Chapter 5

Bracketing Methods
PART II ROOTS OF EQUATIONS

Roots of Equations

Bracketing Methods
- Bisection method
- False Position Method

Open Methods
- Simple fixed point iteration
- Newton Raphson
- Secant
- Modified Newton Raphson

System of Nonlinear Equations

Roots of polynomials
- Muller Method
STUDY OBJECTIVES FOR PART TWO

• Understand the graphical interpretation of a root
• Know the graphical interpretation of the false-position method and why it is usually superior to the bisection method.
• Understand the difference between bracketing and open methods for root location.
• Understand the concepts of convergence and divergence.
• Know why bracketing methods always converge, whereas open methods may sometimes diverge.
• Know the fundamental difference between the false-position and secant methods and how it relates to convergence.
ROOTS OF EQUATIONS

• Root of an equation: is the value of the equation variable which make the equations = 0.0

\[ ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

• But

\[ ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0 \quad \Rightarrow \quad x = ? \]
\[ \sin x + x = 0 \quad \Rightarrow \quad x = ? \]
ROOTS OF EQUATIONS

• Non-computer methods:
  - Closed form solution (not always available)
  - Graphical solution (inaccurate)

• Numerical systematic methods suitable for computers
Graphical Solution

- Plot the function $f(x)$

- The roots exist where $f(x)$ crosses the x-axis.
Graphical Solution: Example

- The parachutist velocity is
  \[ v = \frac{mg}{c} \left( 1 - e^{-\frac{ct}{m}} \right) \]
- What is the drag coefficient \( c \) needed to reach a velocity of 40 m/s if \( m=68.1 \) kg, \( t =10 \) s, \( g= 9.8 \) m/s\(^2\)?

\[ f (c) = \frac{mg}{c} \left( 1 - e^{-\frac{ct}{m}} \right) - v \]

\[ f (c) = \frac{667.38}{c} \left( 1 - e^{-0.146843c} \right) - 40 \]

**Check:** \( F (14.75) = 0.059 \sim 0.0 \)

\[ v (c=14.75) = 40.06 \sim 40 \text{ m/s} \]
Numerical Systematic Methods

I. Bracketing Methods

- For no roots or even number of roots:
  - $f(x_l) = +ve$
  - $f(x_u) = +ve$
  - No roots or even number of roots

- For odd number of roots:
  - $f(x_l) = +ve$
  - $f(x_u) = -ve$
  - Odd number of roots
Bracketing Methods (cont.)

- Two initial guesses \((x_l \text{ and } x_u)\) are required for the root which bracket the root (s).
- If one root of a real and continuous function, \(f(x) = 0\), is bounded by values \(x_l, x_u\) then \(f(x_l)f(x_u) < 0\).

(The function changes sign on opposite sides of the root)
Bracketing Methods

1. Bisection Method

- Generally, if $f(x)$ is real and continuous in the interval $x_l$ to $x_u$ and $f(x_l) \cdot f(x_u) < 0$, then there is at least one real root between $x_l$ and $x_u$ to this function.

- The interval at which the function changes sign is located. Then the interval is divided in half with the root lies in the midpoint of the subinterval. This process is repeated to obtained refined estimates.
**Step 1:** Choose lower $x_l$ and upper $x_u$ guesses for the root such that:

$$f(x_l).f(x_u)<0$$

**Step 2:** The root estimate is:

$$x_r = \frac{x_l + x_u}{2}$$

**Step 3:** Subdivide the interval according to:

- If $(f(x_l).f(x_r)<0)$ the root lies in the lower subinterval; $x_u = x_r$ and go to step 2.
- If $(f(x_l).f(x_r)>0)$ the root lies in the upper subinterval; $x_l = x_r$ and go to step 2.
- If $(f(x_l).f(x_r)=0)$ the root is $x_r$ and stop.
Bisection Method - Termination Criteria

**True relative Error:**

\[ \varepsilon_t = \left| \frac{X_{\text{true}} - X_{\text{approximate}}}{X_{\text{true}}} \right| \times 100\% \]

**Approximate relative Error:**

\[ \varepsilon_a = \left| \frac{X^n_r - X^{n-1}_r}{X^n_r} \right| \times 100\% \]

\[ \varepsilon_a = \left| \frac{X_u - X_l}{X_u + X_l} \right| \times 100\% \text{ (Bisection)} \]

- For the Bisection Method \( \varepsilon_a > \varepsilon_t \)
- The computation is terminated when \( \varepsilon_a \) becomes less than a certain criterion \( (\varepsilon_a < \varepsilon_s) \)
Bisection method: Example

- The parachutist velocity is
  \[ v = \frac{mg}{c} \left( 1 - e^{-\frac{ct}{m}} \right) \]
- What is the drag coefficient \( c \) needed to reach a velocity of 40 m/s if \( m = 68.1 \, \text{kg} \), \( t = 10 \, \text{s} \), \( g = 9.8 \, \text{m/s}^2 \)

\[ f(c) = \frac{mg}{c} \left( 1 - e^{-\frac{ct}{m}} \right) - v \]

\[ f(c) = \frac{667.38}{c} \left( 1 - e^{-0.146843c} \right) - 40 \]
1. **Assume** \( x_l = 12 \) and \( x_u = 16 \)
   \( f(x_l) = 6.067 \) and \( f(x_u) = -2.269 \)

2. **The root**: \( x_r = (x_l + x_u) / 2 = 14 \)
   \( f(14) = 1.569 \)

3. **Check** \( f(12) \cdot f(14) = 6.067 \times 1.569 = 9.517 > 0 \);
   the root lies between 14 and 16.

4. **Set** \( x_l = 14 \) and \( x_u = 16 \), thus the new root
   \( x_r = (14 + 16) / 2 = 15 \)
   \( f(14) = -0.425 \)

5. **Check** \( f(14) \cdot f(15) = 1.569 \times -0.425 = -0.666 < 0 \);
   the root lies bet. 14 and 15.

6. **Set** \( x_l = 14 \) and \( x_u = 15 \), thus the new root
   \( x_r = (14 + 15) / 2 = 14.5 \)

and so on......
Bisection method: Example

- In the previous example, if the stopping criterion is $\varepsilon_t = 0.5\%$; what is the root?

<table>
<thead>
<tr>
<th>Iter.</th>
<th>$X_l$</th>
<th>$X_u$</th>
<th>$X_r$</th>
<th>$\varepsilon_a%$</th>
<th>$\varepsilon_t%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
<td>16</td>
<td>14</td>
<td>-----</td>
<td>5.279</td>
</tr>
<tr>
<td>2</td>
<td>14</td>
<td>16</td>
<td>15</td>
<td>6.667</td>
<td>1.487</td>
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<td>3</td>
<td>14</td>
<td>15</td>
<td>14.5</td>
<td>3.448</td>
<td>1.896</td>
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<td>14.75</td>
<td>1.695</td>
<td>1.204</td>
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<td>14.75</td>
<td>15</td>
<td>14.875</td>
<td>0.84</td>
<td>0.641</td>
</tr>
<tr>
<td>6</td>
<td>14.74</td>
<td>14.875</td>
<td>14.813</td>
<td>0.422</td>
<td>0.291</td>
</tr>
</tbody>
</table>
Bisection method
Bracketing Methods

Bisection

Example;
Use bisection method to find the root of

\[ f(x) = x^3 + 4x^2 - 10 \]

Continue the iterations until the approximate error falls below a stopping criteria \((\varepsilon_s) = 0.5\%\).
Flow Chart – Bisection

Start

Input: $x_l$, $x_u$, $\varepsilon_s$, maxi

$f(x_l) \cdot f(x_u) < 0$

$i = 0$
$\varepsilon_a = 1.1 \varepsilon_s$

while $\varepsilon_a > \varepsilon_s$ & $i < \text{maxi}$

$x_r = \frac{x_l + x_u}{2}$
$i = i + 1$

Print: $x_r$, $f(x_r)$, $\varepsilon_a$, $i$

Stop
\( \varepsilon_u = \left| \frac{x_u - x_f}{x_u + x_f} \right| \times 100\% \)

\( Test = f(x_i). \quad f(x_f) \)

\( Test = 0 \)

\( Test < 0 \)

\( x_i = x_r \)
Bracketing Methods

2. False-position Method

- The bisection method divides the interval $x_l$ to $x_u$ in half not accounting for the magnitudes of $f(x_l)$ and $f(x_u)$. For example if $f(x_l)$ is closer to zero than $f(x_u)$, then it is more likely that the root will be closer to $f(x_l)$. 
2. False-position Method

- False position method is an alternative approach where \( f(x_l) \) and \( f(x_u) \) are joined by a straight line; the intersection of which with the x-axis represents an improved estimate of the root.
False-position Method - Procedure

$$f(x)$$

$$(x_l, f(x_l)), (x_r, f(x_r)), (x_u, f(x_u))$$

$$\frac{f(x_l)}{x_r - x_l} = \frac{f(x_u)}{x_r - x_u}$$

$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$
**False-position Method -Procedure**

**Step 1:** Choose lower $x_l$ and upper $x_u$ guesses for the root such that: $f(x_l).f(x_u)<0$

**Step 2:** The root estimate is:

$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

**Step 3:** Subdivide the interval according to:

- If $(f(x_l).f(x_r)<0)$ the root lies in the lower subinterval; $x_u = x_r$ and go to step 2.
- If $(f(x_l).f(x_r)>0)$ the root lies in the upper subinterval; $x_l = x_r$ and go to step 2.
- If $(f(x_l).f(x_r)=0)$ the root is $x_r$ and stop
The parachutist velocity is

\[ v = \frac{mg}{c} \left( 1 - e^{-\frac{ct}{m}} \right) \]

What is the drag coefficient \( c \) needed to reach a velocity of 40 m/s if \( m = 68.1 \text{ kg} \), \( t = 10 \text{ s} \), \( g = 9.8 \text{ m/s}^2 \)?

\[
f(c) = \frac{mg}{c} \left( 1 - e^{-\frac{ct}{m}} \right) - v
\]

\[
f(c) = \frac{667.38}{c} \left( 1 - e^{-0.146843c} \right) - 40
\]
False position method: Example

1. Assume \( x_l = 12 \) and \( x_u = 16 \)
   \[ f(x_l) = 6.067 \text{ and } f(x_u) = -2.269 \]

2. The root: \( x_r = 14.9113 \)
   \[ f(12) \cdot f(14.9113) = -1.5426 < 0; \]

3. The root lies between 12 and 14.9113.

4. Assume \( x_l = 12 \) and \( x_u = 14.9113 \), \( f(x_l) = 6.067 \) and
   \[ f(x_u) = -0.2543 \]

5. The new root \( x_r = 14.7942 \)

6. This has an approximate error of 0.79%
False position method: Example
Flow Chart –False Position

Start

Input: $x_l$, $x_0$, $\varepsilon_s$, maxi

$f(x_l) \cdot f(x_u) < 0$

$i = 0$
$\varepsilon_a = 1.1 \varepsilon_s$

While
$\varepsilon_a > \varepsilon_s$ & $i < \text{maxi}$

$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$

$i = i + 1$

Stop

Print: $x_r$, $f(x_r)$, $\varepsilon_a$, $i$
\[ i = 1 \quad \text{or} \quad x_r = 0 \]

\[ \varepsilon_a = \left| \frac{x_r - x_{r0}}{x_r} \right| \times 100\% \]

Test = \text{f}(x_i) \cdot \text{f}(x_r)

Test = 0

Test < 0

\begin{align*}
\text{false} & : & x_i &= x_r \\
& & x_{r0} &= x_r
\end{align*}

\begin{align*}
\text{true} & : & x_u &= x_r \\
& & x_{r0} &= x_r
\end{align*}

\begin{align*}
\text{true} & : & \varepsilon_a &= 0.0
\end{align*}
A Case Where Bisection Is Preferable to False Position

Problem Statement. Use bisection and false position to locate the root of
\[ f(x) = x^{10} - 1 \]
between \( x = 0 \) and 1.3.

Solution. Using bisection, the results can be summarized as

<table>
<thead>
<tr>
<th>Iteration</th>
<th>( x_l )</th>
<th>( x_u )</th>
<th>( x_r )</th>
<th>( \varepsilon_a (%) )</th>
<th>( \varepsilon_r (%) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1.3</td>
<td>0.65</td>
<td>100.0</td>
<td>35</td>
</tr>
<tr>
<td>2</td>
<td>0.65</td>
<td>1.3</td>
<td>0.975</td>
<td>33.3</td>
<td>2.5</td>
</tr>
<tr>
<td>3</td>
<td>0.975</td>
<td>1.3</td>
<td>1.1375</td>
<td>14.3</td>
<td>13.8</td>
</tr>
<tr>
<td>4</td>
<td>0.975</td>
<td>1.1375</td>
<td>1.05625</td>
<td>7.7</td>
<td>5.6</td>
</tr>
<tr>
<td>5</td>
<td>0.975</td>
<td>1.05625</td>
<td>1.015825</td>
<td>4.0</td>
<td>1.6</td>
</tr>
</tbody>
</table>

Thus, after five iterations, the true error is reduced to less than 2 percent. For false position, a very different outcome is obtained:

<table>
<thead>
<tr>
<th>Iteration</th>
<th>( x_l )</th>
<th>( x_u )</th>
<th>( x_r )</th>
<th>( \varepsilon_a (%) )</th>
<th>( \varepsilon_r (%) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1.3</td>
<td>0.09430</td>
<td>90.6</td>
<td></td>
</tr>
<tr>
<td>2</td>
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<tr>
<td>3</td>
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<td>0.26287</td>
<td>30.9</td>
<td>73.7</td>
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<td>4</td>
<td>0.26287</td>
<td>1.3</td>
<td>0.33811</td>
<td>22.3</td>
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<tr>
<td>5</td>
<td>0.33811</td>
<td>1.3</td>
<td>0.40788</td>
<td>17.1</td>
<td>59.2</td>
</tr>
</tbody>
</table>
False Position Method - Example 2
Roots of Polynomials: Using Software Packages

MS Excel: Goal seek

\[ f(x) = x - \cos x \]