CHAPTER 9

EELE 3332 – Electromagnetic II
Chapter 9

Maxwell's Equations

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2016

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1. Maxwell equations.
2. Faraday law.
3. Moving loop in static **B** field.
4. Moving loop in time varying field.
5. Displacement current.
6. Maxwell equations in final form.
7. Time varying potentials.
8. Time harmonic field.
9. Summary.

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Review – Electrostatics and Magnetostatics

Electrostatic Fields E(x,y,z)

produced by stationary charges.

Magnetostatic Fields H(x,y,z)

produced by steady (DC) currents or stationary magnet materials.

Electrostatic Fields and Magnetostatic Fields do not vary with time (time invariant).

They are independent of each other.

Time dependent Electromagnetic Fields (Dynamic fields)

produced by time-varying currents. E(x,y,z,t) H(x,y,z,t)

In Summary:

stationary charges → electrostatic fields
steady currents → magnetostatic fields
time-varying currents → electromagnetic fields (or waves)

Maxwell's Equations for *static* fields

Differential form	*Integral form*	*Derived from*
$\vec{\nabla}\cdot\vec{D}=\rho_V$	$\oint_S \vec{D}\cdot d\vec{S}=\int_V \rho_V dV$	*Gauss's Law*
$\vec{\nabla}\cdot\vec{B}=0$	$\oint_S \vec{B}\cdot d\vec{S}=0$	*Nonexistance of magnetic Monopole*
$\vec{\nabla}\times\vec{E}=0$	$\oint_L \vec{E}\cdot d\vec{l}=0$	*Conservativeness of Electrostatic field*
$\vec{\nabla}\times\vec{H}=\vec{J}$	$\oint_L \vec{H}\cdot d\vec{l}=\int_S \vec{J}\cdot d\vec{S}$	*Ampere's Law*

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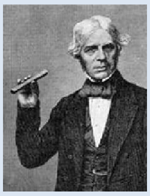
9.2 Faraday's Law



Michael Faraday
(1791-1867)

Michael Faraday's ideas about conservation of energy led him to believe that since an electric current could cause a magnetic field, a magnetic field should be able to produce an electric current. He demonstrated this principle of induction in 1831.

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Faraday's Law

- In 1831, Farady discovered that a time-varying magnetic field would produce an induced voltage (called electromotive force or emf).
- According to Faraday's experiments, a static magnetic field produces no current flow.

Faraday discovered that the induced V_{emf} (in volts) in any closed circuit is equal to the time rate of change of the magnetic flux linkage by the circuit.

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Faraday's Law

$$V_{emf} = -\frac{d\lambda}{dt} = -N\frac{d\Psi}{dt}$$

where $\lambda = N\Psi$ is the flux linkage

N is the number of turns in the circuit.

Ψ is the flux through each turn.

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Faraday's Law

Change flux due to moving permanent magnet

$$V_{emf} = -N\frac{d\Psi}{dt}$$

https://www.youtube.com/watch?v=vwIdZjjd8fo

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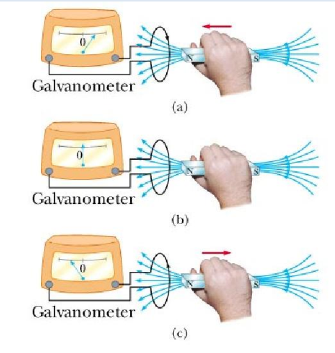
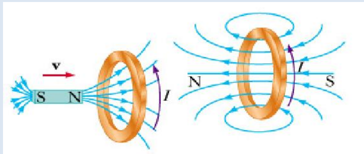
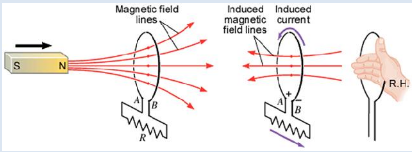
Faraday's Law



Minus Sign? Lenz's Law

Induced EMF is in direction that ***opposes the change*** in flux that caused it





9.3 Transformer and motional electromotive forces

For a circuit with a single turn , N = 1

$$V_{emf} = -\frac{d\Psi}{dt}$$

In terms of E and B

$$V_{\text{emf}} = \oint_L E.dl = -\frac{d}{dt}\int_S B.dS$$

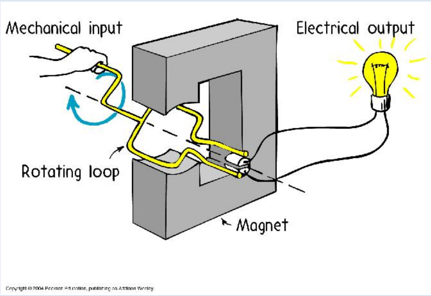
where S is the surface area of the circuit bounded by the closed path L

Notice that in time varying situation, both electric and magnetic fields are present and interrelated.

The variation of flux with time may be caused in three ways:

1) By having a stationary loop in a time-varying B field. (*Transformer induction*)
2) By having a time-varying loop area in static B field. (*motional induction*)
3) By having a time-varying loop area in a time-varying B field. (*general case, transformer and motional induction*)

Application: DC Generators



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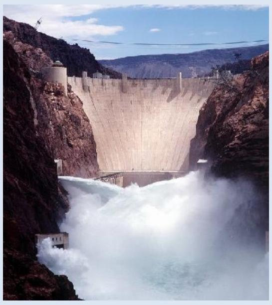
Application: AC Generator

Water turns wheel
→ rotates magnet
→ changes flux
→ induces emf
→ drives current



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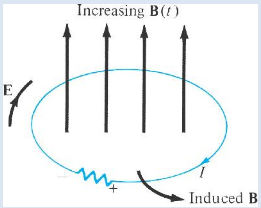
A. Stationary loop in Time-Varying B field (Transformer EMF)



$$V_{emf} = \oint_L E.dl = -\int_S \frac{\partial B}{\partial t}.dS$$

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A. Stationary loop in Time-Varying B field (Transformer EMF)

$$V_{emf} = \oint_L E.dl = -\int_S \frac{\partial B}{\partial t}.dS$$

By applying Stoke's theorem

$$\int_S (\nabla \times \mathrm{E}).dS = -\int_S \frac{\partial B}{\partial t}.dS$$

⇒

$$\nabla \times \mathrm{E} = -\frac{\partial B}{\partial t}$$

This is one of Maxwell's equations for time-varying fields.

Note that $\nabla \times \mathrm{E} \neq 0$ (time-varying E field is *non-conservative*)

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B. Moving loop in static B field (Motional EMF)

Consider a conducting loop moving with uniform velocity **u**, the emf induced in the loop is

$$\mathbf{F}_m = Q\mathbf{u} \times \mathbf{B}$$

$$\mathbf{E}_m = \frac{\mathbf{F}_m}{Q} = \mathbf{u} \times \mathbf{B}$$

$$V_{emf} = \oint_L E_m . dl = \oint_L (u \times B) . dl$$

- This type of emf is called ***motional emf*** or ***flux-cutting emf*** (due to motion action).

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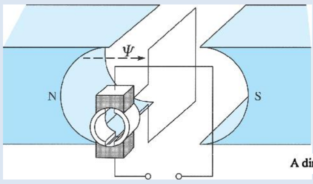
B. Moving loop in static B field (Motional EMF)

- It is kind of emf found in electrical machines such as motors and generators.
- Given below is in example of dc machine , where voltage is generated as the coil rotates within the magnetic field.



A direct-current machine.

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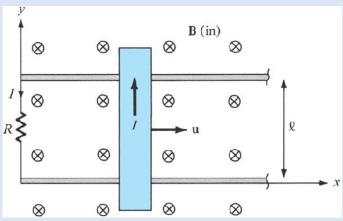
B. Moving loop in static B field (Motional EMF)

- Another example of motional emf is illustrated below, where a conducting bar is moving between a pair of rails.



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B. Moving loop in static B field (Motional EMF)

* Recall that the force on a charge moving with uniform velocity u in a a magnetic field B is: $F_m = Q u \times B$

* We define the motional electric field E_m as

$$E_m = \frac{F_m}{Q} = u \times B$$

$$V_{emf} = \oint_L E_m . dl = \oint_L (u \times B) . dl$$

By applying Stoke's theorem,

$$\int_S (\nabla \times E_m) . dS = \int_S \nabla \times (u \times B) . dS$$

$$or \qquad \nabla \times E_m = \nabla \times (u \times B)$$

$$V_{emf} = \oint_L E_m . dl = \oint_L (u \times B) . dl$$

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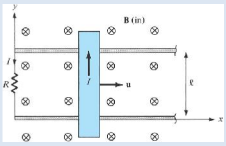
B. Moving loop in static B field (Motional EMF)

$$V_{emf} = \oint_L E_m . dl = \oint_L (u \times B).dl$$



Notes:

- The integral is zero along the portion of the loop where u=0. (e.g. ***dl*** is taken along the rod in the shown figure.
- The direction of the induced current is the same that of Em or uxB. The limits of integration are selected in the direction opposite to the direction of u x B to satisfy Lenz's law. (e.g. induced current flows in the rod along a_y, the integration over L is along $-a_y$).

C. Moving loop in Time Varying Field

Both Transformer emf and motional emf are present.

$$V_{emf} = \oint_L E.dl = -\int_S \frac{\partial B}{\partial t}.dS + \oint_L (u \times B).dl$$

Transformer

Motional

$$or\ \nabla \times E = -\frac{\partial B}{\partial t} + \nabla \times (u \times B)$$

Note that eq. $V_{emf} = -\frac{d\Psi}{dt}$ can always be applied in place of the equations in cases A, B, and C.

Example 9.1

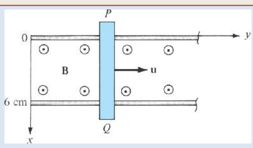
a conducting bar can slide freely over two conducting rails as shown in the figure. Calculate the induced voltage in the bar

(a) If the bar is staioned at y=8 cm and $B = 4\cos 10^6 t\ a_z$ mWb/m²

(b) If the bar slides at a velocity $u = 20\ a_y$ m/s and $B = 4a_z$ mWb/m²

(c) If the bar slided at a velocity $u = 20\ a_y$ m/s and

$B = 4\cos(10^6 t - y)\ a_z$ mWb/m²



(a) we have transformer emf

$$V_{emf} = -\int \frac{\partial B}{\partial t}.dS = \int_{y=0}^{0.08}\int_{x=0}^{0.06} 4(10^{-3})(10^6)\sin 10^6 t\ dx\ dy$$

$$= 4(10^3)(0.08)(0.06)\sin 10^6 t$$

$$= 19.2\ \sin\ 10^6 t\ \text{V}$$

(b) This is motional emf

$$V_{emf} = \int (u \times B).dl = \int_{x=l}^{0} (ua_y \times Ba_z).dx\ a_x$$

$$= -ubl = -20(4.10^{-3})(0.06)$$

$$= -4.8\ \text{mV}$$

(c) Both transformer emf and motional emf are present in this case. this problem can be solved in two ways:

Method 1:

$$\text{Vemf} = -\int \frac{\partial B}{\partial t}.dS + \int (u\times B).dl$$

$$= \int_{x=0}^{0.06}\int_{0}^{y} 4.(10^{-3})(10^{6})\sin(10^{6}t - y)\,dy\,dx$$

$$+ \int_{0.06}^{0}\left[20a_y \times 4.10^{-3}\cos(10^{6}t-y)a_z\right].dx\ a_x$$

$$= 240\cos\left(10^{6}t-y\right)\Big|_0^y - 80(10^{-3})(0.06)\cos(10^{6}t-y)$$

$$= 240\cos\left(10^{6}t-y\right) - 240\cos 10^{6}t - 4.8(10^{-3})\cos\left(10^{6}t-y\right)$$

$$\simeq 240\cos\left(10^{6}t-y\right) - 240\cos 10^{6}t$$

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Method 2:

$$V_{emf} = -\frac{\partial \Psi}{\partial t}, \quad \text{where } \Psi = \int B.dS$$

$$= \int_{y=0}^{y}\int_{x=0}^{0.06} 4\cos(10^{6}t-y)\ dx\ dy$$

$$= -40(0.06)\sin(10^{6}t-y)\Big|_{y=0}^{y}$$

$$= -0.24\sin(10^{6}t-y) + 0.24\sin 10^{6}t \ \text{mWb}$$

But

$$\frac{dy}{dt} = u \rightarrow y = ut = 20t$$

Hence,

$$\Psi = -0.24\sin(10^{6}t-20t) + 0.24\sin 10^{6}t \ \text{mWb}$$

$$V_{emf} = -\frac{\partial \Psi}{\partial t} = 0.24(10^{6}-20)\cos(10^{6}t-20t) - 0.24(10^{6})\cos 10^{6}t \ \text{mV}$$

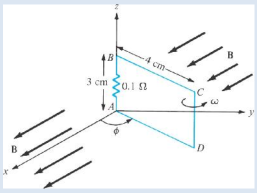
$$\simeq 240\cos\left(10^{6}t-y\right) - 240\cos 10^{6}t$$

Example 9.2

The loop shown in the figure is inside a uniform magnetic field **B**=50 a_x mWb/m². If side DC of the loop cuts the flux lines at the frequency of 50 Hz and the loop lies in the yz-plane at the time t=0, find

(a) The induced emf at at t=1 ms.

(b) The induced current at t=3 ms.



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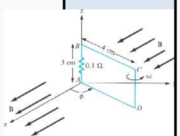
The B field is time invariant, the induced emf is motional

$$V_{emf} = \int (u\times B).dl, \quad \text{where } dl = dza_z, \quad u = \frac{\rho\partial\phi}{\partial t}a_\phi = \rho\omega a_\phi$$

$\rho = AD = 4$ cm, $\omega = 2\pi f = 100\pi$

Transform B into cylindrical corrdinates:



$B = B_0 a_x = B_0\left(\cos\phi a_\rho - \sin\phi a_\phi\right)$, where $B_0 = 0.05$

$$u\times B = \begin{vmatrix} a_\rho & a_\phi & a_z \\ 0 & \rho\omega & 0 \\ B_0\cos\phi & -B_0\sin\phi & 0 \end{vmatrix} = -\rho\omega B_0\cos\phi\ a_z$$

$$(u\times B).dl = -\rho\omega B_0\cos\phi\ dz = -0.04(100\pi)(0.05)\cos\phi\ dz$$

$$= -0.2\pi\cos\phi\ dz$$

$$V_{emf} = \int_{z=0}^{0.03} -0.2\pi\cos\phi\ dz = -6\pi\cos\phi \ \text{mV}$$

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To determine ϕ recall that

$\omega = \frac{d\phi}{dt} \rightarrow \phi = \omega t + C_0$ (C_0 is the integration constant)

At t=0, $\phi = \pi/2$ beacause the loop is in the yz-plane at that time,

$C_0 = \pi/2$. Hence, $\phi = \omega t + \pi/2$

$V_{emf} = -6\pi\cos(\omega t + \pi/2) = 6\pi\sin(100\pi t)$ mV

At t=1 ms, $V_{emf} = 6\pi\sin(0.1\pi) = 5.825$ mV

(b) The current induced is

$i = \frac{V_{emf}}{R} = 60\pi\sin(100\pi t)$ mA

At t=3ms, $i = 60\pi\sin(0.3\pi)$ mA=0.1525 A

Note that for sides AD and BC $(u\times B).dl = 0$ since $a_z.a_\rho = 0$

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9.4 Displacement current

Here we will consider Maxwell's curl equation for magnetic fields (Ampere's Law) for time-varying conditions.

For static EM fields, recall that

$$\nabla\times H = J \quad \cdots \quad (1)$$

But the divergence of the curl is zero:

$$\nabla\cdot(\nabla\times H) = 0 = \nabla\cdot J \quad \cdots \quad (2)$$

However, the equation of continuity requires that

$$\nabla\cdot J = -\frac{\partial\rho_V}{\partial t} \quad \cdots \quad (3)$$

Thus, equations (2) and (3) are incompatible for time varying conditions !!

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Displacement current

Here We must modify equation (1) to consider time-varying situation:

To do this, add a term J_d to eq. (1):

$$\nabla\times H = \mathrm{J} + \mathrm{J_d} \quad \cdots \quad (4)$$

again, taking the divergence, we have:

$$\nabla\cdot(\nabla\times H) = 0 = \nabla\cdot\mathrm{J} + \nabla\cdot\mathrm{J_d} \quad \cdots \quad (5)$$

Since $\nabla\cdot\mathrm{J} = -\frac{\partial\rho_V}{\partial t}$, then, $\nabla\cdot\mathrm{J_d} = \frac{\partial\rho_V}{\partial t}$

Since $\nabla\cdot\mathrm{D} = \rho_V$

$$\rightarrow \nabla\cdot\mathrm{J_d} = \frac{\partial}{\partial t}(\nabla\cdot D) = \nabla\cdot\frac{\partial\mathrm{D}}{\partial t} \quad \rightarrow \quad \mathrm{J_d} = \frac{\partial\mathrm{D}}{\partial t} \quad \cdots \quad (6)$$

Substituting eq (6) into eq (4) results in,

$$\nabla\times H = \mathrm{J} + \frac{\partial\mathrm{D}}{\partial t} \quad \cdots \quad (7)$$

Displacement current

$\nabla\times H = \mathrm{J} + \frac{\partial\mathrm{D}}{\partial t}$ * This is Maxwell's equation (based on Ampere's circuit law) for a time varying field.

The term $\mathrm{J}_d = \frac{\partial\mathrm{D}}{\partial t}$ is known as *displacement current density*. This is the third type of current density we have met:

- ***Conduction current density:*** $\mathrm{J} = \sigma E$

(motion of charge in a conductor)

- ***Convection Current Density:*** $\mathrm{J} = \rho_V \mathrm{u}$

(doesn't involve conductors, current flows through an insulating medium, such as liquid, or vacuum).

- ***Displacement Current Density:*** $\mathrm{J}_d = \frac{\partial\mathrm{D}}{\partial t}$

(is a result of time-varying electric field).

Displacement current

- The insertion of J_d into Ampere's equation was one of the major contributions of Maxwell.
- Without the term J_d, the propagation of electromagnetic waves (e.g., radio or TV) would be impossible.
- Displacement current is the mechanism which allows electromagnetic waves to propagate in a non-conducting medium.
- The displacement current is defined as:

$$I_d = \int J_d . dS = \int \frac{\partial D}{\partial t} . dS$$

$$\nabla \times H = J + J_d$$

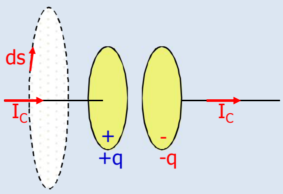
Note

At low frequency J_d is usually neglected compared with J. However, at Radio frequencies they become comparable. At the time of Maxwell, high frequency sources were not available and this equation could not be verified experimentally. It was years later that Hertz succeeded in generating and detecting radio waves. This is one of the rare situation where mathematical argument paved the way for experimental investigation.

Displacement Current

Apply Ampere's Law to a charging capacitor.

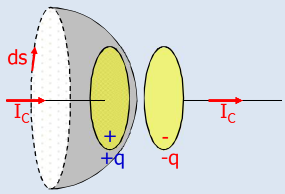
$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_C$$



The shape of the surface used for Ampere's Law shouldn't matter, as long as the "path" is the same.

Imagine a soup bowl surface, with the + plate resting near the bottom of the bowl.

Apply Ampere's Law to the charging capacitor.



$$\oint \vec{B} \cdot d\vec{s} = 0$$

The integral is zero because no current passes through the "bowl."

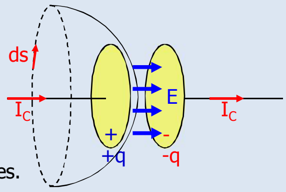
$$\oint \vec{B}\cdot d\vec{s} = \mu_0 I_C \qquad \oint \vec{B}\cdot d\vec{s} = 0$$

(The equation on the right is actually incorrect, and the equation on the left is incomplete.)

As the capacitor charges, the electric field between the plates changes.

$$q = C\Delta V = \left(\varepsilon \frac{A}{d}\right)(Ed)$$

$$= \varepsilon EA = \varepsilon \Phi_E$$



As the charge and electric field change, the electric flux changes.

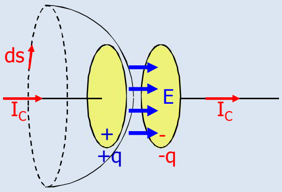
$$\frac{dq}{dt} = \frac{d}{dt}(\varepsilon \Phi_E) = \varepsilon \frac{d}{dt}(\Phi_E)$$

This term has units of current.

We define the displacement current to be

$$I_D = \varepsilon \frac{d}{dt}(\Phi_E).$$

The changing electric flux through the "bowl" surface is equivalent to the current I_C through the flat surface.



The generalized ("always correct") form of Ampere's Law is then

$$\oint \vec{B}\cdot d\vec{s} = \mu_0 (I_C + I_D)_{encl} = \mu_0 I_{encl} + \mu_0 \varepsilon \frac{d\Phi_E}{dt}.$$

Magnetic fields are produced by both conduction currents and time varying electric fields.

The "stuff" inside the gray boxes serves as your official starting equation for the displacement current I_D.

$$\oint \vec{B}\cdot d\vec{s} = \mu_0 (I_C + I_D)_{encl} = \mu_0 I_{encl} + \mu_0 \text{ -------}$$

I_D

Why "displacement?" If you put an insulator in between the plates of the capacitor, the atoms of the insulator are "stretched" because the electric field makes the protons "want" to go one way and the electrons the other. The process of "stretching" the atom involves displacement of charge, and therefore a current.

Displacement current

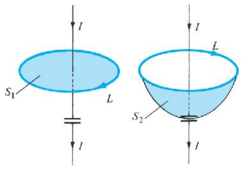
A typical example of displacement current is the current through a **capacitor** when alternating voltage source is applied to its plates.

- The total current density is $J+J_d$.
- Take Amperian path as shown. Consider 2 surfaces bounded by path L.
- **If surface S1 is chosen: J_d=0**

$$\oint_L H.dl = \int_{S_1} J.dS = I_{enc} = I$$

- **If surface S2 is chosen: J=0**

$$\oint_L H.dl = \int_{S_2} J_d.dS = \frac{d}{dt}\int_{S_2} D.dS = \frac{dQ}{dt} = I$$



- So we obtain the same current for either surface.
- A time-varying electric field induces magnetic field inside the capacitor.

Conduction to Displacement Current Ratio

The conduction current density is given by $J_c = \sigma E$

The displacement current density is given by $J_d = \varepsilon \frac{\partial E}{\partial t}$

Assume that the electric field is a sinusoidal function of time:

$$E = E_0 \cos \omega t$$

Then, $J_c = \sigma E_0 \cos \omega t, \quad J_d = -\omega\varepsilon E_0 \sin \omega t$

We have $\left|J_c\right|_{\max} = \sigma E_0$, $\left|J_d\right|_{\max} = \omega\varepsilon E_0$

Therefore $\frac{\left|J_c\right|_{\max}}{\left|J_d\right|_{\max}} = \frac{\sigma}{\omega\varepsilon}$

$\sigma >> \omega\varepsilon$	Good Conductor (I_d neglible)
$\sigma << \omega\varepsilon$	Good Insulator (I_c neglible)

Note: In free space (or other perfect dielectric), the conduction current is zero and only displacement current can exist.

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Example 9.4

A parallel plate capacitor with plate area of 5 cm² and plate separation of 3 mm has a voltage 50 sin 10^3 t V applied to its plates. Calculate the displacement current assuming $\varepsilon = 2\varepsilon_0$.

$$I_d = J_d.S, \quad J_d = \frac{\partial D}{\partial t}$$

$$but \quad D = \varepsilon E = \varepsilon\frac{V}{d} \rightarrow J_d = \frac{\partial D}{\partial t} = \frac{\varepsilon}{d}\frac{dV}{dt}$$

$$\rightarrow I_d = \frac{\varepsilon S}{d}\frac{dV}{dt} = C\frac{dV}{dt} \quad (\text{same as conduction current } I_c).$$

$$\rightarrow I_d = 2.\frac{10^{-9}}{36\pi}.\frac{5\times10^{-4}}{3\times10^{-3}}.10^{-3}\times 50\cos(10^3 t)$$

$$\rightarrow I_d = 147.4\cos(10^3 t) \quad \text{nA}$$

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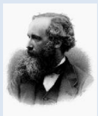
9.5 Maxwell's Equations in Final Forms

Differential form	*Integral form*	*Derived from*
$\vec{\nabla}\cdot\vec{D} = \rho_V$	$\oint_S \vec{D}\cdot\vec{dS} = \int_V \rho_V dV$	*Gauss's Law*
$\vec{\nabla}\cdot\vec{B} = 0$	$\oint_S \vec{B}\cdot\vec{dS} = 0$	*Nonexistance of magnetic Monopole*
$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$	$\oint_L \vec{E}\cdot\vec{dl} = -\frac{d}{dt}\int_S \vec{B}\cdot\vec{dS}$	*Faraday's Law*
$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$	$\oint_L \vec{H}\cdot\vec{dl} = \int_S\left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right)\cdot\vec{dS}$	*Ampere's Law modified by continuity eqn*



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❑The concepts of linearity, isotropy, and homogeneity of a material medium still apply for time varying fields.

❑In a linear, homogeneous, and isotropic medium characterized by σ, ε, and μ, the following relations hold for time varying fields:

$$\mathrm{D} = \varepsilon \mathrm{E} = \varepsilon_0 \mathrm{E} + \mathrm{P}$$
$$\mathrm{B} = \mu \mathrm{H} = \mu_0 (\mathrm{H} + \mathrm{M})$$
$$\mathrm{J} = \sigma \mathrm{E} + \rho_V \mathrm{u}$$

❑Consequently, the boundary conditions remain valid for time varying fields.

$$E_{1t} - E_{2t} = 0, \qquad D_{1n} - D_{2n} = \rho_S$$
$$H_{1t} - H_{2t} = K, \qquad B_{1n} - B_{2n} = 0$$

Electromagnetic flow diagrams

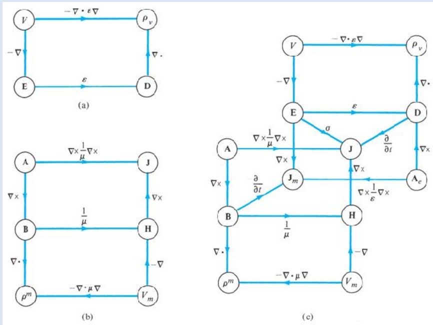

Figure 9.11 Electromagnetic flow diagrams showing the relationship between the potentials and vector fields: (a) electrostatic system, (b) magnetostatic system, (c) electromagnetic system.