Calculus A
Chapter 3

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3.2. The Derivative as a Function

**Definition 1.** The derivative of the function $f(x)$ with respect to the variable $x$ is the function $f'$ whose value at $x$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists.

**Alternative Formula for the Derivative**

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

**Calculating Derivatives from the Definition**

**Example 2.** Use the definition to find $f'(x)$ for the following functions:

1. $f(x) = 2x - 1$

2. $f(x) = \sqrt{x + 2}$

3. $f(x) = \frac{x}{x - 3}$
Notations

There are many ways to denote the derivative of a function \( y = f(x) \), where the independent variable is \( x \) and the dependent variable is \( y \). Some common alternative notations for the derivative are

\[
f'(x), y', \frac{dy}{dx}, \frac{df}{dx}, D_x f(x), \dot{f}(x), \ddot{y}.
\]

To indicate the value of a derivative at a specified number \( x = a \), we use the notation

\[
f'(a), \frac{dy}{dx}|_{x=a}, \frac{df}{dx}|_{x=a}, \frac{d}{dx}f(x)|_{x=a}
\]

Differentiable on an Interval; One-Sided Derivatives

**Definition 3.** A function \( y = f(x) \) is differentiable on an open interval (finite or infinite) if it has a derivative at each point of the interval. It is differentiable on a closed interval \([a, b]\) if it is differentiable on the interior \((a, b)\) and if the limits

\[
\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \quad \text{Right-hand derivative at } a
\]

\[
\lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h} \quad \text{Left-hand derivative at } b
\]

exist at the endpoints.

**Example 4.** Show that the function \( y = |x| \) is differentiable on \((-\infty, 0)\) and \((0, \infty)\) but has no derivative at \( x = 0 \).

**Example 5.** Show that \( f(x) = \sqrt{x} \) is not differentiable at \( x = 0 \).
When Does a Function Not Have a Derivative at a Point?

A function \( f \) can fail to have a derivative at a point for many reasons. Some of these reasons are as follows:

1. If \( f \) has a discontinuity at \( x_0 \).

2. If \( \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \pm \infty \), then the graph has a vertical tangent at \( x_0 \).

3. If \( L = \lim_{h \to 0^+} \frac{f(x+h)-f(x)}{h} \neq \lim_{h \to 0^-} \frac{f(x+h)-f(x)}{h} = M \), then the graph has a corner at \( x_0 \).

4. If \( \pm \infty = \lim_{h \to 0^+} \frac{f(x+h)-f(x)}{h} \neq \lim_{h \to 0^-} \frac{f(x+h)-f(x)}{h} = \pm \infty \), then the graph has a cusp at \( x_0 \).

Example 6. Find the points at which \( f \) is not differentiable,

1. \( f(x) = |x| \)

2. \( f(x) = |x| \)

3. \( f(x) = \sqrt{|x|} \)

Differentiable Functions Are Continuous

A function is continuous at every point where it has a derivative.

Theorem 7. (Differentiability Implies Continuity)
If \( f \) has a derivative at \( x = c \), then \( f \) is continuous at \( x = c \).

Remark 8. The converse of Theorem 7 is false. A function need not have a derivative at a point where it is continuous.

Example 9. The function \( f(x) = |x| \) is continuous at \( x = 0 \) but not differentiable.
Example 10. Show that the following function is continuous at $x = 3$ but not differentiable at $x = 3$.

$$f(x) = \begin{cases} 
  x^2 - 1, & x < 3; \\
  \frac{8x}{3}, & x \geq 3.
\end{cases}$$

3.3. Differentiation Rules

In this section we introduce several rules that allow us to differentiate many functions directly, without using the definition.

Differentiation Rules

Let $f$ and $g$ be differentiable functions, and $k$ is constant, then

1. $\frac{d}{dx} k = 0$.

2. If $n$ is a real number, then $\frac{d}{dx} x^n = nx^{n-1}$ for all $x$ where $x^n$ and $x^{n-1}$ are defined.

3. $\frac{d}{dx} kf(x) = k \frac{d}{dx} f(x)$.

4. $\frac{d}{dx} (f(x) \pm g(x)) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$.

5. $\frac{d}{dx} (f(x)g(x)) = g(x) \frac{d}{dx} f(x) + f(x) \frac{d}{dx} g(x)$.

6. If $g(x) \neq 0$ then

$$\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{g(x) \frac{d}{dx} f(x) - f(x) \frac{d}{dx} g(x)}{[g(x)]^2}$$

Example 11. Find the derivative of the following functions:

- $f(x) = (x^3 + 3x)(4x^2 - 1)$

- $f(x) = \frac{(x-1)(x^2-2x)}{x^4}$
Example 12. Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

Example 13. (Normal to a curve)
Find an equation for the line perpendicular to the tangent to the curve $y = x^3 - 4x + 1$ at the point $(2, 1)$.

Example 14. Find the value of $a$ that makes $f(x) = \begin{cases} 2x^2 - 2, & x \leq -1; \\ a(x + 1), & x > -1 \end{cases}$ differentiable for all $x$-values.

Second and Higher-Order Derivatives

The function $f''$ is called the second derivative of $f$ because it is the derivative of the first derivative $f'$. It is written in several ways:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y''$$

In general, the $n$-th derivative of $y = f(x)$ is

$$f^n(x) = \frac{d^n f(x)}{dx^n} = \frac{d}{dx} y^{n-1} = \frac{d^n y}{dx^n} = y^n$$
Example 15. 1. Find $y^{(4)}$ if $y = x^7 + 3x^4 - 8x^2 + 3x - 1$.

2. Find $y''(2)$ if $y = \frac{3x^{-2} + x^2}{2x-1}$.

3.5. Derivatives of Trigonometric Functions

Derivative of the Sine Function

To calculate the derivative of $f(x) = \sin x$, for $x$ measured in radians, we combine the limits with the angle sum identity for the sine function:

$$\sin(x + h) = \sin x \cos h + \cos x \sin h.$$ 

If $f(x) = \sin x$, then

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x + h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin x (\cos h - 1) + \cos x \sin h}{h}$$

$$= \sin x \cdot \lim_{h \to 0} \left(\frac{\cos h - 1}{h}\right) + \cos x \cdot \lim_{h \to 0} \left(\frac{\sin h}{h}\right)$$

$$= \sin x \cdot 0 + \cos x \cdot 1 = \cos x.$$

Derivatives of the trigonometric functions

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$
\[
\frac{d}{dx}(\cot x) = -\csc^2 x \\
\frac{d}{dx}(\sec x) = \sec x \tan x \\
\frac{d}{dx}(\csc x) = -\csc x \cot x
\]

**Example 16.** Find \( \frac{dy}{dx} \) for the following functions:

1. \( y = \frac{x \tan x}{x \sec x} \)

2. \( y = \frac{\cos^2 x}{\sin x + \cot x} \)

3. \( y = (\sin x + \cos x) \sec x \)

4. \( y = \sqrt{\csc x + x^2 \cot x} \)

5. Find \( y'' \) if \( y = \sec x \)

### 3.6 The Chain Rule

**Derivative of a Composite Function**

**Theorem 17.** If \( f(u) \) is differentiable at the point \( u = g(x) \) and \( g(x) \) is differentiable at \( x \), then the composite function \( (f \circ g)(x) = f(g(x)) \) is differentiable at \( x \), and

\[
(f \circ g)'(x) = f'(g(x)).g'(x).
\]

In Leibniz’s notation, if \( y = f(u) \) and \( u = g(x) \), then

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},
\]

where \( \frac{dy}{du} \) is evaluated at \( u = g(x) \).

**Example 18.** Find the derivative of the following functions:

1. Given \( y = \sin u \) and \( u = x^3 + x \csc x \), find \( \frac{dy}{dx} \).
2. \( y = \frac{1}{2} \sin^{-5} x - x^3 \cos^3 x \).

3. \( y = (2x - 5)^{-1}(x^2 - 5x)^6 \).

4. \( y = \tan^2(\sec^3 t) \).

5. \( y = \sqrt[3]{3t + \sqrt{2 + \sqrt{1 - t}}} \).

6. Let \( f \) and \( g \) be two functions such that \( g(x) = f(\sqrt{x}) + \sqrt{f(x)} \). If \( f(1) = 4 \) and \( f'(1) = 8 \), then find \( g'(1) \).

7. Let \( y = f(x)g^4(x) \) and \( f(3) = 5 \), \( g(3) = 2 \), \( f'(3) = -3 \) and \( g'(3) = 4 \). Find \( \frac{dy}{dx} \) at \( x = 3 \).

8. Find \( Y'' \) if \( y = f(\cos x) \).

9. Let \( f \circ g \) be a composite of the differentiable functions \( y = f(u) \) and \( u = g(x) \). If \( f'(4) = -3 \), \( g(1) = 4 \) and \( g'(1) = 7 \), then find \( (f \circ g)'(1) \).
10. Find $y'$ at $x = 0$ if $y = \cos(5x^3 - 3\tan x + 6\pi)$.

Remark 19.  
1. If $f$ is a differentiable functions and even, then $f'$ is odd.
2. If $f$ is a differentiable functions and odd, then $f'$ is even.

3.7 Implicit Differentiation

When we cannot put an equation $F(x, y) = 0$ in the form $y = f(x)$ to differentiate in the usual way, we may still be able to find $y'$ by implicit differentiation.

Implicit defined functions

The graph of of the equation $x = y^2$ has a well-defined slope at nearly every point because it is the union of the graphs of the two differentiable functions $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$. In order to find the slope of the tangent we treat $y$ as a differentiable function of $x$ and differentiate both sides of the equation $y^2 = x$ with respect to $x$, using the differentiation rules.

Example 20.  
1. Find $\frac{dy}{dx}$ if $x^3 + 5y^4 + xy^2 = 4$.

2. Find $\frac{dy}{dx}$ if $y = 2xy^4 + \cot(xy^3)$.

3. If $y^2 + 3y - 2x = 4$, find $d^2y = dx^2$ at the point $(3, 2)$.

4. Find $d^2y = dx^2$ if $y = x^{\frac{3}{2}} + 4y^{\frac{3}{2}}$. 
5. Verify that the point \((-1, 1)\) is on the curve \(x^2 - xy + 2y^3 = 4\) and find the tangent and normal lines to the curve at this point.

6. Find the tangent and normal lines to the curve \(x \sin(2y) = y \cos(2x)\) at the point \((\frac{\pi}{4}, \frac{\pi}{2})\).

### 3.9 Linearization and Differentials

Linearization is a method to approximate complicated functions with simpler ones.

**Definition 21.** (Linearization) Let \(f\) be differentiable function at \(x = x_0\).

1. The linearization of \(f\) at \(x_0\) is the approximating function
   \[
   L(x) = f(x_0) + f'(x_0)(x - x_0)
   \]

2. The approximation \(f(x) \approx L(x)\) of \(f\) by \(L\) is the standard linear approximation of \(f\) at \(x_0\).

**Remark 22.** If \(L\) is the linear approximation of \(f\) at \(x_0\), then \(x_0\) is the center of the approximation.

**Example 23.** Find the linearization of \(f(x) = \sqrt{x^2 + 9}\) at \(x = -4\) and use it to approximate \(f(-4, 5)\).

**Example 24.** Find the linearization of \(f(x) = 2x^2 + x - 2\) at \(x = 1\).
Example 25. Find a linearization of $f(x) = x + \sqrt{x}$ at suitably chosen integer near $x = 2, 1$ at which $f(x)$ and $f'(x)$ are easy to evaluate. Then use the linearization to approximate $f(2, 1)$

Differentials

Theorem 26. (Differential) Let $y = f(x)$ be differentiable function. The differential $dx$ is an independent variable. The differential $dy$ is $dy = f'(x)dx$.

Example 27. Let $y = x\sqrt{4 - x^2}$. Find $dy$ and the value of $dy$ when $x = 0$ and $dx = 0.1$. 