) = 4\pi \)

But from the divergence theorem we know that \( \int_v \vec{\nabla} \cdot \vec{A} \ d\tau = \int_S \vec{A} \cdot d\vec{S} \)

Her we have a contradiction. The problem is the point \( r=0 \), where the vector blows up.

Her we have a contradiction. The problem is the point \( r=0 \), where the vector blows up. So we write

\[ \vec{\nabla} \cdot \frac{\hat{r}}{r^2} = 4\pi \delta^3(r) \]
Where $\delta(x)$ is called the Dirac delta function with the following properties:

$$
\delta(x-a) = \begin{cases} 
0 & x \neq a \\
\infty & x = a 
\end{cases}
$$

$$
\int_{-\infty}^{\infty} \delta(x-a) \, dx = 1
$$

$$
\int_{-\infty}^{\infty} F(x) \delta(x-a) \, dx = F(a)
$$

$$
\delta^{3}(r) = \delta(x) \delta(y) \delta(z)
$$

Using these properties we have

$$
\int_{v} \vec{\nabla} \cdot \vec{A} \, d\tau = \int_{v} 4\pi \delta^{3}(r) \, d\tau = 4\pi
$$

As expected