Chapter II
Newtonian Mechanics
Single Particle

**Recommended problems:** 2-2, 2-5, 2-6, 2-8, 2-9, 2-11, 2-12, 2-14, 2-16, 2-21, 2-22, 2-24, 2-25, 2-26, 2-27, 2-29, 2-30, 2-32, 2-34, 2-37, 2-38, 2-39, 2-41, 2-42, 2-43, 2-44, 2-47, 2-51, 2-52, 2-53, 2-54.
2.2 Newton’s Laws

The First Law: A body remains at rest or in uniform motion unless acted upon by a force.

The Second Law: A body acted upon by a force moves in such a manner that the force is equal to the time rate of change of momentum, i.e.,

\[ \vec{F} = \frac{d\vec{P}}{dt} \]  

(2.1)

with \( \vec{P} = m\vec{v} \)  

(2.2)

The Third Law: If two bodies exert force on each other, these force are equal in magnitude and opposite in direction.

In another word, if two bodies constitute an ideal, isolated system, then the accelerations of these bodies are always in opposite directions, and the ratio of the magnitudes of the accelerations is constant and equal to the inverse ratio of the masses of the bodies.

\[ \vec{F}_1 = -\vec{F}_2 \]  

(2.3)

Using Eq.(2.1) we get
\[ \frac{d\vec{P}_1}{dt} = -\frac{d\vec{P}_2}{dt} \quad (2.4) \Rightarrow \]

\[ m_1 \frac{d\vec{v}_1}{dt} = -m_2 \frac{d\vec{v}_2}{dt} \quad (\text{with constant mass}) \quad \Rightarrow \]

\[ m_1\ddot{a}_1 = -m_2\ddot{a}_2 \quad \Rightarrow \quad \frac{\ddot{a}_1}{\ddot{a}_2} = -\frac{m_2}{m_1} \quad (2.5) \]

Eq. (2.4) can be rearranged as

\[ \frac{d(\vec{P}_1 + \vec{P}_2)}{dt} = 0 \quad \Rightarrow \quad \vec{P}_1 + \vec{P}_2 = \text{constant} \]

That is, for isolated system, the total momentum is conserved.
2.3 Frames of Reference

For Newton's laws of motion to have meaning, the motion must be measured relative to some reference frame.

A reference frame is called an inertial frame if Newton’s laws are valid in that frame.

If Newton’s laws are valid in one reference frame, then they are also valid in any reference frame in uniform motion (not accelerated) with respect to the first frame.

This is because the Equation $\bar{F} = m\bar{\dot{v}}$ involves a time derivative of velocity: a change of coordinates involving constant velocity doesn’t influence the equation.
2.4 The Equation of Motion of a Particle

Newton’s second law can be expressed as

$$\vec{F} = m\vec{v} = m\vec{r}$$  \hspace{1cm} (2.6)

This is a second order differential equation, which can be solved to find $r$ if the force function $F(v,r,t)$ and the initial values of $r$ and $v$ are known.

**Example 2.1** If a particle slides without friction down a fixed, inclined plane with $\theta=30^\circ$. What is the block’s acceleration.

**Solution** There are two forces acting on the block: The gravitational force and the normal force. If the block is constrained to move on the plane, and taking the +ve $x$-axis down the plane the only direction the block can move is the $x$-direction. Eq.(2.6) now reads

$$\vec{F}_g + \vec{N} = m\vec{r}$$
Applying the last equation in the two directions, we have

\[ y\text{-direction:} \quad -F_g \cos \theta + N = m\ddot{y} = 0 \]

\[ x\text{-direction:} \quad F_g \sin \theta = m\ddot{x} \quad \Rightarrow \quad \ddot{x} = \frac{F_g}{m} \sin \theta \]

\[ mg \sin \theta = m\ddot{x} \quad \Rightarrow \quad \ddot{x} = g \sin \theta = 4.9 \text{ m/s}^2 \]

To find the velocity after time \( t \), we have

\[ \frac{dx}{dt} = g \sin \theta \quad \Rightarrow \quad v = g \sin \theta \int_{t_0}^{t} dt \quad \Rightarrow \quad v - v_o = g \sin \theta t \]

To find the velocity after it moves a distance \( x \) down the plane we have

\[ \frac{dx}{dt} = g \sin \theta \quad \Rightarrow \quad 2x \frac{dx}{dt} = 2g \sin \theta \dot{x} \quad \Rightarrow \quad \frac{dx^2}{dt} = 2g \sin \theta \frac{dx}{dt} \]

\[ \frac{dx^2}{dt} = 2g \sin \theta \frac{dx}{dt} \quad \Rightarrow \quad \int_{t_0}^{t} dx^2 = 2g \sin \theta \int_{x_0}^{x} dx \Rightarrow \]

\[ v^2 - v_o^2 = 2g \sin \theta (x - x_o) \]
\[ v^2 - v_0^2 = 2g \sin \theta (x - x_o) \]

If at \( t=0 \), \( x_o \) and \( v_o \) are zero, then \[ v = \sqrt{2gx \sin \theta} \]

**Example 2.2** If the coefficient of static friction between the block and the plane is \( \mu_s=0.4 \), at what angle \( \theta \) will the block start sliding if it is initially at rest?

**Solution** We have now an additional force, the static frictional force which is parallel to the plane. Applying Newton’s we have

\( y \)-direction: \[-F_g \cos \theta + N = m\ddot{y} = 0\]

\( x \)-direction: \[F_g \sin \theta - f_s = m\ddot{x}\]

Knowing that \( f_s \leq f_{\text{max}} = \mu_s N \)

At the verge of slipping \( f_s \) reaches its maximum value, then we write
\[ F_g \sin \theta - f_{\text{max}} = m\ddot{x} \quad \Rightarrow \quad F_g \sin \theta - \mu_s N = m\ddot{x} \]

Substituting for \( F_g = mg \) and for \( N \) from the first equation we get

\[ mg \sin \theta - mg \mu_s \cos \theta = m\ddot{x} \]

\[ \ddot{x} = g \left( \sin \theta - \mu_s \cos \theta \right) \]

Just before the block starts to slide, the acceleration is zero \( \Rightarrow \)

\[ \sin \theta - \mu_s \cos \theta = 0 \quad \Rightarrow \]

\[ \theta = \tan^{-1}(\mu_s) = \tan^{-1}(0.4) = 22^\circ \]
Example 2.3 After the block in the previous example begin to slide, the coefficient of kinetic friction becomes $\mu_k = 0.3$. Find the acceleration for the angle $\theta = 30^\circ$.

Solution Applying Newton’s again we have

$\gamma$-direction: 
$$-mg \cos \theta + N = m\ddot{y} = 0$$

$x$-direction: 
$$mg \sin \theta - f_k = m\ddot{x}$$

Knowing that 
$$f_k = \mu_k N = mg \cos \theta \quad \Rightarrow \quad \ddot{x} = g (\sin \theta - \mu_k \cos \theta) = 0.24g$$

In general $\mu_s > \mu_k$. This can be observed by lowering the angle $\theta$ below $16.7^\circ$, we find that $\ddot{x} < 0$ and the block stops. If we increase the angle $\theta$ above $16.7^\circ$, the block doesn’t sliding again until $\theta$ exceeds $22^\circ$. This is because now the force that retarding the motion is no longer the kinetic frictional force but rather it is the static frictional force which is greater than the kinetic frictional force.
Effects of Retarding Forces

If a body is acted upon by a resisting force \( F(v) \) in addition, for instance, to the gravitational force, the total force is then.

\[
\bar{F} = \bar{F}_g + \bar{F}_r = \bar{F}_g + \bar{F}(v)
\]  
(2.7)

or \( \bar{F} = mg - mkv^n \frac{\vec{v}}{v} \)  
(2.8)

Where \( k \) is a positive constant that specifies the strength of the retarding force and \( \vec{v}/v \) is a unit vector in the direction of \( \vec{v} \).

Example 2.4 Find the displacement and the velocity of horizontal motion of a particle in a medium in which the retarding force is proportional to the velocity.

Solution  Knowing that the only horizontal force acting on the particle is the retarding force, applying Newton’s again we have

\[ x \text{-direction:} \quad -kmv = m\ddot{x} \quad \Rightarrow \quad \frac{dv}{dt} = -kv \quad \Rightarrow \]

\[
\int \frac{dv}{v} = -k \int dt \quad \Rightarrow \quad \ln v = -kt + C_1
\]
If the initial velocity (at $t=0$) $v=v_0$ then

$$v = v_0 e^{-kt}$$

To find the displacement we have

$$v = \frac{dx}{dt} = v_0 e^{-kt} \quad \Rightarrow \quad x = v_0 \int e^{-kt} dt = -\frac{v_0}{k} e^{-kt} + C_2$$

If $x(t=0)=0$ then $C_2 = \frac{v_0}{t} \Rightarrow x = \frac{v_0}{k} \left(1 - e^{-kt}\right)$

We can find the velocity as a function of displacement by writing

$$\frac{dv}{dx} = \frac{dv}{dt} \frac{dt}{dx} = \frac{dv}{dt} \frac{1}{v} \quad \Rightarrow \quad v \frac{dv}{dx} = \frac{dv}{dt} = -kv \quad \Rightarrow \quad \frac{dv}{dx} = -k \quad \Rightarrow$$

$$\int dv = -k \int dx \quad \Rightarrow \quad v = -kx + C_3$$

If the initial vel $x(t=0)=0$ and $v(t=0)=v_0$ then $C_3 = v_0$

$$v = v_0 - kx$$
Example 2.5 Find the displacement and the velocity of a particle undergoing vertical motion in a medium with a retarding force is proportional to the velocity.

Solution  Considering the particle is falling downward with an initial velocity $v_0$ from a height $h$ in a constant gravitational field. The equation of motion is

$$F = m \frac{dv}{dt} = -mg - kmv \quad \Rightarrow$$

The minus sign in the retarding force (which is upward force) is due to the fact that the velocity is downward. The last equation can be written as

$$\frac{dv}{kv + g} = -dt \quad \Rightarrow$$

Knowing that (at $t=0$) $v=v_0$ then integrating the last equation we get

$$\frac{1}{k} \ln(kv + g) = -t + C \quad \Rightarrow \quad \frac{1}{k} \ln \left( \frac{kv + g}{k v_0 + g} \right) = -t$$

$$kv + g = (kv_0 + g)e^{-kt} \quad \Rightarrow \quad v = \frac{dz}{dt} = -\frac{g}{k} + \frac{(kv_0 + g)}{k^2} e^{-kt}$$

It is clear that as $t \to \infty$, the velocity approaching the terminal value $(-g/k)$. At this value the net force vanish.
If \( v_0 \) exceeds the terminal velocity in magnitude, then the body begins to slow down and \( v \) approaches the terminal speed from the opposite direction.

To find the displacement we integrate again, with (at \( t=0 \)) \( z=h \) to get

\[
z = h - \frac{gt}{k} + \frac{(kv_0 + g)}{k}(1 - e^{-kt})
\]
Example 2.6 Let us study the projectile motion in 2-dimensions without considering air resistance.

**Solution** The equations of motion are

- **x-direction:** \( 0 = m\ddot{x} \)
- **y-direction:** \( -mg = m\ddot{y} \)

Assuming \( x(t=0) = y(t=0) \) we get

\[
\begin{align*}
\dot{x} &= v_o \cos \theta \\
\dot{y} &= v_o \sin \theta - gt
\end{align*}
\]

\[
\begin{align*}
x &= (v_o \cos \theta)t \\
y &= (v_o \sin \theta)t - \frac{1}{2}gt^2
\end{align*}
\]

Eliminating \( t \) from the above 2-equations we get

\[
y = (v_o \tan \theta)x - \left( \frac{g}{2v_o^2 \cos^2 \theta} \right)x^2
\]

Which is the equation of a parabola.
The speed and the total displacement are found to be

\[ v = \sqrt{x^2 + y^2} = \sqrt{v_o^2 + g^2 t^2 - 2v_o gt \sin \theta} \]

\[ r = \sqrt{x^2 + y^2} = \sqrt{v_o^2 t^2 + \frac{1}{4} g^2 t^2 - v_o g t^3 \sin \theta} \]

The range can be found by determining the value of \( x \) when the projectile falls back to ground, i.e., \( y = 0 \)

\[ y = t(v_o \sin \theta - \frac{1}{2} g t) = 0 \quad \Rightarrow \quad t = 0 \quad \& \quad t = T = \frac{2v_o \sin \theta}{g} \]

Now the range \( R \) is found by

\[ R = x(t = T) = \frac{2v_o^2 \sin \theta \cos \theta}{g} = \frac{v_o^2 \sin 2\theta}{g} \]

It is easy to show that the maximum range occurs at \( \theta = 45^\circ \).
Example 2.7 Let us study the effect of the air resistance to the projectile motion in the previous example, assuming that the retarding force is directly proportional to the projectile’s velocity ($F_r=-kmv$).

Solution The equations of motion are in this case

$x$-direction: 
\[-km\ddot{x} = m\dddot{x}\]

$y$-direction: 
\[-mg - km\dot{y} = m\ddot{y}\]

Assuming again $x(t=0)= y(t=0)$ we get

\[
x = \frac{v_o \cos \theta}{k} \left(1 - e^{-kt}\right) \quad y = -\frac{gt}{k} + \frac{(kv_o \sin \theta + g)}{k^2} \left(1 - e^{-kt}\right)
\]

To find the range we need the time $T$ when $y=0$ \[\Rightarrow \]

\[
T = \frac{(kv_o \sin \theta + g)}{gk} \left(1 - e^{-kT}\right)
\]

This is a transcendental equation so we can’t obtain an analytical expression for $T$. It can be solved by approximation (perturbation) or by numerical technique.
To apply the perturbation method we assume that $k$ is relatively small. Now rewrite the transcendental equation as

$$T = \frac{(k\nu_0 \sin \theta + g)}{gk} \left(kT - \frac{1}{2}k^2T^2 + \frac{1}{6}k^3T^3 + \cdots \right) \Rightarrow$$

$$T = (k\nu_0 \sin \theta / g + 1)T - \frac{1}{2}(k\nu_0 \sin \theta / g + 1)kT^2 + \frac{1}{6}(k\nu_0 \sin \theta / g + 1)k^2T^3 \Rightarrow$$

$$0 = (k\nu_0 \sin \theta / g) - \frac{1}{2}(k\nu_0 \sin \theta / g + 1)T + \frac{1}{6}(k\nu_0 \sin \theta / g + 1)kT^2 \Rightarrow$$

$$T = \frac{2\nu_0 \sin \theta / g}{1 + k\nu_0 \sin \theta / g} + \frac{1}{3}kT^2$$

Using the identity $(1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + \cdots \Rightarrow$

$$\frac{1}{1 + k\nu_0 \sin \theta / g} = 1 - k\nu_0 \sin \theta / g + (k\nu_0 \sin \theta / g)^2 \cdots \Rightarrow$$

To first order of $k$ we then have

$$T = \frac{2\nu_0 \sin \theta}{g} + \left(\frac{T^2}{3} - \frac{2\nu_0^2 \sin^2 \theta}{g^2}\right)k + O(k^2)$$
With no air resistance ($k=0$) we recover the same result as in the previous example, i.e.,

$$T = T_o = \frac{2v_o \sin \theta}{g}$$

If $k$ is small (but nonvanishing), the flight time will be approximately equal to $T_o$. Using this approximated value we get

$$T = \frac{2v_o \sin \theta}{g} + \left( \frac{4v_o^2 \sin^2 \theta}{3g^2} - \frac{2v_o^2 \sin^2 \theta}{g^2} \right) k \quad \Rightarrow \quad T \approx \frac{2v_o \sin \theta}{g} \left( 1 - \frac{kv_o \sin \theta}{3g} \right)$$

Now to find the range we have

$$x = \frac{v_o \cos \theta}{k} \left( kt - \frac{1}{2} k^2 t^2 + \frac{1}{6} k^3 t^3 + \cdots \right)$$

To first order of $k$ the range is obtained from

$$R = v_o \cos \theta \left( T - \frac{1}{2} kT^2 \right)$$

Substituting for $T$ in the last equation
\[ R \approx \frac{v_o^2 \sin 2\theta}{g} \left( 1 - \frac{4kv_o \sin \theta}{3g} \right) = R_o \left( 1 - \frac{4kv_o \sin \theta}{3g} \right) \]

With \( R_o \) is the range without air resistance.
Example 2.9 Atwood's machine consists of a smooth pulley with 2-masses suspended from a light string at each end. Find the acceleration of the masses and the tension of the string (a) when the pulley center is at rest and (b) when the pulley is descending in an elevator with constant acceleration $\alpha$.

Solution (a) The equations of motion, for each mass, are

$$m_1 g - T = m_1 \ddot{x}_1$$
$$m_2 g - T = m_2 \ddot{x}_2$$

If the string is inextensible, i.e.,

$$x_1 + x_2 = \text{constant} \quad \Rightarrow$$

$$\ddot{x}_1 = -\ddot{x}_2 \quad \Rightarrow$$

$$m_1 g - m_2 g = (m_2 + m_1) \ddot{x}_1 \quad \Rightarrow$$

$$\ddot{x}_1 = \frac{(m_1 - m_2) g}{(m_2 + m_1)} = -\ddot{x}_2$$
Solving the first two equations for $T$ we get

$$T = \frac{2m_1m_2g}{(m_2 + m_1)}$$

(b) The coordinate system with the origin at the pulley center is no longer an inertial. So we select the origin of the coordinates to be at the top of the elevator shaft. The equations of motions in such a system are

$$m_1g - T = m_1\ddot{x}_1$$
$$m_2g - T = m_2\ddot{x}_2$$

But, as it is clear from the figure, $x_1'' = x_1' + x_1$, $x_2'' = x_2' + x_2$ \Rightarrow

$$m_1g - T = m_1(\ddot{x}_1' + \ddot{x}_1)$$
$$m_2g - T = m_2(\ddot{x}_2' + \ddot{x}_2)$$

Knowing that $\ddot{x}_1 = -\ddot{x}_2$ & $\ddot{x}_1' = \ddot{x}_2' = \alpha$ \Rightarrow

$$m_1g - T = m_1(\alpha + \ddot{x}_1)$$
$$m_2g - T = m_2(\alpha - \ddot{x}_1)$$
Solving the last 2-equations for the acceleration and the tension we get

\[ \ddot{x}_1 = -\ddot{x}_2 = \frac{(m_1 - m_2)}{(m_2 + m_1)}(g - \alpha) \]

\[ T = \frac{2m_1 m_2 (g - \alpha)}{(m_2 + m_1)} \]

Note that the result are just as if the acceleration of gravity were reduced by an amount of the elevator acceleration. If the elevator is ascending rather than descending we expect

\[ \ddot{x}_1 = -\ddot{x}_2 = \frac{(m_1 - m_2)}{(m_2 + m_1)}(g + \alpha) \]

\[ T = \frac{2m_1 m_2 (g + \alpha)}{(m_2 + m_1)} \]
Example 2.10 Consider a charged particle entering a region of uniform magnetic field. Determine its subsequent motion.

Solution Let the magnetic field be parallel to the $y$-axis. The magnetic force is

$$\vec{F} = q\vec{v} \times \vec{B}$$

The equation of motion reads

$$qB(\dot{x}\hat{k} - \dot{z}\hat{i}) = m(\ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}) \quad \Rightarrow$$

$$m\ddot{x} = -qB\dot{z} \quad (1)$$

$$m\ddot{y} = 0 \quad (2)$$

$$m\ddot{z} = qB\dot{x} \quad (3)$$

Integrating Eq. (2) we get

$$\dot{y} = \text{constant} = \dot{y}_o \quad (4) \quad \Rightarrow$$

$$y = \dot{y}_o t + y_o \quad (5)$$

Integrating Eq. (1&3) we get
\[ \dot{x} = -\frac{qB}{\omega} z + C_1 = -\alpha z + C_1 \quad (6) \]
\[ \dot{z} = \alpha x + C_2 \quad (7) \]

Substituting for \( \dot{z} \) Eq. (7) into Eq. (1) we get

\[ \dot{x} = -\alpha^2 x + C_3 \quad \Rightarrow \quad \dot{x} + \alpha^2 x = \alpha^2 A \quad \Rightarrow \]
\[ \dot{x} + \alpha^2 x = \alpha^2 A \quad (8) \quad \Rightarrow \]
\[ x = a + R \cos(\omega t + \theta_o) \quad (9) \]

Differentiating Eq. (9) w.r.t time and substituting into Eq. (6) \( \Rightarrow \)

\[ -\alpha R \sin(\omega t + \theta_o) = -\alpha z + C_1 \Rightarrow \]
\[ z = b - R \sin(\omega t + \theta_o) \quad (10) \]

Squaring Eqs (9+10) and then adding we get

\[ (x-a)^2 + (z-b)^2 = R^2 \quad (11) \]
Then the path of the motion is a circle of radius $R$ and centered at $(a, b)$.

Now for $\dot{z} = \text{constant}$ the motion is helix with its axis in the direction of $B$.

Now from Eqs.(10&11) we have

$$\dot{x} = -R \omega \sin(\omega t + \theta_o) \quad \dot{y} = -R \omega \cos(\omega t + \theta_o)$$

Squaring the above two equations and then adding we get

$$\dot{x}^2 + \dot{y}^2 = R^2 \quad w^2 = v^2$$

$$\Rightarrow \quad R = \frac{v}{\omega} = \frac{mv}{qB}$$
2.5 Conservation Theorems

Recalling Eq.(1) and assuming that the net force is zero we get

\[ \vec{F} = \frac{d\vec{P}}{dt} = 0 \quad \Rightarrow \]

\[ \vec{P} = \text{constant} \quad (2.9) \]

I. The total linear momentum of a particle is conserved when the total force on it is zero.

Let \( s \) be some constant vector such that \( \vec{F} \cdot \vec{s} = 0 \). Then

\[ \vec{F} \cdot \vec{s} = \vec{P} \cdot \vec{s} = 0 \quad \Rightarrow \]

\[ \vec{P} \cdot \vec{s} = \text{constant} \quad (2.10) \]

The component of linear momentum in a direction in which the force vanishes is constant in time.
Defining the angular momentum of a particle with respect to origin as

\[ \vec{L} = \vec{r} \times \vec{P} \]  \hspace{1cm} (2.11)

The torque with respect to the same origin is defined as

\[ \vec{N} = \vec{r} \times \vec{F} \]  \hspace{1cm} (2.12)

Where \( \vec{r} \) is the position vector from the origin to the point where the force acts.

Now substituting for \( \vec{F} \) from Eq.(2.1) into Eq.(2.12) we get

\[ \vec{N} = \vec{r} \times \vec{P} \]  \hspace{1cm} (2.13)

Now from Eq.(2.12) we have

\[ \vec{L} = \frac{d(\vec{r} \times \vec{P})}{dt} = \dot{\vec{r}} \times \vec{P} + \vec{r} \times \dot{\vec{P}} \]

But

\[ \dot{\vec{r}} \times \vec{P} = m(\dot{\vec{r}} \times \dot{\vec{r}}) = 0 \]

\[ \Rightarrow \]

\[ \vec{L} = \vec{r} \times \vec{P} = \vec{N} \]  \hspace{1cm} (2.14)

If no torque acting on a particle then the angular momentum is constant, i.e.,
II. The angular momentum of a particle subject to no torque is conserved.

Defining the work done on a particle by a force in transforming the particle from point 1 to point 2 as

\[ W = \int_{1}^{2} \vec{F} \cdot d\vec{r} \quad (2.15) \]

Now \( \vec{F} \cdot d\vec{r} = m \frac{d\vec{v}}{dt} \cdot \frac{d\vec{r}}{dt} dt = m \frac{d\vec{v}}{dt} \cdot \vec{v} dt \quad \Rightarrow \]

\[ \vec{F} \cdot d\vec{r} = \frac{1}{2} m \frac{d(\vec{v} \cdot \vec{v})}{dt} dt = d\left( \frac{1}{2} mv^2 \right) \quad \Rightarrow \]

\[ W = \frac{1}{2} m \left( v_2^2 - v_1^2 \right) = T_2 - T_1 \quad (2.16) \]

With \( T = \frac{1}{2} mv^2 \) is the kinetic energy of the particle.

If the work of a force is independent on the path, such a force is called conservative. For every conservative force we associate a potential energy according to
\[ W = -\Delta U \quad \Rightarrow \int_{1}^{2} \bar{F} \cdot d\bar{r} = U_1 - U_2 \quad (2.17) \]

From the last equation we conclude that the force can be written as

\[ \bar{F} = -\nabla U \quad (2.18) \]

To prove Eq. (2.18) we have from Eq. (2.17)

\[ \int_{1}^{2} \bar{F} \cdot d\bar{r} = \nabla U \cdot d\bar{r} = -\int_{1}^{2} dU = U_1 - U_2 \]

Potential energy has no absolute meaning; only differences of potential energy are physically meaningful.

Now defining the total energy as the sum of kinetic and potential energies, i.e.,

\[ E = T + U \quad (2.19) \]

\[ \Rightarrow \quad \frac{dE}{dt} = \frac{dT}{dt} + \frac{dU}{dt} \quad (2.20) \]
But $dT = d\left(\frac{1}{2}mv^2\right) = \vec{F} \cdot d\vec{r}$ \quad \Rightarrow

$$\frac{dT}{dt} = \vec{F} \cdot \vec{r} \quad (2.21)$$

And $\frac{dU}{dt} = \sum_i \frac{\partial U}{\partial x_i} \frac{\partial x_i}{\partial t} + \frac{\partial U}{\partial t} = \sum_i \frac{\partial U}{\partial x_i} \dot{x}_i + \frac{\partial U}{\partial t} = (\vec{\nabla} U) \cdot \vec{r} + \frac{\partial U}{\partial t} \quad (2.22)$

Substituting Eqs. (2.21 & 2.22) into Eq. (2.20) we get

$$\frac{dE}{dt} = (\vec{F} + \vec{\nabla} U) \cdot \vec{r} + \frac{\partial U}{\partial t}$$

Since $\vec{F} = -\vec{\nabla} U \quad \Rightarrow \quad (\vec{F} + \vec{\nabla} U) \cdot \vec{r} = 0 \quad \Rightarrow \quad \frac{dE}{dt} = \frac{\partial U}{\partial t}$

If $U$ is not an explicit function of time then the total energy is constant, i.e.,

III. The total energy $E$ of a particle is a conservative field is conserved.
Example 2.11 A mouse of mass \( m \) jumps on the outside edge of a freely turning ceiling fan of rotational inertia \( I \) and radius \( R \). By what ratio does the angular velocity change?

**Solution** Here the angular momentum is conserved before and after the mouse's jumping. Recalling that the angular momentum can be written as

\[
L = I \omega
\]

\[
L_i = L_f = I \omega_o = I \omega + mvR
\]

Knowing that \( v = \omega R \)

\[
I \omega_o = I \omega + m \omega R^2 = \omega (I + mR^2)
\]

\[
\frac{\omega}{\omega_o} = \frac{I}{I + mR^2}
\]
2.6 Energy

In today’s physics, energy is more popular than Newton’s laws. Most of physical problems are solved by means of energy.

Consider a particle under the influence of a conservative, 1-dimensional, force. The total energy is written as

\[ E = T + U = \frac{1}{2}mv^2 + U(x) \]  \hspace{1cm} (2.23)

\[ \Rightarrow v = \frac{dx}{dt} = \pm \sqrt{\frac{2}{m}} [E - U(x)] \]  \hspace{1cm} (2.24)

\[ \Rightarrow t - t_o = \frac{dx}{dt} = \int_{x_o}^{x} \frac{\pm dx}{\sqrt{\frac{2}{m}} [E - U(x)]} \]  \hspace{1cm} (2.25)

If we know \( U \) we can solve Eq.(2.25) to get \( x \) as a function of time.

We can know a lot about the motion of a particle by examining the plot of \( U(x) \). Let consider the plot of the following figure.

It is clear, from Eq.(2.19), and since \( T \) is always positive \( \Rightarrow \)

\[ E = T + U \geq U(x) \]  \hspace{1cm} (2.26)
The motion is bounded for $E_1$ & $E_2$, i.e., it can't move off to $\infty$.

For $E_1$ the motion is periodic between $x_a$ & $x_b$, i.e., $x_a \leq x \leq x_b$.

For $E_2$ the motion is periodic in two possible regions: between $x_c \leq x \leq x_d$ and $x_e \leq x \leq x_f$.

For $E_0$ the particle is at rest since here $E=U$.

For $E_3$ the particle comes from $\infty$, stops and turns at $x = x_g$ and returns to $\infty$. Here the motion is unbounded.

For $E_4$ the motion is unbounded and the particle may be at any position.

The point $x = x_o$ is called an equilibrium point. In general, the equilibrium state is characterized by

$$\frac{dU(x)}{dx} = 0 \quad (2.27)$$
The equilibrium is said to be stable if \[ \left( \frac{d^2U(x)}{dx^2} \right)_{x=x_o} > 0 \quad (2.28) \]

The equilibrium is said to be unstable if \[ \left( \frac{d^2U(x)}{dx^2} \right)_{x=x_o} < 0 \quad (2.29) \]

An equilibrium is considered **stable** if the system always returns to equilibrium after small disturbances. If the system moves away from the equilibrium after small disturbances, then the equilibrium is **unstable**.
Example 2.12 Consider the system of light pulleys, masses, and string shown. A light string of length $b$ is attached at point $A$, passes over a pulley at point $B$ located a distance $2d$ away, and finally attaches to mass $m_1$. Another pulley with mass $m_2$ attached passes over the string, pulling it down between $A$ & $B$. Calculate the distance $x_1$ when the system is in equilibrium, and determine whether the equilibrium is stable or unstable.

Solution Let $U=0$ along the line $AB$. \[ U = -m_1 g x_1 - m_2 g (x_2 + c) \]

But $x_2 = \sqrt{(b - x_1)^2 / 4 - d^2} \Rightarrow U = -m_1 g x_1 - m_2 g \sqrt{(b - x_1)^2 / 4 - d^2} - m_2 g c$

To determine the equilibrium position we set \[ \frac{dU}{dx_1} = 0 \Rightarrow -m_1 g - \frac{m_2 g (b - x_1)}{4 \sqrt{(b - x_1)^2 / 4 - d^2}} = 0 \Rightarrow \]
\[ 4m_1 \sqrt{(b - x_1)^2/4 - d^2} = m_2 (b - x_1) \implies \]
\[ 4m_1^2 (b - x_1)^2 - 16m_1^2 d^2 = m_2^2 (b - x_1)^2 \implies (b - x_1)^2 \left(4m_1^2 - m_2^2\right) = 16m_1^2 d^2 \implies \]
\[ (b - x_1)^2 = \frac{16m_1^2 d^2}{4m_1^2 - m_2^2} \implies x_0 = x_1 = b - \frac{4m_1 d}{\sqrt{4m_1^2 - m_2^2}} \]

Notice that a real solution exists only when \(4m_1^2 > m_2^2\)

Now to determine whether the equilibrium is stable or not we have

\[ \frac{d^2 U}{dx_1^2} = \frac{m_2 g}{4 \sqrt{(b - x_1)^2/4 - d^2}} + \frac{m_2 g (b - x_1)^2}{16 \left\{(b - x_1)^2/4 - d^2\right\}^{3/2}} \implies \]

Now substituting for \(x_1 = x_0\) we get

\[ \frac{d^2 U}{dx_1^2} = g \left(\frac{4m_1^2 - m_2^2}{4m_2 d}\right)^{3/2} > 0 \text{ for } 4m_1^2 > m_2^2 \implies \]

The equilibrium is stable for real solution.
Example 2.12 Consider the potential

\[ U(x) = -Wd^2 \left( x^2 + d^2 \right) \]

\[ x^4 + 8d^4 \]

Sketch the potential and discuss the motion at various values of \( x \). Is the motion bounded or unbounded? Where are the equilibrium values? Are they stable or unstable? Find the turning points for \( E = -W/8 \). \( W \) is a +ve constant.

Solution Rewrite the potential as

\[ Z(y) = \frac{U(x)}{W} = -\left( \frac{y^2 + 1}{y^4 + 8} \right) \]

with \( y = \frac{x}{d} \)

Let us first find the equilibrium points using Eq. (2.27)

\[ \frac{dZ}{dy} = -\frac{2y}{y^4 + 8} + \frac{4y^3(y^2 + 1)}{(y^4 + 8)^2} = 0 \]

\[ \Rightarrow \]

\[ y(y^4 + 8) + 4y^3(y^2 + 1) \]

\[ (y^4 + 8)^2 \]

\[ y(y^4 + 2y^2 - 8) \]

\[ (y^4 + 8)^2 \]

\[ y(y^4 + 2y^2 - 8) = y(y^2 + 4)(y^2 - 2) = 0 \]

\[ \Rightarrow \]

\[ y_o^2 = 0.2 \]

\[ \Rightarrow \]
Using \( y = \frac{x}{d} \) \( \Rightarrow \) \( x_{o1} = 0 \) \( x_{o2} = \sqrt{2}d \) \( x_{o3} = -\sqrt{2}d \)

We have 3-equilibrium points. Now sketch \( Z(y) \) versus \( y \) we get

As it is clear from the figure, the equilibrium is stable at \( x_{o2} \) & \( x_{o3} \) but unstable fro \( x_{o1} \). The motion is bounded for all energies \( E<0 \).

AT the turning points the speed is zero. So \( T=0 \), i.e.

\[
E = U \quad \Rightarrow \\
E = U(y) = \frac{-W(y^2 + 1)}{y^4 + 8} = -\frac{W}{8} \\
y^4 + 8 = 8y^2 + 8 \quad \Rightarrow \\
y^2(y^2 - 8) = 0 \quad \Rightarrow \quad y = 0, \pm 2\sqrt{2}
\]