<table>
<thead>
<tr>
<th>Section</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
</tr>
<tr>
<td>Feedback linearization: types, applications and warnings</td>
</tr>
<tr>
<td>motivation</td>
</tr>
<tr>
<td>Feedback linearizable system: advantages and disadvantages</td>
</tr>
<tr>
<td>Calculation of T(·)</td>
</tr>
<tr>
<td>Input-Output Linearization Procedure</td>
</tr>
<tr>
<td>Examples</td>
</tr>
<tr>
<td>Conclusion</td>
</tr>
<tr>
<td>References</td>
</tr>
</tbody>
</table>
Introduction

Feedback linearization is an approach to algebraically transform nonlinear systems into (fully or partly) linear ones, so linear control techniques can be applied.

Differs from Jacobian linearization, because feedback linearization is achieved by exact state transformation and feedback, and linear approximations of dynamics.

The basic idea of simplifying the form of a system by choosing a different state representation; the choice of coordinate systems.

Feedback linearization = ways of transforming original system models into equivalent models of a simpler form.
Feedback linearization

Applications

- Helicopters, high-performance aircraft, industrial robots, biomedical devices, vehicle control.

Warnings

- There are a number of shortcomings and limitations associated with the feedback linearization approach. These problems were very much topics of research (2005).
Terminology

**Feedback Linearization**
- Control techniques in which the input is used to linearize all or part of the system’s differential equations.

**Input/output Linearization**
- The output $y$ is differentiated until the input $u$ appears in the $r$th derivative of $y$. Then $u$ is chosen to yield a transfer function from the “synthetic input”, $v$, to, the relative degree, is less than $n$, the order of the system.

**Input/State Linearization**
- A control technique where some new output $y_{new} = h_{new}(x)$ is chosen so that WRT $y_{new}$, the relative degree of the system is $n$. Then the design procedure using $y_{new}$ is the same as for I/O linearization.
Feedback(Exact) Linearization

• Motivation

Consider
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -a[\sin(x_1 + \delta) - \sin \delta] - bx_2 + cu
\end{align*}
\] pendulum

Choose \( u = \frac{a}{c}[\sin(x_1 + \delta) - \sin \delta] + \frac{v}{c} \)

\[
\begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = -bx_2 + v
\end{cases}
\] : linear system

Choose \( v = k_1x_1 + k_2x_2 \) so that the closed loop system is asymptotically stable.

Thus \( u = \frac{a}{c}[\sin(x_1 + \delta) - \sin \delta] + \frac{1}{c}(k_1x_1 + k_2x_2) \)
Feedback linearizable system

Advantages

- The nonlinear control reduces the system to a linear one (exactly).
- The nonlinear control results in global asymptotically stability of the resulting linear behavior.

Disadvantages

- The real system might be different from the nominal system. Thus the behavior might be nonlinear.
- Not every system can be treated this way.
Thus the feedback linearizable systems are of the form:

\[
\dot{x} = Ax + B\beta^{-1}(x)[u - \alpha(x)] \\
u = \alpha(x) + \beta(x)v
\]
Feedback linearizable system (Cont.)

\[ \dot{x} = Ax + B \beta^{-1}(x)[u - \alpha(x)] \]

where \((A, B)\) controllable \quad \rightarrow \quad (1)

Choose \[ u = \alpha(x) + \beta(x)v \]

\[ \begin{align*}
\dot{x} &= Ax + B \beta^{-1}(x)[\alpha(x) + \beta(x)v - \alpha(x)] \\
&= Ax + Bv
\end{align*} \]

Choose \( v = Kx \) so that \((A + BK) : Hurwitz\)

Then \( u = \alpha(x) + \beta(x)Kx \)

Ex:

\[ \begin{align*}
\dot{y}_1 &= a \sin y_2 \\
\dot{y}_2 &= -y_1^2 + u
\end{align*} \]

Is it of the form (1) ?

Introduce

\[ \begin{align*}
x_1 &= y_1 \\
x_2 &= a \sin y_2 = \dot{y}_1, \quad -\frac{\pi}{2} < y_2 < \frac{\pi}{2}
\end{align*} \]
Feedback linearizable system (Cont.)

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= a \cos y_2 (\gamma y_1^2 + u)
\end{align*}
\]

Choose \( u = y_1^2 + \frac{1}{a \cos y_2} \). Then \( \dot{x}_1 = x_2, \quad \dot{x}_2 = v \).

The transformed dynamic equations are

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= a \cos \left[ \sin^{-1} \left( \frac{x_2}{a} \right) \right] \left( \gamma x_1^2 + u \right)
\end{align*}
\]

which are of the form (*) with

\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

\[
\alpha(x) = x_1^2, \quad \beta(x) = \frac{1}{a \cos \left[ \sin^{-1} \left( \frac{x_2}{a} \right) \right]}
\]

The linearizing controller is

\[
u = x_1^2 + \frac{1}{a \cos \left[ \sin^{-1} \left( \frac{x_2}{a} \right) \right]} v
\]
So to determine whether a system can be reduced to state space form (1), we need the following substitution.

**key:**

\[ x = T(y) \]

Diffeomorphism

where \( T \) should be

- differentiable,
- have a unique inverse \( T^{-1}(x) \) (in the area of interest),
- the inverse has to be differentiable as well.

**Definition:** A nonlinear system

\[
\begin{align*}
\dot{y} &= f(y) + G(y)u & (1) \\
 f : Y &\to \mathbb{R}^n, \quad y \in \mathbb{R}^n, \quad u \in \mathbb{R}^p \quad f &\text{ & } G \Rightarrow \text{sufficiently smooth on } Y. \\
 G : Y &\to \mathbb{R}^{n \times p}
\end{align*}
\]

is said to be feedback (or exact) linearizable if \( \exists \) a diffeomorphism \( T : Y \to \mathbb{R}^n \) such that \( D = T(Y) \) contains the origin and the change of variable \( x = T(y) \) transform (1) to the form

\[
\dot{x} = Ax + B\beta^{-1}(x)[u - \alpha(x)] \quad (2)
\]

with \((A, B)\) controllable, \(\beta(x)\) nonsingular, \(\forall x \in D\).
Calculation of $T(\cdot)$

Suppose $\dot{y} = f(y) + G(y)u$ is feedback linearizable.

$$x = T(y)$$

$$\dot{x} = \frac{\partial T}{\partial y} \dot{y} = \frac{\partial T}{\partial y} [f(y) + G(y)u]$$

$$\Rightarrow \dot{x} = Ax + B \beta^{-1}(x)[u - \alpha(x)]$$

Also

$$= AT(y) + B \beta^{-1}(T(y))[u - \alpha(T(y))]$$

Thus the following equations should be satisfied in the domain of interest.

$$\frac{\partial T}{\partial y} [f(y) + G(y)u] = AT(y) + B \beta^{-1}(T(y))[u - \alpha(T(y))]$$

$$\Rightarrow \begin{cases} \frac{\partial T}{\partial y} f(y) = AT(y) - B \beta^{-1}(T(y)) \alpha(T(y)) \\ \frac{\partial T}{\partial y} G(y) = B \beta^{-1}(T(y)) \end{cases} \quad \Rightarrow \quad (3)$$
Calculation of $T(\cdot)$ (cont.)

- Two PDEs are the necessary & sufficient conditions for feedback linearization.
- $T$ is not unique

Let $z = Mx$, $\det M \neq 0$

$$\dot{z} = MAM^{-1}z + MB \beta^{-1}(M^{-1}z)[u - \alpha(M^{-1}z)]$$  : another form

So $x = T(y)$, $z = Mx$, then $M^{-1}z = T(y)$. Thus $z = MT(y) \otimes \hat{T}(y)$.

Let $p = 1$

$$MAM^{-1} = A'_c = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ -a_n & -a_{n-1} & \ldots & -a_1 \end{bmatrix}, \quad MB = B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The non-uniquesness is exploited to simplify (3)
Calculation of $T(\cdot)$ (cont.)

\[ A'_c = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots \\ 0 & \cdots & 1 \\ 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -a_n & -a_{n-1} & \cdots & -a_1 \end{bmatrix} \]

\[ T(y) = \begin{bmatrix} T_1(y) \\ \vdots \\ T_n(y) \end{bmatrix} \]

\[ \dot{x} = A_c x + B_c (ax) + B_c \beta^{-1}(x)[u - \alpha(x)] \]

\[ = A_c x + B_c \beta^{-1}(x)[u - (\alpha(x) - \beta(x)(ax)))] \]
Thus we can have the following canonical form.

\[ A_c T(y) - B_c \beta^{-1}(T(y))\alpha(T(y)) = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ \vdots \\ T_n \end{bmatrix} - \begin{bmatrix} \beta^{-1}(T(y))\alpha(T(y)) \end{bmatrix} = \begin{bmatrix} T_2(y) \\ \vdots \\ T_n(y) \\ -\alpha(T(y))/\beta(T(y)) \end{bmatrix} \]

\[ B_c \beta^{-1}(T(y)) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} \beta^{-1}(T(y)) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1/\beta(T(y)) \end{bmatrix} \]
Thus the first PDE simplifies to

\[
\begin{bmatrix}
\frac{\partial T_1}{\partial y} \\
\vdots \\
\frac{\partial T_n}{\partial y}
\end{bmatrix} f(y) =
\begin{bmatrix}
T_2(y) \\
T_3(y) \\
\vdots \\
-\alpha/\beta
\end{bmatrix}
\]

\[
\begin{aligned}
\frac{\partial T_1}{\partial y} f(y) &= T_2(y) \\
\frac{\partial T_2}{\partial y} f(y) &= T_3(y) \\
&\vdots \\
\frac{\partial T_{n-1}}{\partial y} f(y) &= T_n(y) \\
\frac{\partial T_n}{\partial y} f(y) &= -\alpha/\beta
\end{aligned}
\]

find \( T_1 \) that meets these conditions.
The second PDE becomes

\[ \frac{\partial T_1}{\partial y} G(y) = 0 \]

(ii)

\[ \frac{\partial T_2}{\partial y} G(y) = 0 \]

\[ \vdots \]

\[ \frac{\partial T_{n-1}}{\partial y} G(y) = 0 \]

\[ \frac{\partial T_n}{\partial y} G(y) = \frac{1}{\beta} \neq 0 \]
Calculation of $T(\cdot)$ (cont.)

Thus we have to search for a function $T_i$ that satisfies

$$\frac{\partial T_i}{\partial y} G(y) = 0, \quad i = 1, \ldots, n-1$$

$$\frac{\partial T_n}{\partial y} G(y) \neq 0$$

where $T_{i+1} = \frac{\partial T_i}{\partial y} f(y), \quad i = 1, 2, \ldots, n-1$ \hspace{1cm} (5)

If there exists $T_i$ that satisfies these equations, then $\alpha$ and $\beta$ are given by

$$\beta = \begin{bmatrix} 1 \\ \frac{\partial T_n}{\partial y} G(y) \end{bmatrix}, \quad \alpha = \begin{bmatrix} \frac{\partial T_n}{\partial y} f(y) \\ \frac{\partial T_n}{\partial y} G(y) \end{bmatrix}$$
Example

In solving (4), (5), it is convenient to choose $T_1(y)$ such that

$$T_1(y^*) = 0$$

where $y^*$ is the equilibrium point of the open-loop system, i.e., $f(y^*) = 0$.

Then, as it follows from (5)

$$T_i(y^*) = 0, \quad i = 2, \cdots, n$$

i.e., $T(y)$ maps the equilibrium $y^*$ into the origin $x^* = 0$

Ex:

$$\begin{aligned}
\begin{cases}
\dot{y}_1 &= a \sin y_2 \\
\dot{y}_2 &= -y_1^2 + u
\end{cases}
\end{aligned}$$

$$y^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad f(y) = \begin{bmatrix} a \sin y_2 \\ -y_1^2 \end{bmatrix}, \quad G(y) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Find $T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$ s.t., $\frac{\partial T_1}{\partial y} G(y) = 0$, $\frac{\partial T_2}{\partial y} G(y) \neq 0$, where $T_1(0) = 0$, $T_2(y) = \frac{\partial T_1}{\partial y} f(y)$.  

10-19
Example (cont.)

From the first equation,

$$\frac{\partial T_1}{\partial y_1} G_1 + \frac{\partial T_1}{\partial y_2} G_2 = 0 \quad \Rightarrow \quad \frac{\partial T_1}{\partial y_2} = 0 \Rightarrow T_1 = T_1(y_1)$$

Now,

$$T_2(y_1, y_2) = \frac{\partial T_1}{\partial y_1} f_1 + \frac{\partial T_1}{\partial y_2} f_2 = \frac{\partial T_1}{\partial y_1} a \sin y_2$$

Finally,

$$0 \neq \frac{\partial T_2}{\partial y} G = \frac{\partial T_2}{\partial y_1} G_1 + \frac{\partial T_2}{\partial y_2} G_2$$

$$\frac{\partial T_2}{\partial y_2} = a \cos y_2 \frac{\partial T_1}{\partial y_1} \neq 0$$

Thus we can choose,

$$x_1 = T_1(y_1) = y_1, \quad x_2 = T_2(y_1, y_2) = a \sin y_2$$
Example (cont.)

Now,
\[
\beta = \left[ \frac{1}{\frac{\partial T_2}{\partial y} G} \right] = \frac{1}{\frac{\partial T_2}{\partial y_1} G_1 + \frac{\partial T_2}{\partial y_2} G_2} = \frac{1}{a \cos y_2} = \frac{1}{a \cos[\sin^{-1}(\frac{x_2}{a})]} \\
\]

\[
\alpha = -\left( \frac{\partial T_2}{\partial y_1} f_1 + \frac{\partial T_2}{\partial y_2} f_2 \right) = \frac{-(a \cos y_2)(-y_1^2)}{a \cos y_2} = y_1^2 = x_1^2 \\
\]

\[
\begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = a \cos[\sin^{-1}(\frac{x_2}{a})][u - x_1^2]
\end{cases}
\]

where
\[
u = \alpha(x) + \beta(x)v = x_1^2 + \frac{v}{a \cos[\sin^{-1}(\frac{x_2}{a})]}
\]
Input-Output Linearization

Let’s consider the tracking problem

\[
\begin{aligned}
\dot{x} &= f(x,u) \\
y &= h(x)
\end{aligned}
\]

where \( y_d(t) \) and its derivative up to a sufficiently high order are assumed to be known and bounded.

Objective: To make \( y(t) \) tracks \( y_d(t) \) while keeping the whole state bounded.

An apparent difficulty with this system is that the output \( y \) is only indirectly related to the input \( u \), through the state variable \( x \) and the nonlinear state equation.

One might guess that the difficulty of the tracking control design can be reduced if we can find a direct and simple relation between the system output \( y \) and the control input \( u \).
Input-Output Linearization Procedure

\[
\begin{align*}
\dot{x} &= f(x, u) \\
y &= h(x)
\end{align*}
\]

a) Differentiate \( y \) until \( u \) appears in one of the equations for the derivatives of \( y \)

\[
\begin{align*}
y' &= \dot{y} \\
y'' &= \ddot{y} \\
y^{(r)} &= \alpha(x) + \beta(x)u \\
& \quad \text{after } r \text{ steps } u \text{ appears}
\end{align*}
\]

b) Choose \( u \) to give \( y(r) = v \), where \( v \) is the synthetic input

\[
u = \alpha(x) + \beta(x)v
\]

c) Then sys. form \( y^{(r)} = v \), Design a linear control law for this \( r \)-integrator linear system.

d) Check internal dynamics.
Example

\[
\begin{align*}
\dot{x}_1 &= \sin x_2 + (x_2 + 1)x_3 \\
\dot{x}_2 &= x_1^5 + x_3 \\
\dot{x}_3 &= x_1^2 + u \\
y &= x_1
\end{align*}
\]

Let’s differentiate the output \( y \).

\[
\begin{align*}
\dot{y} &= \dot{x}_1 = \sin x_2 + (x_2 + 1)x_3 & (1) \\
\ddot{y} &= (\cos x_2)\dot{x}_2 + \dot{x}_2 x_3 + (x_2 + 1)\dot{x}_3 \\
&= (x_2 + 1)u + f_1(x) & (2)
\end{align*}
\]

where \( f_1(x) = (x_1^5 + x_3)(x_3 + \cos x_2) + (x_2 + 1)x_1^2 \)

(2) represents an explicit relationship between \( y \) and \( u \).

Choose 

\[
u = \frac{1}{x_2 + 1} (v - f_1)
\]

where \( v \) is a new input to be determined.
Example (cont.)

Then \[ \ddot{y} = v \]

Let \[ e = y(t) - y_d(t) \] then choosing the new input \( v \) as

\[ v = \dot{y}_d - k_1 e - k_2 \dot{e} \]

(where \( k_1, k_2 \) are positive)

Then the tracking error of closed loop is

\[ \ddot{e} + K_2 \dot{e} + K_1 e = 0 \quad (4) \]

which represents an exponentially stable error dynamics.

Note:

(i) The control law is defined everywhere except that \( x_2 = -1 \).

(ii) Full state measurement is necessary in implementing the control law because (3) requires the value of \( x \)

One must remember that (4) only accounts for part of the closed loop dynamics, because it has only order 2, while the whole dynamics has order 3. Thus, a part of the system dynamics has been rendered unobservable in the input-output linearization. We call it internal dynamics.
Example (cont.)

For the above example, the internal dynamics is represented by

\[ \dot{x}_3 = x_1^2 + \frac{1}{x_2 + 1}(\dot{y}_d - k_1 e - k_2 \dot{e} - f) \]

If this internal dynamics is stable (by which we actually mean stability in BIBO), our tracking control problem has indeed been solved.

Otherwise, it will end up with burning-up fuses or violent vibration of mechanical members.

Therefore, the effectiveness of the above control design, based on the reduced order model, hinges upon the stability of the internal dynamics.
Assume that the control objective is to make $y$ track $y_d$.

Since we know $\dot{y} = \dot{x}_1 = x_2^3 + u$, we choose the control law

$$u = -x_2^3 - e(t) + \dot{y}_d(t)$$

(1)

Then $\dot{e} + e = 0$  

(2)

The same control input is applied to the second equation leading to the internal dynamics

$$\dot{x}_2 + x_2^3 = \dot{y}_d(t) - e$$

We know that $e$ is guaranteed to be bounded by (2) and $\dot{y}_d$ is assumed to be bounded. Then

$$|\dot{y}_d(t) - e| \leq D$$
Example (cont.)

Thus we conclude that \( |x_2| \leq D^{1/3} \)

since \( \dot{x}_2 < 0 \), when \( x_2 > D^{1/3} \)
\( \dot{x}_2 > 0 \), when \( x_2 < -D^{1/3} \)

Therefore (1) represents a satisfactory tracking control law.

If we need to differentiate the output \( y \) \( r \)-times to generate an explicit relationship between the output \( y \) and the input \( u \), the system is said to have relative degree \( r \).

If the relative degree of a system is the same as its order, then there is no internal dynamics. In this case, input-output linearization leads to input-state linearization and output tracking can be achieved easily.
Feedback linearization is an approach to transform nonlinear systems into (fully or partly) linear ones, so linear control techniques can be applied.

Feedback linearization types, applications, warnings, advantages, and disadvantages were presented.

Terminologies such as: relative degree, internal dynamics have been recognized.

How to calculate $T(.)$, and several different examples were discussed.
References


Thank you for your attention