Energy can be transported from one point (where a transmitter is located) to another point (with a receiver) by means of EM waves. The rate of energy transportation can be obtained from Maxwell's equations:

Using Maxwell equation: \( \nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \) 

Dotting both sides with \( \mathbf{E} \):

\[ \mathbf{E} \cdot (\nabla \times \mathbf{H}) = \mathbf{E} \cdot (\sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}) \]

But from vector identities: \( \mathbf{E} \cdot (\nabla \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{H} \cdot (\nabla \times \mathbf{E}) \)

\[ \mathbf{E} \cdot (\nabla \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) + \mathbf{H} \cdot (\nabla \times \mathbf{E}) \]

\[ \mathbf{H} \cdot (\nabla \times \mathbf{E}) = \sigma \mathbf{E}^2 + \frac{1}{2} \varepsilon \frac{\partial \mathbf{E}^2}{\partial t} \quad ... \ (1) \]

### 10.7 Power and the Poynting Vector

**Total power leaving the volume**

**Rate of decrease in energy stored in electric and magnetic fields**

**Ohmic power dissipated**

The **Poynting vector** (Watts/m²) is defined as:

\[ S = \mathbf{E} \times \mathbf{H} \]

It represents the instantaneous power density vector associated with the EM field at a given point.
Power and the Poynting Vector

**Poynting theorem:** states that the net power flowing out of a given volume \( v \) is equal to the time rate of decrease in the energy stored with \( v \) minus the ohmic losses.

Illustration of power balance for EM fields.

Note that \( \mathbf{E} \times \mathbf{H} \) is normal to both \( \mathbf{E} \) and \( \mathbf{H} \) and is therefore along the direction of propagation \( \mathbf{a} \).

The time-average Poynting vector \( \mathbf{\hat{v}}_{av}(z) \) over the period \( T=2\pi/\omega \) is:

\[
\mathbf{\hat{v}}_{av}(z) = \frac{1}{T} \int_{0}^{T} \mathbf{\hat{v}}(z,t) \, dt
\]

It can also be found by:

\[
\mathbf{\hat{v}}_{av}(z) = \frac{1}{2} \text{Re}(\mathbf{E} \times \mathbf{H}^*)
\]

For \( \mathbf{\hat{v}}(z,t) = \frac{E_z^2}{2|\mathbf{H}|} e^{-2\omega z} \left[ \cos \theta \gamma + \cos(2\omega t - \beta z - \theta) \right] \mathbf{a}_\gamma 
\]

\[
\mathbf{\hat{v}}_{av}(z) = \frac{E_z^2}{2|\mathbf{H}|} e^{-2\omega z} \cos \theta \gamma \mathbf{a}_\gamma
\]

The total time-average power crossing a given surface \( S \) is given by:

\[
P_{av} = \int \mathbf{\hat{v}}_{av} \cdot dS
\]

Assume that

\[
\mathbf{E}(z,t) = E_z e^{-\omega z} \cos(\omega t - \beta z) \mathbf{a}_\gamma
\]

then

\[
\mathbf{H}(z,t) = \frac{E_z}{|\mathbf{H}|} e^{-\omega z} \cos(\omega t - \beta z - \theta) \mathbf{a}_\gamma
\]

and

\[
\mathbf{\hat{v}}(z,t) = \mathbf{E} \times \mathbf{H} = \frac{E_z^2}{2|\mathbf{H}|} e^{-2\omega z} \cos(\omega t - \beta z) \cos(\omega t - \beta z - \theta) \mathbf{a}_\gamma
\]

\[
\mathbf{\hat{v}}_{av}(z) = \frac{E_z^2}{2|\mathbf{H}|} e^{-2\omega z} \left[ \cos \theta \gamma + \cos(2\omega t - 2\beta z - \theta) \right] \mathbf{a}_\gamma
\]

since

\[
\cos \alpha \cos \beta = \frac{1}{2} \left[ \cos(\alpha - B) + \cos(\alpha + B) \right]
\]

The total time-average power through a surface \( S \) is given by:

\[
P_{av} = \int \mathbf{\hat{v}}_{av} \cdot dS
\]

Power and the Poynting Vector

Assume that

\[
E(z,t) = E_z e^{-\omega z} \cos(\omega t - \beta z) \mathbf{a}_\gamma
\]

then

\[
H(z,t) = \frac{E_z}{|\mathbf{H}|} e^{-\omega z} \cos(\omega t - \beta z - \theta) \mathbf{a}_\gamma
\]

and

\[
\mathbf{\hat{v}}(z,t) = \mathbf{E} \times \mathbf{H} = \frac{E_z^2}{|\mathbf{H}|} e^{-2\omega z} \cos(\omega t - \beta z) \cos(\omega t - \beta z - \theta) \mathbf{a}_\gamma
\]

\[
\mathbf{\hat{v}}_{av}(z) = \frac{E_z^2}{2|\mathbf{H}|} e^{-2\omega z} \left[ \cos \theta \gamma + \cos(2\omega t - 2\beta z - \theta) \right] \mathbf{a}_\gamma
\]

since

\[
\cos \alpha \cos \beta = \frac{1}{2} \left[ \cos(\alpha - B) + \cos(\alpha + B) \right]
\]
Example 10.7

In a nonmagnetic medium
\[ E = 4 \sin (2\pi \times 10^7 t - 0.8x) \ a_x \ \text{V/m} \]

Find
(a) \( \alpha \), \( \eta \)
(b) The time-average power carried by the wave
(c) The total power crossing 100 cm² of plane \( 2x + y = 5 \)

Example 10.7 - Solution

(a) Since \( \alpha = 0 \) and \( \beta \neq \omega c \), the medium is not free space but a lossless medium.
\[ \beta = 0.8 \ \text{,} \ \omega = 2\pi \times 10^7 \ \text{,} \ \mu = \mu_0 \text{(nonmagnetic)}, \ \epsilon = \epsilon_0 \epsilon_r \]
Hence
\[ \beta = \omega \sqrt{\mu \epsilon} = \omega \sqrt{\mu_0 \epsilon_0 \epsilon_r} = \frac{\omega}{c} \sqrt{\epsilon_r} \]

or
\[ \sqrt{\epsilon_r} = \frac{\beta c}{\omega} = \frac{0.8 \times (3 \times 10^8)}{2\pi \times 10^7} \times \frac{12}{\pi} \]
\[ \epsilon_r = 14.59 \]

\[ \eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{120\pi}{120\pi \frac{\pi}{12} \times 10^3}} = 98.7 \ \Omega \]

Example 10.7 - Solution

(b) \( \varphi = E \times H = \frac{E^2}{\eta} \sin^2(\omega t - \beta x) \ a_x \)
\[ \varphi_{ave} = \frac{\int \varphi \ dt}{T} = \frac{E^2}{2\eta} \ a_x = \frac{16}{2 \times 10^6} \ a_x = 81 \ a_x \ \text{mW/m}^2 \]

(c) On plane \( 2x + y = 5 \) (see Example 3.5 or 8.5),
\[ a_y = 2a_x + a_x \]
Hence the total power is
\[ P_{ave} = \int \varphi_{ave} \ dS = \varphi_{ave} S a_x = (81 \times 10^{-3} a_x) \ (100 \times 10^{-3}) \left( \frac{2a_x + a_x}{\sqrt{5}} \right) \]
\[ = \frac{162 \times 10^{-3}}{\sqrt{5}} = 724.5 \ \mu W \]

10.8 Reflection of a plane wave at normal incidence

- When a plane wave from one medium meets a different medium, it is partly reflected and partly transmitted.
- The proportion of the incident wave that is reflected or transmitted depends on the parameters (\( \epsilon, \mu, \sigma \)) of the two media involved.
- Normal incidence (plane wave is normal to the boundary) and oblique incidence will be studied.
Reflection of a plane wave at normal incidence

Suppose a plane wave propagating along the +z direction is incident normally on the boundary z=0 between medium 1 (z<0) characterised by $\varepsilon_1,\mu_1,\sigma_1$ and medium 2 (z>0) characterised by $\varepsilon_2,\mu_2,\sigma_2$.

**Incident Wave**
(E, H) is traveling along +a in medium 1.

Assume the electric and magnetic field (in phasor form) as follows:

If $E_{i0}(z) = E_0 e^{j\omega z} a_y$, then

$$E_{i0}(z) = E_0 e^{j\omega z} a_y$$

If $H_{i0}(z) = H_0 e^{j\omega z} a_y$, then

$$H_{i0}(z) = H_0 e^{j\omega z} a_y$$

$E_{i0}$ is magnitude of the incident electric field at z=0

$H_{i0}$ is magnitude of the incident magnetic field at z=0

**Reflected Wave**
(E, H) is traveling along -a in medium 1.

$E_{r0}(z) = E_0 e^{j\omega z} a_y$

$H_{r0}(z) = H_0 e^{j\omega z} a_y$

$E_{r0}$ is magnitude of the reflected electric field at z=0

$H_{r0}$ is magnitude of the reflected magnetic field at z=0

**Transmitted Wave**
(E, H) is traveling along +a in medium 2.

If $E_{t0}(z) = E_0 e^{j\omega z} a_y$, then

$$E_{t0}(z) = E_0 e^{j\omega z} a_y$$

If $H_{t0}(z) = H_0 e^{j\omega z} a_y$, then

$$H_{t0}(z) = H_0 e^{j\omega z} a_y$$

$E_{t0}$ is magnitude of the transmitted electric field at z=0

$H_{t0}$ is magnitude of the transmitted magnetic field at z=0
Reflection of a plane wave at normal incidence

Field in medium 1: \( E_i = E_e + E_r \), \( H_i = H_r + H_e \)
Field in medium 2: \( E_i = E_e \), \( H_i = H_e \)

→ Since the waves are transverse, E and H fields are entirely tangential to the interface.

→ Applying the boundary conditions at the interface \( z = 0 \): \( (E_{i1} = E_{i2} \) and \( H_{i1} = H_{i2} ) \)
then:
\[
E_i(0) + E_{i1}(0) = E_{i2}(0) \rightarrow E_{i2} + E_{i1} = E_{i0}
\]
\[
H_i(0) + H_{i1}(0) = H_{i2}(0) \rightarrow \frac{1}{\eta_i}(E_{i0} - E_{i1}) = \frac{E_{i0}}{\eta_2}
\]

Reflection of a plane wave at normal incidence

When medium 1 is a perfect dielectric (lossless, \( \sigma_1 = 0 \)), and medium 2 is a perfect conductor (\( \sigma_2 = \infty \)):

For conductor \( n = \sqrt{\frac{\omega \mu \epsilon}{\sigma}} \):

\( \eta_2 = 0 \rightarrow \Gamma = -1 \rightarrow \tau = 0 \)
The wave is totally reflected and there is no transmitted wave \( (E_2 = 0) \).

The totally reflected wave combines with the incident wave to form a standing wave.

A standing wave "stands" and does not travel; it consists of two travelling waves \( (E_i \) and \( E_r) \) of equal amplitudes but in opposite directions.

Reflection of a plane wave at normal incidence

The standing wave in medium 1 is:
\[
E_{x_i} = E_{x_0} + E_{x_1} = E_{x_0} e^{-j\alpha x} + E_{x_1} e^{j\beta x} a_x
\]
But \( \Gamma = \frac{E_{x_0}}{E_{x_1}} = -1 \), \( \sigma_1 = 0, \alpha_z = 0, \gamma_i = j\beta_i \)
\[
E_{x_1} = -E_{x_0} e^{j\beta x} a_x
\]
or \( E_{x_1} = -2jE_{x_0} \sin \beta z a_x \) (since \( \sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \))
Thus \( E_i = \text{Re} \left( E_{x_0} e^{j\omega t} \right) \), or \( E_i = 2E_{x_0} \sin \beta z \sin \omega t a_x \)

Similarly, it can be shown that:
\[
H_i = \frac{2E_{x_0}}{\eta_2} \cos \beta z \cos \omega t a_y
\]
Reflection of a plane wave at normal incidence

Standing waves $E = 2E_0 \sin \beta z \sin \omega t$. The curves $0, 1, 2, 3, 4, \ldots$, are, respectively, at times $t = 0, T/8, T/4, 3T/8, T/2, \ldots$; $\lambda = 2\pi/\beta$.

Standing waves $E = 2E_0 \sin \beta z \sin \omega t$. The curves $0, 1, 2, 3, 4, \ldots$, are, respectively, at times $t = 0, T/8, T/4, 3T/8, T/2, \ldots$; $\lambda = 2\pi/\beta$.

Reflection of a plane wave at normal incidence

Medium 1: perfect dielectric $\sigma_1 = 0$

Medium 2: perfect dielectric $\sigma_2 = 0$

- $\eta_1$ and $\eta_2$ are real and so are $\Gamma$ and $\tau$.
- There is a standing wave in medium 1 but there is also a transmitted wave in medium 2. (incident wave is partly reflected and partly transmitted).
- However, the incident and reflected waves have amplitudes that are not equal in magnitude.
- Two cases:
  - case 1: when $\eta_2 > \eta_1$
  - case 2: when $\eta_2 < \eta_1$

CASE 1

Medium 1: perfect dielectric $\sigma_1 = 0$, Medium 2: perfect dielectric $\sigma_2 = 0$

If $\eta_2 > \eta_1$, $\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$, $\Gamma > 0$,

$\Gamma = |E_i| = |E_r| = 0$

$\Gamma = |E_i| = |E_r| = 0$

$\eta_1$ and $\eta_2$ are real

$\Gamma$ and $\Gamma$ are real

$E_i = E_0 + E_r = E_0(e^{-i\beta z} + \Gamma e^{i\beta z})$

$E_i = E_0 + E_r = E_0(e^{-i\beta z} + \Gamma e^{i\beta z})$

$E_i$ is maximum when $e^{2i\beta z} = 1$ $\rightarrow |E_i|_{\text{max}} = |E_0| (1 + |\Gamma|)$

$E_i$ is maximum when $e^{2i\beta z} = 1$ $\rightarrow |E_i|_{\text{max}} = |E_0| (1 + |\Gamma|)$

$-2\beta x_{\text{max}} = 0, 2\pi, 4\pi, 6\pi, \ldots$

$-2\beta x_{\text{max}} = 0, 2\pi, 4\pi, 6\pi, \ldots$

$\rightarrow x_{\text{max}} = \frac{n\pi}{\beta}$

$\rightarrow x_{\text{max}} = \frac{n\pi}{\beta}$

$n = 0, 1, 2, 3$

$|E_i|_{\text{max}} = E_0 (1 + |\Gamma|)$

$|E_i|_{\text{max}} = E_0 (1 + |\Gamma|)$

- $2\beta x_{\text{max}} = \pi, 3\pi, 5\pi, \ldots$

- $2\beta x_{\text{max}} = \pi, 3\pi, 5\pi, \ldots$

$\rightarrow |E_i|_{\text{min}} = E_0 (1 - |\Gamma|)$

$\rightarrow |E_i|_{\text{min}} = E_0 (1 - |\Gamma|)$

$-2\beta x_{\text{min}} = \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$-2\beta x_{\text{min}} = \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$n = 0, 1, 2, 3$

$\frac{2\beta x_{\text{min}}}{2\beta} = \frac{(2n+1)\pi}{4}$

$\frac{2\beta x_{\text{min}}}{2\beta} = \frac{(2n+1)\pi}{4}$

$\rightarrow |E_i|_{\text{min}} = E_0 (1 - |\Gamma|)$

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$\frac{2\beta x_{\text{min}}}{2\beta} = \frac{(2n+1)\pi}{4}$

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$\rightarrow |E_i|_{\text{min}} = E_0 (1 - |\Gamma|)$

$\rightarrow |E_i|_{\text{min}} = E_0 (1 - |\Gamma|)$

http://www.walter-fendt.de/ph14e/stwaverefl.htm
CASE 2

Medium 1: perfect dielectric \( \sigma_1 = 0 \), Medium 2: perfect dielectric \( \sigma_2 = 0 \)

If \( \eta_1 < \eta_2 \), \( \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \), \( \Gamma < 0 \), \( \Gamma = |\Gamma|e^{i\angle\Gamma} \), \( \angle \Gamma > 180^\circ \), \( \gamma \) and \( \Gamma \) are real

\[
|E_1| \text{ is maximum when } e^{i\angle\Gamma} = -1 \rightarrow |E_1|_{\text{max}} = |E_0| (1 + |\Gamma|)
\]

\[
-2\beta z_{\text{max}} = \pi, 3\pi, 5\pi, \ldots
\]

or \( -2\beta z_{\text{max}} = \frac{\pi, 3\pi, 5\pi, \ldots}{2\beta} \)

\[
z_{\text{max}} = -\left(\frac{2n + 1)\pi}{2\beta}\right)
\]

\( n = 0, 1, 2, 3, \ldots \)

\[
|E_1| \text{ is minimum when } e^{-i\angle\Gamma} = 1 \rightarrow |E_1|_{\text{min}} = |E_0| (1 - |\Gamma|)
\]

\[
-2\beta z_{\text{min}} = 0, 2\pi, 4\pi, 6\pi, \ldots
\]

or \( -2\beta z_{\text{min}} = \frac{\pi, 3\pi, 5\pi, \ldots}{2\beta} \)

\[
z_{\text{min}} = -\left(\frac{n\pi}{2}\right) = -\left(\frac{n\lambda}{2}\right)
\]

\( n = 0, 1, 2, 3, \ldots \)

Standing Wave Ratio, SWR

- Measures the amount of reflections, the more reflections, the larger the standing wave that is formed.
- The ratio of \( |E_{\text{max}}| \) to \( |E_{\text{min}}| \)

\[
s = \frac{|E_{\text{max}}|}{|E_{\text{min}}|} = \frac{|H_1|_{\text{max}}}{|H_1|_{\text{min}}} = 1 + |\Gamma|
\]

or \( |\Gamma| = \frac{s - 1}{s + 1} \)

\( s \) is dimensionless, expressed in decibels (dB) as: \( s \text{dB} = 20\log_{10}s \)

Example 10.8

In freespace \( z \geq 0 \), a plane wave with \( \mathbf{H}_0 = 10 \cos(10^3 t - \beta z) \mathbf{a}_z \) mA/m is incident normally on a lossless medium \( (\epsilon = 2, \mu = 4\mu_0) \) in region \( z \geq 0 \).

Determine the reflected wave \( \mathbf{H}_r, \mathbf{E}_r \) and the transmitted wave \( \mathbf{H}_t, \mathbf{E}_t \)

Solution:

\[
\beta_1 = \frac{\omega}{c} = \frac{10^3}{3 \times 10^8} = \frac{1}{3}
\]

\( \eta_1 = \eta_2 = 120\pi \)

\[
\beta_2 = \frac{\omega}{c} = \frac{10^3}{4 \times 10^8} = \frac{25}{3}
\]

\[
\eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}} = \sqrt{\frac{4\mu_0}{4\epsilon_0}} = \frac{2\eta_0}{3}
\]
Example 10.8 – solution continued

Given that \( \mathbf{H} = 10 \cos(10t - \beta z) \mathbf{a}_z \), we expect that
\[ \mathbf{E} = \mathbf{E}_0 \cos(10t - \beta z) \mathbf{a}_z, \]

where \( \mathbf{a}_z = \mathbf{a}_m \times \mathbf{a}_n = \mathbf{a}_x \times \mathbf{a}_y = -\mathbf{a}_y \) and \( E_n = \eta H_m = 10 \eta \).

Hence,
\[ E_x = -10 \eta \cos(10t - \beta z) \mathbf{a}_x \text{ mV/m}. \]

Now \( \frac{E_m}{E_n} = \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{2\eta_2 - \eta_m - \eta_n}{2\eta_2 + \eta_m + \eta_n} = \frac{1}{3} \), \( E_m = \frac{1}{3} E_n \).

Thus
\[ E_x = -\frac{10}{3} \eta \cos \left(10t + \frac{1}{3}z\right) \mathbf{a}_x \text{ mV/m}. \]

from which we easily obtain \( \mathbf{H}_x \) as
\[ \mathbf{H}_x = -\frac{10}{3} \cos \left(10t + \frac{1}{3}z\right) \mathbf{a}_x \text{ mA/m}. \]

Example 10.8 – solution continued

Similarly,
\[ \frac{E_n}{E_m} = \tau = 1 + \Gamma = \frac{4}{5} \text{ or } E_m = \frac{4}{3} E_n. \]

Thus
\[ E_x = E_m \cos \left(10t + \beta z\right) \mathbf{a}_x. \]

where \( \mathbf{a}_{m} = \mathbf{a}_{n} = -\mathbf{a}_{z} \). Hence,
\[ E_x = -\frac{40}{3} \eta \cos \left(10t - \frac{4}{3}z\right) \mathbf{a}_x \text{ mV/m}. \]

from which we obtain
\[ \mathbf{H}_x = \frac{20}{3} \cos \left(10t - \frac{4}{3}z\right) \mathbf{a}_x \text{ mA/m}. \]

Example 10.9

Given a uniform plane wave in air as
\[ \mathbf{E}_x = 40 \cos(\omega t - \beta z) \mathbf{a}_x + 30 \sin(\omega t - \beta z) \mathbf{a}_y \text{ V/m}. \]

(a) Find \( \mathbf{H}_x \).
(b) If the wave encounters a perfectly conducting plate normal to the \( z \) axis at \( z = 0 \), find the reflected wave \( \mathbf{E}_x \) and \( \mathbf{H}_x \).
(c) What are the total \( \mathbf{E} \) and \( \mathbf{H} \) fields for \( z \leq 0 \) ?
(d) Calculate the time-average Poynting vectors for \( z \leq 0 \) and \( z \geq 0 \).

Example 10.9 - solution

Solution

(a) This is similar to the problem in Example 10.3.
We may treat the wave as consisting of two waves \( \mathbf{E}_1 \) and \( \mathbf{E}_2 \), where
\[ \mathbf{E}_1 = 40 \cos(\omega t - \beta z) \mathbf{a}_x, \quad \mathbf{E}_2 = 30 \sin(\omega t - \beta z) \mathbf{a}_y. \]

At atmospheric pressure, air has \( \varepsilon_r = 1.0006 \pm 1 \).

Thus air may be regarded as free space.

Let \( \mathbf{H}_1 = \mathbf{H}_1 + \mathbf{H}_2 \)
\[ \mathbf{H}_1 = H_{1n} \cos(\omega t - \beta z) \mathbf{a}_n. \]

\[ H_{1n} = \frac{E_n}{\eta} = \frac{40}{120\pi} \frac{1}{3\pi} \]
\[ \mathbf{a}_n = \mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z \times \mathbf{a}_y = \mathbf{a}_y \]

Hence \[ \mathbf{H}_1 = \frac{1}{3\pi} \cos(\omega t - \beta z) \mathbf{a}_y. \]
Example 10.9 -solution

Similarly,
\[ \mathbf{H}_z = H_{z,0} \sin(\omega t - \beta z) \mathbf{a}_n \]
where
\[ H_{z,0} = \frac{E_{z,0}}{\eta_0} = \frac{30}{120\pi} = \frac{1}{4\pi} \]
\[ \mathbf{a}_n = \mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_x \times \mathbf{a}_y = -\mathbf{a}_z \]

Hence
\[ \mathbf{H}_z = \frac{1}{4\pi} \sin(\omega t - \beta z) \mathbf{a}_z \]
and
\[ \mathbf{H}_z = \mathbf{H}_z + \mathbf{H}_x = -\frac{1}{4\pi} \sin(\omega t - \beta z) \mathbf{a}_x + \frac{1}{3\pi} \cos(\omega t - \beta z) \mathbf{a}_y \text{ mA/m} \]

This problem can also be solved using Method 2 of Example 10.3.

Example 10.9 -solution

(b) Since medium 2 is perfectly conducting,
\[ \sigma_z >> 1 \rightarrow \eta_z << \eta_i \]
that is \( \Gamma = -1 \), \( \tau = 0 \)
showing that the incident \( \mathbf{E} \) and \( \mathbf{H} \) fields are totally reflected.\[ E_{inc} = \mathbf{E}_{inc} \mathbf{E}_{inc} = -E_w \]

Hence,
\[ \mathbf{E}_a = -40 \cos(\omega t + \beta z) \mathbf{a}_y - 30 \sin(\omega t + \beta z) \mathbf{a}_y \text{ V/m} \]
\[ \mathbf{H}_a = \frac{1}{3\pi} \cos(\omega t + \beta z) \mathbf{a}_y - \frac{1}{4\pi} \sin(\omega t + \beta z) \mathbf{a}_y \text{ A/m} \]

(c) The total fields in air
\[ \mathbf{E}_i = \mathbf{E} + \mathbf{E}_a \text{ and } \mathbf{H}_i = \mathbf{H} + \mathbf{H}_a \]
can be shown to be standing wave.
The total fields in the conductor are \( \mathbf{E}_z = \mathbf{E}_i = 0 \), \( \mathbf{H}_z = \mathbf{H}_a = 0 \).
Oblique incidence

\[ \theta_i \text{ is angle of incidence.} \]

The plane defined by propagation vector \( \mathbf{k} \) and a unit normal vector \( \mathbf{a}_n \) to the boundary is called \textit{plane of incidence}.

Parallel Polarization

It's defined as \( \mathbf{E} \parallel \) to incidence plane (\( \mathbf{E} \)-field lies in the xz-plane)

Oblique incidence

\[
\begin{align*}
E_i &= E_{i0} \cos(k_{ix}x + k_{iy}y + k_{iz}z - \omega t) \\
E_r &= E_{r0} \cos(k_{rx}x + k_{ry}y + k_{rz}z - \omega t) \\
E_t &= E_{t0} \cos(k_{tx}x + k_{ty}y + k_{tiz}z - \omega t)
\end{align*}
\]

where
\[
\begin{align*}
k_i &= k_z = \beta_i = \omega \sqrt{\mu_i \epsilon_i} \\
k_r &= \beta_r = \omega \sqrt{\mu_r \epsilon_r} \\
k_t &= \beta_t \cos \theta_r \\
k_{tiz} &= \beta_t \sin \theta_r
\end{align*}
\]

Parallel Polarization

\[
\begin{align*}
E_{ix} &= E_{i0} \cos \theta_i \; \mathbf{a}_x - \sin \theta_i \; \mathbf{a}_z e^{-j\beta_i (\sin \theta_i + \cos \theta_i)} \\
H_{ix} &= \frac{E_{i0}}{\eta_1} e^{-j\beta_i (\sin \theta_i + \cos \theta_i)} \; \mathbf{a}_y \\
E_{ry} &= E_{r0} \cos \theta_r \; \mathbf{a}_x + \sin \theta_r \; \mathbf{a}_z e^{-j\beta_r (\sin \theta_r - \cos \theta_r)} \\
H_{ry} &= \frac{E_{r0}}{\eta_1} e^{-j\beta_r (\sin \theta_r - \cos \theta_r)} \; \mathbf{a}_y
\end{align*}
\]
Tangential components of E and H should be continuous at the boundary $z=0$, 

$$E_{in} \cos \theta_i + E_{in} \cos \theta_i = E_{in} \cos \theta_i \quad \text{(x-components of E)}$$

$$\frac{E_{in}}{\eta_1} - \frac{E_{in}}{\eta_2} = \frac{E_{in}}{\eta_1}$$  \quad \text{(y-component of H)}

Reflection coefficient

$$\Gamma_i = \frac{E_{in}}{E_{in}} = \frac{\eta_1 \cos \theta_i - \eta_2 \cos \theta_i}{\eta_1 \cos \theta_i + \eta_2 \cos \theta_i}, \quad E_{in} = \Gamma_i E_{in}$$

Transmission coefficient

$$t_i = \frac{E_{in}}{E_{in}} = \frac{2\eta_1 \cos \theta_i}{\eta_1 \cos \theta_i + \eta_2 \cos \theta_i}, \quad E_{in} = t_i E_{in}$$

where \( t_i = (1 + \Gamma_i) \cos \theta_i \cos \theta_i \)

Parallel Polarization - Brewster angle, $\theta_B$

- Defined as the incidence angle at which the reflection coefficient is 0 (all transmission).

By setting $\theta = \theta_B$:

$$\Gamma_i = \eta_1 \cos \theta_i - \eta_2 \cos \theta_i = 0$$

$$\eta_2 \cos \theta_i = \eta_1 \cos \theta_i$$

or

$$\eta_1 \left(1 - \sin^2 \theta_i\right) = \eta_2 \left(1 - \sin^2 \theta_i\right)$$

Since

$$\sin \theta_i = \frac{\mu_2}{\mu_1} \quad \text{and} \quad \theta_i = \theta_{\text{B}}$$

$$\sin^2 \theta_B = \frac{1 - (\epsilon_2/\epsilon_1)}{1 - (\epsilon_1/\epsilon_2)}$$
Perpendicular Polarization

In this case, the \( E \) field is **perpendicular** to the plane of incidence (the \( xz \)-plane)

\[ E_z = E_{\text{in}} e^{-j \beta \left( \sin \theta_i + \cos \theta_i \right)} \mathbf{a}_y \]

\[ H_z = \frac{E_{\text{in}}}{\eta_2} \left( -\cos \theta_i \mathbf{a}_x + \sin \theta_i \mathbf{a}_z \right) e^{-j \beta \left( \sin \theta_i + \cos \theta_i \right)} \]

**Tangential** components of \( E \) and \( H \) should be continuous at the boundary \( z=0 \), and by setting \( \theta_i = \theta_r \):

\[ E_{\text{in}} + E_{\text{tr}} = E_{\text{in}} \]  \( \text{y-component of E} \)

\[ \left( \frac{E_{\text{in}}}{\eta_i}, \frac{E_{\text{in}}}{\eta_i} \right) \cos \theta_i = \frac{E_{\text{in}}}{\eta_2} \cos \theta_r \]  \( \text{x-component of H} \)

**Reflection coefficient**

\[ \Gamma = \frac{E_{\text{in}}}{E_{\text{in}}} = \frac{\eta_2 \cos \theta_i - \eta_i \cos \theta_i}{\eta_2 \cos \theta_i + \eta_i \cos \theta_i} \]

**Transmission coefficient**

\[ \tau = \frac{E_{\text{tr}}}{E_{\text{in}}} = \frac{2\eta_2 \eta_i \cos \theta_i}{\eta_2 \cos \theta_i + \eta_i \cos \theta_i} \]

where \( 1 + \Gamma = \tau \)
For no reflection (total transmission):

By setting \( \theta_1 = \theta_2 \):

\[
\Gamma = \frac{\eta_2 \cos \theta_2 - \eta_1 \cos \theta_1}{\eta_2 \cos \theta_2 + \eta_1 \cos \theta_1} = 0
\]

or

\[
\eta_1 (1 - \sin^2 \theta_2) = \eta_2 (1 - \sin^2 \theta_1)
\]

Since

\[
\frac{\sin \theta_1}{\sin \theta_2} = \sqrt{\frac{\mu_2 \varepsilon_1}{\mu_1 \varepsilon_2}} \quad \text{and} \quad \theta_2 = \theta_2
\]

\[
\sin^2 \theta_2 = \frac{1 - (\varepsilon_2 \mu_1 / \varepsilon_1 \mu_2)}{1 - (\mu_1 / \mu_2)^2}
\]

### Summary

<table>
<thead>
<tr>
<th>Property</th>
<th>Normal Incidence</th>
<th>Perpendicular</th>
<th>Parallel</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reflection coefficient</td>
<td>( r = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} )</td>
<td>( r = \frac{\eta_2 \cos \theta_2 - \eta_1 \cos \theta_1}{\eta_2 \cos \theta_2 + \eta_1 \cos \theta_1} )</td>
<td>( r = \frac{\eta_2 \cos \theta_2 - \eta_1 \cos \theta_1}{\eta_2 \cos \theta_2 + \eta_1 \cos \theta_1} )</td>
</tr>
<tr>
<td>Transmission coefficient</td>
<td>( \tau = \frac{2\eta_1}{\eta_1 + \eta_2} )</td>
<td>( \tau = \frac{2\eta_1 \cos \theta_2}{\eta_2 \cos \theta_2 + \eta_1 \cos \theta_1} )</td>
<td>( \tau = \frac{2\eta_2 \cos \theta_2}{\eta_2 \cos \theta_2 + \eta_1 \cos \theta_1} )</td>
</tr>
<tr>
<td>Relation</td>
<td>( r = 1 + \Gamma )</td>
<td>( \tau = 1 + \Gamma )</td>
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</tr>
</tbody>
</table>

### Example 10.10

An EM wave travels in free space with the electric field component

\[
E_x = 100e^{j(0.866y + 0.5z)} \mathbf{a}_x \text{ V/m}
\]

Determine

(a) \( \omega \) and \( \lambda \)

(b) The magnetic field component

(c) The time average power in the wave

### Example 10.10 - solution

(a) Comparing the given \( E \) with

\[
E_x = E_o e^{j(k_x x + k_y y + k_z z)} \mathbf{a}_x
\]

it is clear that

\[
k_x = 0 , \quad k_y = 0.866 , \quad k_z = 0.5
\]

\[
k = \sqrt{k_x^2 + k_y^2 + k_z^2} = \sqrt{(0.866)^2 + (0.5)^2} = 1
\]

But in free space,

\[
k = \frac{\omega}{c} = \frac{2\pi}{\lambda}
\]

Hence,

\[
\omega = kc = 3 \times 10^8 \text{ rad/s}
\]

\[
\lambda = \frac{2\pi}{k} = 2\pi = 6.283 \text{ m}
\]
Example 10.10 - solution

(b) The corresponding magnetic field is given by
\[ \mathbf{H} = \frac{\mathbf{a} \times \mathbf{E}}{\eta} \]
\[ a_x = \frac{0.866a_y + 0.5a_z}{\sqrt{0.866^2 + 0.5^2}} = 0.866a_y + 0.5a_z \]
\[ H_x = \frac{0.866a_y + 0.5a_z}{\eta} \times 100 \mathbf{a}_x e^{j(0.866x - 0.5z)} \quad \eta = (120\pi) \]
\[ H_z = (0.132a_y - 0.23a_x) e^{j(0.866x - 0.5z)} \text{ A/m} \]
(c) The time average power is
\[ \rho_{av} = \frac{1}{2} \text{Re} \left( \mathbf{E} \times \mathbf{H}^* \right) = \frac{E_y^2 - a_y}{2\eta} = \frac{(100)^2}{2(120\pi)} (0.866a_y + 0.5a_z) \]
= 11.49a_y + 6.631a_z \text{ W/m}^2

Example 10.11

A uniform plane wave in air with \( \mathbf{E} = 8 \cos (\omega t - 4x - 3z) \mathbf{a}_y \text{ V/m} \)
is incident on a dielectric slab \( (z \geq 0) \) with \( \mu_1 = 1, \varepsilon_\infty = 2.5, \sigma = 0. \)
Find
(a) The polarization of the wave
(b) The angle of incidence
(c) The reflected field
(d) The transmitted field

Example 10.11 - solution

(a) From the incident field, it is evident that the propagation vector is
\[ \mathbf{k}_i = 4\mathbf{a}_y + 3\mathbf{a}_z \rightarrow k_i = 5 = \omega\sqrt{\mu_1 \varepsilon_\infty} = \frac{\omega}{c} \]
Hence,
\[ \omega = 5c = 15 \times 10^8 \text{ rad/s} \]
A unit vector normal to the interface \( (z = 0) \) is \( \mathbf{a}_z \).
The plane containing \( \mathbf{k} \) and \( \mathbf{a}_z \) is \( y = \text{constant} \), which is the \( xz \)-plane, the plane of incidence. Since \( \mathbf{E}_i \) is normal to this plane, we have perpendicular polarization (similar to Figure 10.17).

(b) From the figure, \( \tan \theta' = \frac{k_x}{k_z} = \frac{4}{3} \rightarrow \theta' = 53.13^\circ \)
Alternatively, we can obtain \( \theta' \) from the fact that \( \theta' \)
is the angle between \( \mathbf{k} \) and \( \mathbf{a}_z \), that is, \( \cos \theta' = a_z \cdot a_z \)
\[ = \left( 4a_y + 3a_z \right) \cdot a_z = \frac{3}{5} \]
or \[ \theta' = 53.13^\circ \]
Example 10.11 - solution

c) Let \( \mathbf{E}_t = E_0 \cos(\omega t - \mathbf{k} \cdot \mathbf{r}) \mathbf{a}_y \),
which is similar to form to the given \( \mathbf{E}_0 \). The unit vector \( \mathbf{a}_y \) is chosen in
view of the fact that the tangential component of \( \mathbf{E} \) must be continuous
at the interface. From the Figure:
\[ \mathbf{k}_x = \mathbf{k} \cdot \mathbf{a}_x - \mathbf{k} \cdot \mathbf{a}_y \]
\[ \mathbf{k}_z = k_x \mathbf{a}_x + k_y \mathbf{a}_y \]
But \( \theta_t = \theta \) and \( k_x = 5 \) because both \( k_x \) and \( k_y \) are in
the same medium. Hence
\[ k_x = 4a_x - 3a_y \]

Example 10.11 - solution

d) Similarly, let the transmitted electric field be
\[ \mathbf{E}_t = E_0 \cos(\omega t - \mathbf{k} \cdot \mathbf{r}) \mathbf{a}_y \]
where \( k_x = \beta = \omega \sqrt{\mu_x \varepsilon_x} = \frac{\omega}{c} \sqrt{\mu_x \varepsilon_x \varepsilon_z} = \frac{15 \times 10^8}{3 \times 10^8} \sqrt{1 \times 2.5} = 7.906 \)
From the Figure,
\[ k_y \sin \theta = 4 \]
\[ k_y = k \cos \theta = 6.819 \]
\[ k_y = 4a_x + 6.819a_y \]
Notice that \( k_x = k_y - k_z \)
\[ \Gamma = \frac{E_{\text{in}}}{E_{\text{out}}} = \frac{2 \eta_1 \cos \theta}{\eta_1 \cos \theta + \eta_2 \cos \theta} = \frac{2 \times 238.4 \cos 53.13^\circ}{238.4 \cos 53.13^\circ + 377 \cos 30.39^\circ} = 0.611 \]

Example 10.11 - solution

to find \( E_{\text{in}} \), we need \( \theta_t \). From Snell's law
\[ \sin \theta_t = \eta_2 \sin \theta = \frac{c \sqrt{\mu_x \varepsilon_x}}{c \sqrt{\mu_2 \varepsilon_2}} \sin \theta = \frac{\sin 53.13^\circ}{\sqrt{2.5}} \]
or \( \theta_t = 30.39^\circ \)
\[ \Gamma = \frac{E_{\text{out}}}{E_{\text{in}}} = \frac{\eta_2 \cos \theta - \eta_1 \cos \theta}{\eta_1 \cos \theta + \eta_2 \cos \theta} \]
where \( \eta_1 = \eta_2 = 377 \), \( \eta_2 = \frac{\mu_x \mu_2}{\sqrt{\varepsilon_x \varepsilon_2}} = \frac{377}{\sqrt{2.5}} = 238.4 \)
\[ \Gamma = \frac{E_{\text{out}}}{E_{\text{in}}} = \frac{238.4 \cos 53.13^\circ - 377 \cos 30.39^\circ}{238.4 \cos 53.13^\circ + 377 \cos 30.39^\circ} = -0.389 \]
Hence, \( E_{\text{in}} = \Gamma E_{\text{out}} = -0.389(8) = -3.112 \)
\[ E_t = -3.112 \cos(15 \times 10^8 t - 4x + 3z) \mathbf{a}_y \text{ V/m} \]

Example 10.11 - solution

The same result could be obtained from the relation \( r = 1 + \Gamma \). Hence, \( E_{\text{in}} = \Gamma E_{\text{out}} = 0.611 \times 8 = 4.888 \)
\[ E_t = 4.888 \cos(15 \times 10^8 t - 4x - 6.819z) \mathbf{a}_y \]
From \( E_x, H_z \) is easily obtained as
\[ H_z = \frac{\mathbf{a}_z \times E_t}{\eta_2} = \frac{4a_z + 6.819a_y}{7.906(238.4)} \times 4.888 \cos(\omega t - \mathbf{k} \cdot \mathbf{r}) \]
\[ H_z = (-17.69a_x + 10.37a_z) \cos(15 \times 10^8 t - 4x - 6.819z) \text{ mA/m} \]