CHAPTER 9

EELE 3332 – Electromagnetic II
Chapter 9

Maxwell's Equations

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1

9.6 Time Varying Potentials

For Electrostatic fields, we defined electric scalar potential (V).

$$\nabla \times E = 0 \rightarrow E = -\nabla V, \quad \nabla^2 V = -\frac{\rho_V}{\varepsilon}$$
$$V = \int_v \frac{\rho_V dv}{4\pi R}$$

For Magnetostatic fields, we defined magnetic vector potential (A).

$$\nabla \cdot B = 0 \rightarrow B = \nabla \times A, \quad \nabla^2 A = -\mu_0 J$$
$$A = \int_v \frac{\mu J dv}{4\pi R}$$

For Electromagnetic Fields: We need to define (A and V) for time varying fields……

Time Varying Potentials

Since $\nabla \cdot B = 0$ holds for time varying field, then the relation

$$B = \nabla \times A$$

also holds for time varying field.

Using Faraday's law, $\nabla \times E = -\frac{\partial B}{\partial t}$

$$\rightarrow \nabla \times E = -\frac{\partial}{\partial t}(\nabla \times A) \quad or \quad \nabla \times \left(E + \frac{\partial A}{\partial t}\right) = 0$$

Since the curl of the gradient of a scalar is zero, then

$$E + \frac{\partial A}{\partial t} = -\nabla V$$

or
$$E = -\nabla V - \frac{\partial A}{\partial t} \qquad \cdots \quad (1)$$

3

Time Varying Potentials

So, we can determine the vector fields **B** and **E** from the potentials **A** and **V**:

$$B = \nabla \times A$$

$$E = -\nabla V - \frac{\partial A}{\partial t}$$

However, we still need to find some expressions for A and V suitable for time-varying fields.

4

Time Varying Potentials

We know that $\nabla \cdot \mathrm{D}=\rho_V$ is valid for time-varying conditions.

$$\rightarrow \nabla \cdot \mathrm{E}=\frac{\rho_V}{\varepsilon}$$

using eq (1) $\mathrm{E}=-\nabla V-\frac{\partial A}{\partial t}$

$$\rightarrow \nabla \cdot \mathrm{E}=-\nabla \cdot (\nabla V)-\frac{\partial}{\partial t}(\nabla \cdot \mathrm{A})=\frac{\rho_V}{\varepsilon}$$

$$\rightarrow \nabla \cdot \mathrm{E}=-\nabla^2 \mathrm{V}-\frac{\partial}{\partial t}(\nabla \cdot \mathrm{A})=\frac{\rho_V}{\varepsilon}$$

or

$$\nabla^2 \mathrm{V}+\frac{\partial}{\partial t}(\nabla \cdot \mathrm{A})=-\frac{\rho_V}{\varepsilon} \quad \cdots \quad (2)$$

5

Time Varying Potentials

Now take the curl of $\mathrm{B}=\nabla \times \mathrm{A} \rightarrow \nabla \times \mathrm{B}=\nabla \times \nabla \times \mathrm{A}=\mu(\nabla \times \mathrm{H})$

since $\nabla \times \mathrm{H}=\mathrm{J}+\frac{\partial \mathrm{D}}{\partial t}$, and $\mathrm{D}=\varepsilon \mathrm{E}$, and $\mathrm{E}=-\nabla V-\frac{\partial A}{\partial t}$

$$\rightarrow \nabla \times \nabla \times \mathrm{A}=\mu\left(\mathrm{J}+\frac{\partial \mathrm{D}}{\partial t}\right)=\mu\left(\mathrm{J}+\varepsilon \frac{\partial}{\partial t}\left(-\nabla V-\frac{\partial A}{\partial t}\right)\right)$$

$$\rightarrow \nabla \times \nabla \times \mathrm{A}=\mu \mathrm{J}+\mu\varepsilon \frac{\partial}{\partial t}\left(-\nabla V-\frac{\partial A}{\partial t}\right)=\mu \mathrm{J}-\mu\varepsilon \nabla\left(\frac{\partial V}{\partial t}\right)-\mu\varepsilon \frac{\partial^2 A}{\partial t^2}$$

Recall that Laplacian of a vector is given by:

$$\nabla^2 \mathrm{A}=\nabla(\nabla \cdot \mathrm{A})-\nabla \times \nabla \times \mathrm{A} \rightarrow \nabla \times \nabla \times \mathrm{A}=\nabla(\nabla \cdot \mathrm{A})-\nabla^2 \mathrm{A}$$

$$\rightarrow \nabla(\nabla \cdot \mathrm{A})-\nabla^2 \mathrm{A}=\mu \mathrm{J}-\mu\varepsilon \nabla\left(\frac{\partial V}{\partial t}\right)-\mu\varepsilon \frac{\partial^2 A}{\partial t^2}$$

$$\rightarrow \quad \nabla\left(\nabla \cdot \mathrm{A}+\mu\varepsilon \frac{\partial V}{\partial t}\right)-\nabla^2 \mathrm{A}=\mu \mathrm{J}-\mu\varepsilon \frac{\partial^2 A}{\partial t^2} \quad \cdots \quad (3)$$

6

Time Varying Potentials

We have now reduced the set of four Maxwell eqs. to two eqs. But they are still coupled.

$$\nabla^2 \mathrm{V}+\frac{\partial}{\partial t}(\nabla \cdot \mathrm{A})=-\frac{\rho_V}{\varepsilon} \quad \cdots \quad (2)$$

$$\nabla\left(\nabla \cdot \mathrm{A}+\mu\varepsilon \frac{\partial V}{\partial t}\right)-\nabla^2 \mathrm{A}=\mu \mathrm{J}-\mu\varepsilon \frac{\partial^2 A}{\partial t^2} \quad \cdots \quad (3)$$

The uncoupling can be achieved by using Lorenz condition for potentials.

Lorenz condition for potentials (relates V and A):

$$\nabla \cdot \mathrm{A}=-\mu\varepsilon \frac{\partial V}{\partial t} \quad \cdots \quad (4)$$

7

Time Varying Potentials

Using equ (4) $\quad \nabla \cdot \mathrm{A}=-\mu\varepsilon \frac{\partial V}{\partial t}$

Substitute (4) in equations (2) and (3),

$$\nabla^2 V-\mu\varepsilon \frac{\partial^2 V}{\partial t^2}=-\frac{\rho_V}{\varepsilon} \quad \cdots \quad (5)$$

$$\nabla^2 \mathrm{A}-\mu\varepsilon \frac{\partial^2 A}{\partial t^2}=-\mu \mathrm{J} \quad \cdots \quad (6)$$

which are the wave equations to be discussed in the next chapter...

Recall that E and B can ve obtained by:

$$\mathrm{E}=-\nabla V-\frac{\partial A}{\partial t}, \qquad \mathrm{B}=\nabla \times \mathrm{A}$$

Notice that for static conditions (special cases of time varying potentials):

$$\mathrm{E}=-\nabla V, \quad \mathrm{B}=\nabla \times \mathrm{A}, \quad \text{and} \quad \nabla^2 V=-\frac{\rho_V}{\varepsilon}, \quad \nabla^2 \mathrm{A}=-\mu \mathrm{J}$$

9.7 Time Harmonic Fields

A time-harmonic field is one that varies periodically or sinusoidally with time.

Sinosoids are easily expressed in phasors, which are more convenient to work with.
A ***phasor*** is a complex number that contains the amplitude and the phase of a sinusoidal oscillation.

$z = x + jy = r\angle\phi = re^{j\phi} = r\left(\cos\phi + j\sin\phi\right)$

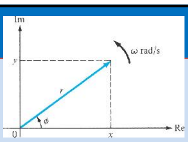
where

$j = \sqrt{-1}$ is , x is the real part of z, y is the imaginary part of z

r is the magnitude of z given by $r = |z| = \sqrt{x^2 + y^2}$

and ϕ is the phase of z, given by: $\phi = \tan^{-1}\dfrac{y}{x}$

Time Harmonic Fields



$z = x + jy$ Rectangular Form

$z = r\angle\phi = re^{j\phi}$ Polar Form

- Addition and subtraction are better performed in rectangular form.
- Multiplication and division are better done in polar form.

Given $z = x + jy = r\angle\phi,\ z_1 = x_1 + jy_1 = r_1\angle\phi_1\ ,\ z_2 = x_2 + jy_2 = r_2\angle\phi_2$

Addition	$z_1 + z_2 = (x_1 + x_1) + j(y_1 + y_2)$
Subtraction	$z_1 - z_2 = (x_1 - x_1) + j(y_1 - y_2)$
Multiplication	$z_1 z_2 = r_1 r_2\angle(\phi_1 + \phi_2)$
Division	$\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\angle(\phi_1 - \phi_2)$
Square root	$\sqrt{z} = \sqrt{r}\angle(\phi / 2)$
Complex Conjugate	$z^* = x - jy = r\angle(-\phi) = re^{-j\phi}$

Time Harmonic Fields

To introduce the time element, we let:

$\phi = \omega t + \theta$

where θ may be a function of time or space coordinates or a constant.

The real and imginary part of $re^{j\phi} = re^{j(\omega t + \theta)} = r\cos(\omega t + \theta) + jr\sin(\omega t + \theta)$ are given by:

$$\mathrm{Re}(re^{j\phi}) = r\cos(\omega t + \theta)$$
$$\mathrm{Im}(re^{j\phi}) = r\sin(\omega t + \theta)$$

Example : current $I(t) = I_0\cos(\omega t + \theta)$ equals real part of $I_0 e^{j\theta} e^{j\omega t}$

Example : current $I(t) = I_0\sin(\omega t + \theta)$ equals imaginary part of $I_0 e^{j\theta} e^{j\omega t}$

also equals real part of $I_0 e^{j\theta} e^{j\omega t} e^{-j90}$ because $\sin\alpha = \cos(\alpha - 90)$

11

Time Harmonic Fields

The Phasor is defined by dropping the time factor $e^{j\omega t}$

For example, $I(t) = I_0\cos(\omega t + \theta) = \mathrm{Re}(I_0 e^{j\theta} e^{j\omega t})$

The phasor current is :

$$I_s = I_0 e^{j\theta} = I_0\angle\theta$$

and hence,

$$I(t) = \mathrm{Re}(I_s e^{j\omega t})$$

If a vector $A(x, y, z, t)$ is a time-harmonic field, the phasor form of A is $A_s(x, y, z)$. The two quantities are related by

$$\mathrm{A} = \mathrm{Re}(\mathrm{A_s} e^{j\omega t})$$

12

Time Harmonic Fields

$$A = \text{Re}(A_s e^{j\omega t})$$

Notice that $\frac{\partial A}{\partial t} = \frac{\partial}{\partial t}\text{Re}(A_s e^{j\omega t}) = \text{Re}(j\omega A_s e^{j\omega t})$

Hence, taking the derivative of the instataneous quantity is equivalent to multiplying its phasor form by $j\omega$

$$\boxed{\frac{\partial A}{\partial t} \to j\omega A_s}$$

Notes :

instantaneous $A(x, y, z, t) \to$ time dependent

phasor $A_s(x, y, z) \to$ time invariant

It is easier to work with A_s and obtain A from A_s whenever necessary using $A = \text{Re}(A_s e^{j\omega t})$

13

Maxwell's Equations – phasor form

Point Form	Integral Form
$\nabla \cdot \mathbf{D}_s = \rho_{vs}$	$\oint \mathbf{D}_s \cdot d\mathbf{S} = \int \rho_{vs}\, dv$
$\nabla \cdot \mathbf{B}_s = 0$	$\oint \mathbf{B}_s \cdot d\mathbf{S} = 0$
$\nabla \times \mathbf{E}_s = -j\omega \mathbf{B}_s$	$\oint \mathbf{E}_s \cdot d\mathbf{l} = -j\omega \int \mathbf{B}_s \cdot d\mathbf{S}$
$\nabla \times \mathbf{H}_s = \mathbf{J}_s + j\omega \mathbf{D}_s$	$\oint \mathbf{H}_s \cdot d\mathbf{l} = \int (\mathbf{J}_s + j\omega \mathbf{D}_s) \cdot d\mathbf{S}$

14

Example 9.5

Evaluate the complex numbers:

$$(a)\ z_1 = \frac{j(3-j4)^*}{(-1+j6)(2+j)^2}$$

$$(b)\ z_2 = \left[\frac{1+j}{4-j8}\right]^{1/2}$$

(a) Method 1 (Working in rectangular form)

$$z_1 = \frac{j(3-j4)^*}{(-1+j6)(2+j)^2} = \frac{j(3+j4)}{(-1+j6)(4+4j-1)} = \frac{-4+j3}{(-1+j6)(3+4j)} = \frac{-4+j3}{-27+j14}$$

$$z_1 = \frac{(-4+j3)(-27-j14)}{(-27+j14)(-27-j14)} = \frac{150-j25}{27^2+14^2}$$

$$z_1 = 0.1622 - j0.027 = 0.1644\angle -9.46$$

15

Example 9.5 - Continued

Method 2 (Working in polar form)

$$z_1 = \frac{(1\angle 90)(5\angle -\tan^{-1}(-4/3))}{\left(\sqrt{1+36}\angle \tan^{-1}(-6)\right)\left(\sqrt{4+1}\angle \tan^{-1}(-1/2)\right)^2}$$

$$z_1 = \frac{(1\angle 90)(5\angle 53.13)}{\left(\sqrt{37}\angle 99.46\right)\left(\sqrt{5}\angle 26.265\right)^2} \to \left(\sqrt{5}\angle 26.265\right)^2 = (5\angle 53.13)$$

$$z_1 = \frac{(1)(5)}{\left(\sqrt{37}\right)(5)}\angle(90+53.13-99.46-53.13) \to z_1 = \frac{1}{\sqrt{37}}\angle(-9.46)$$

$$z_1 = \frac{1}{\sqrt{37}}(\cos -9.46 + j\sin -9.46)$$

$$z_1 = 0.1622 - j0.027$$

16

Example 9.5 - Continued

$$(b)\ z_2=\left[\frac{1+j}{4-j8}\right]^{1/2}$$

$$z_2=\left[\frac{\sqrt{2}\angle\tan^{-1}(1)}{\sqrt{80}\angle\tan^{-1}(-2)}\right]^{1/2}=\left[\sqrt{\frac{2}{80}}\angle(45-(-63.43)\right]^{1/2}$$

$$z_2=\sqrt{0.15811}\angle(108.4/2)$$

$$z_2=0.3976\angle(54.2)$$

17

Example 9.6

Given that $A=10\cos(10^8 t - 10x+60^o)a_z$ and $B_s=(20/j)a_x+10e^{j2\pi x/3}a_y$

Express A in phasor form and B_s in instantaneous form.

$$A=\text{Re}\left[10e^{j(\omega t-10x+60^o)}a_z\right],\quad \text{where } \omega=10^8.$$

$$\text{Hence, } A=\text{Re}\left[10e^{j(60^o-10x)}a_z e^{j\omega t}\right]$$

$$\text{or}\quad A_s=10e^{j(60^o-10x)}a_z$$

$$B_s=(20/j)a_x+10e^{j2\pi x/3}a_y=-j\,20a_x+10e^{j2\pi x/3}a_y$$

$$B_s=20e^{-j\pi/2}a_x+10e^{j2\pi x/3}a_y$$

$$B=\text{Re}[B_s e^{j\omega t}]=\text{Re}[20e^{j(\omega t-\pi/2)}a_x+10e^{j(\omega t+2\pi x/3)}a_y]$$

$$B=20\cos(\omega t-\pi/2)a_x+10\cos(\omega t+2\pi x/3)a_y$$

$$B=20\sin\omega t\ a_x+10\cos(\omega t+2\pi x/3)a_y$$

18

Example 9.7

The electric field and msgnetic field in free space are given by

$$E=\frac{50}{\rho}\cos(10^6 t+\beta z)a_\phi\ \text{V/m},\quad H=\frac{H_0}{\rho}\cos(10^6 t+\beta z)a_\rho\ \text{A/m}$$

Express these in phasor form and determine the constants H_0 and β such that the fields satisfy Maxwell's equations

The instantaneous forms of E and H are written as

$E=\text{Re}(E_s e^{j\omega t})$, $H=\text{Re}(H_s e^{j\omega t})$, where $\omega=10^6$

$$E_s=\frac{50}{\rho}e^{j\beta z}a_\phi,\quad H_s=\frac{H_0}{\rho}e^{j\beta z}a_\rho \quad \cdots \quad (1)$$

In free space, $\rho_v=0$, $\sigma=0$, $\varepsilon=\varepsilon_0$ and $\mu=\mu_0$, So Maxwell's equations become

$$\nabla\cdot D=\varepsilon_0\nabla\cdot E=0\to\nabla\cdot E_s=0 \quad \cdots \quad (2)$$

$$\nabla\cdot B=\mu_0\nabla\cdot H=0\to\nabla\cdot H_s=0 \quad \cdots \quad (3)$$

$$\nabla\times H=\sigma E+\varepsilon_0\frac{\partial E}{\partial t}\to\nabla\times H_s=j\omega\varepsilon_0 E_s \quad \cdots \quad (4)$$

$$\nabla\times E=-\mu_0\frac{\partial H}{\partial t}\to\nabla\times E_s=-j\omega\mu_0 H_s \quad \cdots \quad (5)$$

19

Example 9.7 continued

Substitution eq (1) into (2) and (3), it is verified two Maxwell's equation are verified.

$$\nabla\cdot E_s=\frac{1}{\rho}\frac{\partial}{\partial\phi}\left(E_{\phi s}\right)=0$$

$$\nabla\cdot H_s=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho H_{\rho s}\right)=0$$

$$\nabla\times H_s=\nabla\times\left(\frac{H_0}{\rho}e^{j\beta z}a_\rho\right)=\frac{jH_0\beta}{\rho}e^{j\beta z}a_\phi \quad \cdots \quad (6)$$

Substituting equations (1) and (4) into equation (6):

$$\nabla\times H_s=\frac{jH_0\beta}{\rho}e^{j\beta z}a_\phi=j\omega\varepsilon_0 E_s=j\omega\varepsilon_0\frac{50}{\rho}e^{j\beta z}a_\phi$$

$$\to H_0\beta=50\omega\varepsilon_0 \quad \cdots \quad (7)$$

Similarly, substituting equation (1) into equation (5):

$$\nabla\times E_s=j\omega\mu_0 H_s\to\quad -j\beta\frac{50}{\rho}e^{j\beta z}a_\rho=-j\omega\mu_0\frac{H_0}{\rho}e^{j\beta z}a_\rho$$

$$\to 50\beta=\omega\mu_0 H_0\ \to\ \frac{H_0}{\beta}=\frac{50}{\omega\mu_0} \quad \cdots \quad (8)$$

20

Example 9.7 continued

$$H_0\beta = 50\omega\varepsilon_0 \quad \cdots \quad (7) \quad , \quad \frac{H_0}{\beta} = \frac{50}{\omega\mu_0} \quad \cdots \quad (8)$$

Multiplying equations (7) and (8):

$$H_0^2 = (50)^2 \frac{\varepsilon_0}{\mu_0} \rightarrow H_0 = \pm 50\sqrt{\varepsilon_0 / \mu_0} = \pm \frac{50}{120\pi} = \pm 0.1326$$

Dividing equation (7) by (8):

$$\beta^2 = \omega^2 \mu_0 \varepsilon_0$$

$$\beta = \pm\,\omega\sqrt{\mu_0\varepsilon_0} = \pm\frac{\omega}{c} = \pm\frac{10^6}{3\times 10^8} = \pm 3.33\times 10^{-3}$$

21

Example 9.8

In a medium characterised by $\sigma=0,\ \mu=\mu_0, \varepsilon=4\varepsilon_0$ and
$E=20\sin(10^8 t - \beta z)a_y$ V/m
Calculate β and H.

Method 1 (time domain)--(the harder way!)

It is evident that gauss law is satisfied: $\nabla\cdot E = \frac{\partial E_y}{\partial y} = 0$

From Faraday's law, $\nabla\times E = -\mu\frac{\partial H}{\partial t} \rightarrow H = -\frac{1}{\mu}\int(\nabla\times E)dt$

$$But \quad \nabla\times E = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix} = -\frac{\partial E_y}{\partial z}a_x + \frac{\partial E_y}{\partial x}a_z$$

$$\nabla\times E = 20\beta\cos(10^8 t - \beta z)a_x + 0$$

$$Hence \ \ H = -\frac{20\beta}{\mu}\int\cos(10^8 t - \beta z)\ dt\ a_x = -\frac{20\beta}{\mu 10^8}\sin(10^8 t - \beta z)\ a_x \quad \cdots (1)$$

22

Example 9.8 continued

It is verified that $\nabla\cdot H = \frac{\partial H_x}{\partial x} = 0$

$$\nabla\times H = \sigma E + \varepsilon_0\frac{\partial E}{\partial t} \rightarrow E = \frac{1}{\varepsilon}\int(\nabla\times H)\,dt \quad (\sigma=0)$$

$$But \quad \nabla\times H = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & 0 & 0 \end{vmatrix} = -\frac{\partial H_x}{\partial z}a_y - \frac{\partial H_x}{\partial y}a_z = \frac{20\beta^2}{\mu 10^8}\cos(10^8 t - \beta z)a_y + 0$$

$$\rightarrow E = \frac{20\beta^2}{\mu\varepsilon 10^8}\int\cos(10^8 t - \beta z)dt\ a_y = \frac{20\beta^2}{\mu\varepsilon 10^{16}}\sin(10^8 t - \beta z)\ a_y$$

Since the given is $E=20\sin(10^8 t - \beta z)\ a_y$

$$\rightarrow \frac{20\beta^2}{\mu\varepsilon 10^{16}} = 20 \rightarrow \beta = \pm 10^8\sqrt{\mu\varepsilon} = \pm 10^8\sqrt{\mu_0\cdot 4\varepsilon_0} = \pm\frac{10^8(2)}{c} \rightarrow \beta = \pm\frac{2}{3}$$

From equ (1): $H = \pm\frac{20(2/3)}{4\pi.10^{-7}(10^8)}\sin(10^8 t \pm \frac{2z}{3})a_x$

$$\rightarrow H = \pm\frac{1}{3\pi}\sin(10^8 t \pm \frac{2z}{3})a_x \quad \text{A/m}$$

23

Example 9.8 continued

Method 2 : (Using Phasors)

$E=\mathrm{Im}(E_s e^{j\omega t}) \rightarrow E_s = 20e^{-j\beta z}a_y$ where $\omega=10^8$

Again $\nabla\cdot E_s = \frac{\partial E_{ys}}{dy} = 0$

$$\nabla\times E_s = -j\omega\mu H_s \rightarrow H_s = \frac{\nabla\times E_s}{-j\omega\mu} = \frac{1}{-j\omega\mu}\left[-\frac{\partial E_{ys}}{\partial z}a_x\right] = -\frac{20\beta}{\omega\mu}e^{-j\beta z}a_x$$

$$\nabla\times H_s = j\omega\varepsilon E_s \rightarrow E_s = \frac{\nabla\times H_s}{j\omega\varepsilon} = \frac{1}{j\omega\varepsilon}\frac{\partial H_{xs}}{dz}a_y = \frac{20\beta^2 e^{-j\beta z}}{\omega^2\mu\varepsilon}a_y$$

Comparing this with E_s, we have:

$$\frac{20\beta^2 e^{-j\beta z}}{\omega^2\mu\varepsilon}a_y = 20e^{-j\beta z}a_y \ \rightarrow \frac{20\beta^2}{\omega^2\mu\varepsilon} = 20 \rightarrow \beta = \pm\omega\sqrt{\mu\varepsilon} = \pm\omega\sqrt{\mu\varepsilon} = \pm\frac{2}{3}$$

$$H_s = \pm\frac{20(2/3)e^{\pm j\beta z}}{10^8(4\pi\times 10^{-7})}a_x = \pm\frac{1}{3\pi}e^{\pm j\beta z}a_x$$

$$H = \mathrm{Im}(H_s e^{j\omega t}) = \pm\frac{1}{3\pi}\sin(10^8 t \pm \beta z)a_x \quad \text{A/m}$$

24