Chapter 8
SOURCES OF MAGNETIC FIELD

Recommended Problems:

3, 5, 7, 9, 12, 15, 17, 21, 23, 29, 31, 34, 35, 36, 49, 55, 71, 72.
The Biot and Savart Law

As mentioned before, a current produces a magnetic field. It is proved experimentally that the magnetic field at a nearby point \( P \) due to a small element \( dl \) of the wire is given by

\[
dB = \frac{\mu_o \, I dl \times \hat{r}}{4\pi \, r^2}
\]

where \( r \) is the distance between the element \( dl \) and the point \( P \), and the unit vector directed from the element to the point. The direction of \( dl \) is the direction of the current.

The constant \( \mu_o \) is called the permeability constant with the value

\[
\mu_o = 4\pi \times 10^{-7} \text{ T.m/A}
\]

The total magnetic field due to the entire wire is

\[
B = \frac{\mu_o I}{4\pi} \int \frac{dl \times \hat{r}}{r^2}
\]
It is important to note that the integrand in the last equation is a vector quantity and so it must be handled accordingly.

This means that the integration should be done for each component separately since each component has a specific direction.

If a point along the axis of a wire $r$ and $dl$ are parallel or antiparallel, i.e., $\sin \theta = 0 \Rightarrow$, the magnetic field at a point along the axis of a wire is zero.

The direction of $\mathbf{B}$ is always perpendicular to both $\mathbf{l}$ and $\mathbf{r}$. That is, the field lines of the magnetic field produced by a current element are concentric circles around the element.
Their direction is given by the right hand rule: **Grasp the element in your right hand with the thumb pointing in the direction of the current. Your fingers will curl in the direction of the field.**

The magnetic field lines make concentric circles around the wire such that the direction of $B$ at a point is tangent to these circles at that point.

For a current directed out of page the direction of the m.field is directed as shown.
Consider the current in the length of wire shown. Rank the points $A$, $B$, $C$, and $D$, in terms of magnitude of the magnetic field due to the current in the length element shown, from greatest to least.

$B$, then $C$, then $A=D=0$
Example 30.1 Calculate the magnetic field due to a straight wire of length $2L$ carrying a current $I$ at a point on its perpendicular bisector, at a distance $a$ from the wire.

**Solution**

We begin by identifying an element of the wire, having length $dl=dx$. Now

$$dB = \frac{\mu_0 I}{4\pi} \frac{dx \sin \theta}{r^2}$$

$$B = \frac{\mu_0 I}{4\pi} \frac{2L}{a\sqrt{a^2 + L^2}}$$

If the wire is infinitely long ($L \to \infty$), the term under the square root in the last expression reduces simply to $L$ and the result becomes

$$B = \frac{\mu_0 I}{2\pi a}$$
The direction of the magnetic field, by the right-hand rule, is out of the page for all the elements of the wire. Thus in integrating over the wire we can add just their magnitudes, i.e.,

\[ B = \frac{\mu_0 I}{4\pi} \int \frac{dx \sin \theta}{r^2} \]

From the geometry of Figure we obtain

\[ r^2 = x^2 + a^2 \quad \text{and} \quad \sin \theta = \frac{a}{r} \quad \Rightarrow \]

\[ B = \frac{\mu_0 I a}{4\pi} \int_{-L}^{L} dx \frac{dx}{(a^2 + x^2)^{3/2}} \quad \Rightarrow \]

\[ B = \frac{\mu_0 I}{4\pi} \frac{2L}{a \sqrt{a^2 + L^2}} \]

If the wire is infinitely long \( (L \to \infty) \), the term under the square root in the last expression reduces simply to \( L \) and the result becomes

\[ B = \frac{\mu_0 I}{2\pi a} \]
**Example 30.2**  A wire is bent to form an arc of radius $R$ and central angle $\theta$, as shown in Figure 13.8. If the wire carries a current $I$, calculate the magnetic field at point $O$, the center of the arc.

**Solution**

The wire consists of three parts: two straight portions and a curved one.

Since $dl$ and $r$ are parallel or antiparallel for the straight portions, we deduce that $B$ is zero for them.

Now for the curved portion we have $dl$ tangent to the arc

$$dB = \frac{\mu_o I \, dl \, \sin \theta}{4\pi \, r^2}$$

Note that each element on the arc wire gives a contribution $dB$, which is directed into the page and $\sin\theta=1$. $\Rightarrow$
Using the fact that the length of the arc is \( l = R\theta \), the result reduces to

\[
B = \frac{\mu_0 I \int dl \sin \theta}{2\pi R} = \frac{\mu_0 I}{2\pi R} \int dl
\]

If the wire is a circular ring, then the magnetic field at its center is found from the last formula by letting \( \theta = 2\pi \) as

\[
B = \frac{\mu_0 I}{2R}
\]
Example 30.3  Consider a circular loop of radius \( R \) lying in the \( x\)-\( z \) plane. If the loop carries a current \( I \) as shown, find the magnetic field at point \( P \), a distance \( y \) from the center.

Solution

Let us take an element \( dl \) tangent to the loop and therefore, the angle between \( dl \) and \( r \) is \( 90^\circ \). Now,

\[
dB = \frac{\mu_0 I}{4\pi} \frac{dl \sin \theta}{r^2} = \frac{\mu_0 I}{4\pi} \frac{dl}{r} = \frac{\mu_0 I}{4\pi} \frac{dl}{r} \left( \frac{R^2}{R^2 + y^2} \right)
\]

It is clear that the direction of \( B \) differs from one element to the other. From the symmetry about the \( y \)-axis we conclude that the \( x\)-components cancel and only the \( y\)-components sum up. \( \Rightarrow \)
Note that the magnetic field at its center is found from the last formula by letting \( y=0 \) as

\[
B = \frac{\mu_0 I}{2R}
\]
The figure shows the ends of two long wires carry equal current $I$ but in opposite directions, on out of page, and the other into the page, as shown. The direction of the magnetic field at point $A$ is

a) up  b) down  c) right  d) left

And the direction of the magnetic field at point $B$ is

a) up  b) down  c) right  d) left
FORCE BETWEEN TWO PARALLEL WIRES

Consider two long, and straight, parallel wires separated by a distance $a$ and carrying currents $I_1$ and $I_2$.

Each wire should be affected by a magnetic force due to its presence in the magnetic field of the other wire.

To calculate the force on wire 1 due to the wire 2, we need to find the magnetic field $B_2$ produced by the wire 2 at the site of wire 1.

For a long wire we have, from Example 30.1

$$B_2 = \frac{\mu_o I_2}{2\pi d}$$

The right hand rule tells us that the direction of $B_2$ is out of the plane of the page ⇒
the force acting on wire 1 is

$$\vec{F}_1 = I_1 \vec{L} \times \vec{B}_2 = \frac{\mu_0 I_1 I_2 L}{2\pi a} (i \times \hat{k}) = \frac{\mu_0 I_1 I_2 L}{2\pi a} (-\hat{j})$$

That is, the direction of the force $\vec{F}_1$ is directly toward wire 2, that is, the force between the two wires is attractive.

It is easy to show that if the two currents are in opposite directions, the force will be repulsive. Generally speaking

The wires attract each other if they have currents in the same direction and vice versa.
Example 30.7  Find the m. force exerted on the upper side of the rectangular loop shown.

Solution

The wire $I_1$ creates a magnetic field given by

$$\vec{B}_1 = \frac{\mu_0 I_1}{2\pi x} \mathbf{\hat{\times}}$$

As the m. field is not constant along the top side of the loop, we have to divide it into small elements each of length $dx$, the force on one of these elements is

$$d\vec{F} = I_2 \, d\mathbf{x} \times \vec{B}_1 = B_1 = \frac{\mu_0 I_1 I_2}{2\pi x} \left( \mathbf{i} \times -\mathbf{\hat{k}} \right) = \frac{\mu_0 I_1 I_2}{2\pi x} \left( \mathbf{j} \right)$$

$$\vec{F} = \frac{\mu_0 I_1 I_2}{2\pi} \int_a^{a+b} \frac{dx}{x} \left( \mathbf{j} \right) = \frac{\mu_0 I_1 I_2}{2\pi} \ln \left( \frac{a+b}{a} \right) \mathbf{j}$$
Ampere’s Law

The line integral of $\mathbf{B} \cdot d\mathbf{l}$ around any closed path equals $\mu_0 I_{\text{enc}}$, where $I_{\text{enc}}$ is the total steady current passing through the surface bounded by the closed path, i.e.,

$$\oint \mathbf{B} \cdot d\mathbf{L} = \mu_0 I_{\text{enc}}$$

The integral is over a closed loop, called the Amperian loop.

The m. field due to the whole current distribution.

The length of the Amperian loop.

The total current crossing the surface bounded by the Amperian loop.
Like Gauss’ law, the useful of Ampere’s law is in calculating the m.field. To do that, the following remarks should be noticed.

(i) The closed loop must be chosen such that that is, either $B$ is constant around the loop or the integrand is zero.

(ii) For the above note to be satisfied, Ampere’s law is applicable only for current distribution of high degree of symmetry.

(iii) The point at which $B$ to be calculated must lie on the circumference of that loop.
The figure shows four closed paths around three current-carrying wires. Rank the magnitudes of \( \oint B \cdot dl \) for the four closed paths, from least to greatest.

(b) \( \mu_0 \) then (d) 3 \( \mu_0 \) then (a) 4 \( \mu_0 \) and then (c) 6 \( \mu_0 \)
In Ampere’s law $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$, which of the following statements is correct:

a) $\vec{B}$ is the magnetic field due to the current $I_{enc}$ only.

b) $\vec{B}$ is the magnetic field due to the whole current.

c) $d\vec{l}$ is an element of the wire carrying the current.

d) The integration is to be performed over any loop.
Example 30.4  A long straight wire of radius $R$ carries a steady current $I$, uniformly distributed across the wire. Find the magnetic field outside and inside the wire.

Solution  For the outside region, let us choose a circular loop of radius $r$ around the wire.

\[ \oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} \]

From the symmetry we see that $\vec{B}$ must be constant in magnitude and tangent to the loop at every point on the loop. Thus at every point the angle $\theta$ between $\vec{B}$ and $d\vec{l}$ is $0^\circ$. Then

\[ B \oint dl = B(2\pi r) = \mu_0 I_{enc} \quad \Rightarrow \quad B = \frac{\mu_0 I}{2\pi r} \]

Note that this result agree with that of example 8.1 in the limit of long wire.
For a point inside the wire the circular loop in this case has a radius $r<R$.

Again by symmetry, $B$ is constant in magnitude and tangent to the loop at every point on the loop.

To find the current enclosed by the loop, we note that

$$J = \frac{I}{\pi R^2} = \frac{I_{enc}}{\pi r^2} \implies I_{enc} = I \frac{r^2}{R^2}$$

Ampere’s law now gives

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} \implies$$

$$B\oint dl = B(2\pi r) = \mu_0 I_{enc} = I \frac{r^2}{R^2} \implies B = \mu_0 \frac{Ir}{2\pi R^2}$$

At $r=0$, the magnetic field inside the conductor is zero, Explain?
A solenoid is a long wire wound in the form of a helix around the surface of a cylindrical form. Usually the turns are so closed that each one can be regarded as circular ring. For ideal solenoid, its length is much greater than its diameter.

For such a solenoid, the m.field is zero outside and uniform inside.

To calculate the internal magnetic field, we can apply Ampere’s law to the rectangular loop $abcd$ of length $l$ and width $w$. 
\[
\int \vec{B} \cdot d\vec{l} = \mu_o I_{enc} \quad \Rightarrow \quad \int_a^b \vec{B} \cdot d\vec{l} + \int_c^d \vec{B} \cdot d\vec{l} + \int_b^c \vec{B} \cdot d\vec{l} + \int_d^a \vec{B} \cdot d\vec{l} = \mu_o I_{enc}
\]

For the portions \(bc\) and \(da\) \(\vec{B}\) is \(\perp\) to \(d\vec{l}\) and for the portion \(cd\) \(\vec{B}=0\) \(\Rightarrow\)
\[
\int_a^b \vec{B} \cdot d\vec{l} = \mu_o I_{enc}
\]

For the portion \(ab\) \(\vec{B}\) is \(\parallel\) to \(d\vec{l}\) and its magnitude is constant along it \(\Rightarrow\)
\[
B l = \mu_o I_{enc}
\]

But the current crossing the surface enclosed by the loop is \(I\) multiplied by the No. of turns bounded by the loop \(\Rightarrow\)
\[
B l = \mu_o N I \Rightarrow B = \mu_o \frac{N}{l} I = \mu_o n I
\]

With \(n\) is the No. of turns per unit length.
II- TOROID

A toroid is a long wire wound around a doughnut-shaped structure. It can be considered as a solenoid bent into the shape of a doughnut.

To find the magnetic field inside a toroid, we can apply amperes’s law to a circular loop of radius $r$ as shown.

From the symmetry, the magnetic field lines form concentric circles inside the toroid. This means that $B$ is constant in magnitude along the Amperian loop and directed tangent to it. Therefore, at a point inside the coil we have
\[ \int \vec{B} \cdot d\vec{l} = \mu_o I_{enc} \]

As \( B \) is // to \( dl \) \( \Rightarrow \) \( B(2\pi r) = \mu_o NI \) \( \Rightarrow B = \frac{\mu_o NI}{2\pi r} \)

Note that the current crossing the surface several times equal to the No. of turns.

This result shows that \( B \) is not uniform inside a toroid. The magnetic field outside a toroid is zero, why?

**MAGNETIC FLUX**

As for the electric flux, the magnetic flux through a surface is defined as

\[ \Phi_B = \int \vec{B} \cdot d\vec{A} \]

The SI unit of magnetic flux is T.m² or weber (Wb).

Since no monopole can be isolated we can conclude that

\[ \Phi_B = \int \vec{B} \cdot d\vec{A} = 0 \]

That is, the m. flux through any closed surface is always zero
Consider the hemispherical closed surface shown in the figure. The hemisphere is in a uniform magnetic field that makes an angle $\theta$ with the vertical. The magnetic flux through the hemispherical surface $S_2$ is:

- a) zero
- b) $B\pi R^2$
- b) $B\pi R^2 \cos \theta$
- b) $-B\pi R^2 \cos \theta$
Example 30.4 Find the m. flux through the rectangular loop shown which lies on the right of a long wire of current $I$.

Solution \[ \Phi_B = \int \vec{B} \cdot d\vec{A} \]

With $\vec{B}$ is the m. field created by the wire given by

\[ B = \frac{\mu_0 I}{2\pi r} \]

and directed into the page in the region right to the wire.

As the m. field is not constant over the surface of the loop, we have to divide the surface into small elements each of area $dA = bdr$.

\[ B = \frac{\mu_0 I}{2\pi} \int \frac{dA}{r} \]

\[ B = \frac{\mu_0 Ib}{2\pi} \int_c^{c+a} \frac{dr}{r} = \frac{\mu_0 Ib}{2\pi} \ln \left( \frac{a+c}{a} \right) \]
Problem 30.21  Four long, parallel conductors carry equal currents of $I = 5.0 \, \text{A}$. The current direction is into the page at points $A$ and $B$ and out of the page at $C$ and $D$. Calculate the magnitude and direction of the magnetic field at point $P$, located at the center of the square of edge length 0.2 m.

$$B = \mu_o \frac{I}{2\pi r} = \frac{4\pi \times 10^{-7}}{2\pi} \frac{5}{0.1\sqrt{2}} = 7.1 \, \mu T$$

$$B_A + B_D = 14.2 \, \mu T$$

$$B_B + B_C = 14.2 \, \mu T$$

$$B_P = \sqrt{2} \times 14.2 = 20 \, \mu T \text{ down}$$