CHAPTER 10

EELE 3332 – Electromagnetic II Chapter 10

Electromagnetic Wave Propagation

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1

10.7 Power and the Poynting Vector

* Energy can be trasported from one point (where a transmitter is located) to another point (with a receiver) by means of EM waves.
* The rate of energy transportation can be obtined from Maxwell's equations:

Using Maxwell equation: $\nabla \times \mathrm{H} = \sigma \mathrm{E} + \varepsilon \frac{\partial \mathrm{E}}{\partial t}$

Dotting both sides with E:

$$\mathrm{E} \cdot (\nabla \times \mathrm{H}) = \sigma E^2 + \mathrm{E} \cdot \varepsilon \frac{\partial \mathrm{E}}{\partial t}$$

But from vector identities: $\nabla \cdot (\mathrm{H} \times \mathrm{E}) = \mathrm{E} \cdot (\nabla \times \mathrm{H}) - \mathrm{H} \cdot (\nabla \times \mathrm{E})$

$$\mathrm{E} \cdot (\nabla \times \mathrm{H}) = \mathrm{H} \cdot (\nabla \times \mathrm{E}) + \nabla \cdot (\mathrm{H} \times \mathrm{E})$$

$$\rightarrow \mathrm{H} \cdot (\nabla \times \mathrm{E}) + \nabla \cdot (\mathrm{H} \times \mathrm{E}) = \sigma E^2 + \frac{1}{2} \varepsilon \frac{\partial E^2}{\partial t} \quad \ldots \ (1)$$

2

Using Maxwell equation : $\nabla \times \mathrm{E} = -\mu \frac{\partial H}{\partial t}$, Dotting both sides with H:

$$\mathrm{H} \cdot (\nabla \times \mathrm{E}) = \mathrm{H} \cdot \left(-\mu \frac{\partial \mathrm{H}}{\partial t} \right) = -\frac{\mu}{2} \frac{\partial H^2}{\partial t}$$

Substitute in equation (1): $\mathrm{H} \cdot (\nabla \times \mathrm{E}) + \nabla \cdot (\mathrm{H} \times \mathrm{E}) = \sigma E^2 + \frac{1}{2} \varepsilon \frac{\partial E^2}{\partial t}$

$$-\frac{\mu}{2} \frac{\partial H^2}{\partial t} - \nabla \cdot (\mathrm{E} \times \mathrm{H}) = \sigma E^2 + \frac{1}{2} \varepsilon \frac{\partial E^2}{\partial t}$$

$$\rightarrow \nabla \cdot (\mathrm{E} \times \mathrm{H}) = -\sigma E^2 - \frac{1}{2} \varepsilon \frac{\partial E^2}{\partial t} - \frac{\mu}{2} \frac{\partial H^2}{\partial t}$$

Take volume integral of both sides:

$$\int_v \nabla \cdot (\mathrm{E} \times \mathrm{H}) dv = -\frac{\partial}{\partial t} \int_v \left[\frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2 \right] dv - \int_v \sigma E^2 dv$$

Applying the divergence theorem to the left hand side:

$$\oint_S (\mathrm{E} \times \mathrm{H}) \cdot dS = -\frac{\partial}{\partial t} \int_v \left[\frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2 \right] dv - \int_v \sigma E^2 dv$$

Power and the Poynting Vector

Poynting's theorem

$$\oint_S (\mathrm{E} \times \mathrm{H}) \cdot dS = -\frac{\partial}{\partial t} \int_v \left[\frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2 \right] dv - \int_v \sigma E^2 dv$$

Total power leaving the volume

Rate of decrease in energy stored in electric and magnetic fields

Ohmic power dissipated

the *PoyntingVector* (Watts/m^2) is defined as:

$$\wp = \mathrm{E} \times \mathrm{H}$$

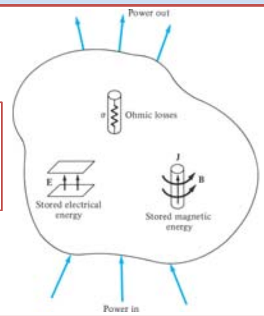
It represents the instantaneous power density vector associated with the EM field at a given point.

4

Power and the Poynting Vector

Poynting theorem: states that the net power flowing out of a given volume *v* is equal to the time rate of decrease in the energy stored with *v* minus the ohmic losses.

Illustration of power balance for EM fields.



Note that $\wp = E \times H$ is normal to both E and H and is therefore along the direction of propagation a_k

5

Power and the Poynting Vector

Assume that

$$E(z,t) = E_0 e^{-\alpha z} \cos(\omega t - \beta z) a_x$$

then $$H(z,t) = \frac{E_0}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta) a_y$$

and $$\wp(z,t) = E \times H = \frac{E_0^2}{|\eta|} e^{-2\alpha z} \cos(\omega t - \beta z) \cos(\omega t - \beta z - \theta_\eta) a_z$$

$$\wp(z,t) = \frac{E_0^2}{2|\eta|} e^{-2\alpha z} \left[\cos\theta_\eta + \cos(2\omega t - 2\beta z - \theta_\eta)\right] a_z$$

since $\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$

6

Power and the Poynting Vector

The time-average Poynting vector $\wp_{ave}(z)$ over the period $T = 2\pi/\omega$ is:

$$\wp_{ave}(z) = \frac{1}{T}\int_0^T \wp(z,t)\,dt$$

It can also be found by:

$$\wp_{ave}(z) = \frac{1}{2}\mathrm{Re}\left(E_s \times H_s^*\right)$$

For $$\wp(z,t) = \frac{E_0^2}{2|\eta|} e^{-2\alpha z}\left[\cos\theta_\eta + \cos(2\omega t - 2\beta z - \theta_\eta)\right] a_z$$

$$\rightarrow \quad \wp_{ave}(z) = \frac{E_0^2}{2|\eta|} e^{-2\alpha z} \cos\theta_\eta \; a_z$$

The total time-average power crossing a given surface S is given by:

$$P_{ave} = \int_S \wp_{ave} \cdot dS$$

7

Power and the Poynting Vector

$\wp(x,y,z,t) \rightarrow$ Poynting vector (time-varying vector) $\rightarrow$ (watts/m^2)

$\wp(x,y,z,t) = E \times H$

$\wp_{ave}(x,y,z) \rightarrow$ time-average of Poynting vector

(time-invariant vector) $\rightarrow$ (watts/m^2)

$$\wp_{ave} = \frac{1}{T}\int_0^T \wp(x,y,z,t)\,dt$$

$$\wp_{ave} = \frac{1}{2}\mathrm{Re}\left(E_s \times H_s^*\right)$$

$$\wp_{ave}(z) = \frac{E_0^2}{2|\eta|} e^{-2\alpha z}\cos\theta_\eta \; a_z \quad \left(\textit{for } E = E_0 e^{-\alpha z}\cos(\omega t - \beta z) a_x\right)$$

$P_{ave} \rightarrow$ total time-average power through a surface (scalar) $\rightarrow$ watts

$$P_{ave} = \int_S \wp_{ave} \cdot dS$$

8

Example 10.7

In a nonmagnetic medium

$E = 4 \sin(2\pi \times 10^7\ t - 0.8x)\ \mathbf{a}_z$ V/m

Find

(a) ε_r , η

(b) The time-average power carried by the wave

(c) The total power crossing 100 cm^2 of plane 2x + y = 5

Example 10.7 - Solution

(a) Since α=0 and $\beta \neq \omega$/c, the medium is not free space but a lossless medium.

$$\beta = 0.8 \quad, \omega = 2\pi \times 10^7 \quad, \mu = \mu_o(\text{nonmagnetic}), \ \varepsilon = \varepsilon_o\varepsilon_r$$

Hence

$$\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_o\varepsilon_o\varepsilon_r} = \frac{\omega}{c}\sqrt{\varepsilon_r}$$

or

$$\sqrt{\varepsilon_r} = \frac{\beta c}{\omega} = \frac{0.8\left(3\times10^8\right)}{2\pi\times10^7} = \frac{12}{\pi}$$

$$\varepsilon_r = 14.59$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_o}{\varepsilon_o\varepsilon_r}} = \frac{120\pi}{\sqrt{\varepsilon_r}} = 120\pi.\frac{\pi}{12} = 10\pi^2 = 98.7\ \Omega$$

Example 10.7 - Solution

(b) $\wp = \mathbf{E}\times\mathbf{H} = \dfrac{E_o^{\ 2}}{\eta}\sin^2(\omega t - \beta x)\ \mathbf{a}_x$

$$\wp_{ave} = \frac{1}{T}\int_0^T \wp\, dt = \frac{E_o^{\ 2}}{2\eta}\ \mathbf{a}_x = \frac{16}{2\times10\pi^2}\ \mathbf{a}_x = 81\ \mathbf{a}_x\ \text{mW/m}^2$$

(c) On plane 2x + y = 5 (see Example 3.5 or 8.5),

$$\mathbf{a}_n = \frac{2\mathbf{a}_x + \mathbf{a}_y}{\sqrt{5}}$$

Hence the total power is

$$P_{ave} = \int \wp_{ave}.dS = \wp_{ave}.S\mathbf{a}_n = \left(81\times10^{-3}\mathbf{a}_x\right).\left(100\times10^{-4}\right)\left[\frac{2\mathbf{a}_x + \mathbf{a}_y}{\sqrt{5}}\right]$$

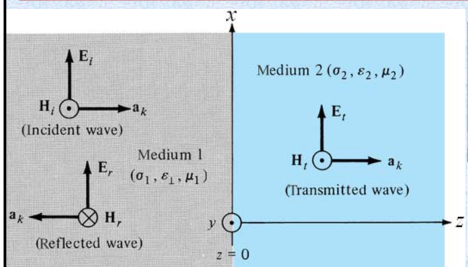
$$= \frac{162\times10^{-5}}{\sqrt{5}} = 724.5\mu\text{W}$$

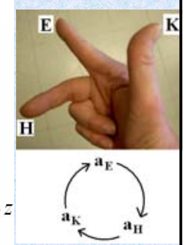
10.8 Reflection of a plane wave at normal incidence

- When a plane wave from one medium meets a different medium, it is partly **reflected** and partly **transmitted**.
- The proportion of the incident wave that is reflected or transmitted depends on the parameters (ε,μ,σ) of the two media involved.
- **Normal** incidence (plane wave is normal to the boundary) and **oblique** incidence will be studied.

Reflection of a plane wave at normal incidence

Suppose a plane wave propagating along the +z direction is incident normally on the boundary z=0 between medium 1 (z<0) characterised by ε1,μ1,σ1 and medium 2 (z>0) characterised by ε2,μ2,σ2.





13

Reflection of a plane wave at normal incidence

Incident Wave

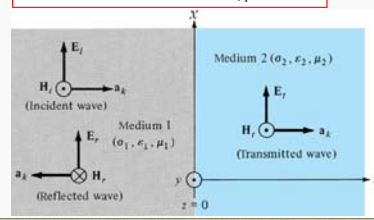
(E_i, H_i) is traveling along $+a_z$ in medium 1.

Assume the electric and megnetic filed (in phasor form) as follows:

$$E_{is}(z) = E_{i0} e^{-\gamma_1 z} a_x, \quad then$$

$$H_{is}(z) = H_{i0} e^{-\gamma_1 z} a_y = \frac{E_{i0}}{\eta_1} e^{-\gamma_1 z} a_y$$

E_{i0} is magnitude of the incident electric field at z=0



14

Reflection of a plane wave at normal incidence

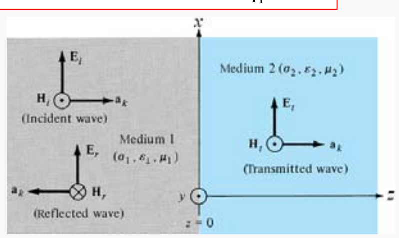
Reflected Wave

(E_r, H_r) is traveling along $-a_z$ in medium 1.

$$\text{If } E_{rs}(z) = E_{r0} e^{\gamma_1 z} a_x, \quad then$$

$$H_{rs}(z) = H_{r0} e^{\gamma_1 z} \left(-a_y\right) = -\frac{E_{r0}}{\eta_1} e^{\gamma_1 z} a_y$$

E_{r0} is magnitude of the reflected electric field at z=0



15

Reflection of a plane wave at normal incidence

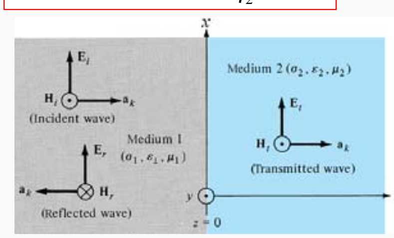
Transmitted Wave

(E_t, H_t) is traveling along $+a_z$ in medium 2.

$$\text{If } E_{ts}(z) = E_{t0} e^{-\gamma_2 z} a_x, \quad then$$

$$H_{ts}(z) = H_{t0} e^{-\gamma_2 z} a_y = \frac{E_{t0}}{\eta_2} e^{-\gamma_2 z} a_y$$

E_{i0} is magnitude of the transmitted electric field at z=0



16

Reflection of a plane wave at normal incidence

Field in medium 1: $E_1 = E_i + E_r$, $H_1 = H_i + H_r$
Field in medium 2: $E_2 = E_t$, $H_2 = H_t$

$\rightarrow$ Since the waves are transverse, E and H fields are entirely tangential to the interface.

$\rightarrow$ Applying the boundary conditions at the interface $z = 0$:
($E_{1t} = E_{2t}$ and $H_{1t} = H_{2t}$)

then :

$E_i(0) + E_{r0}(0) = E_{t0}(0) \rightarrow E_{i0} + E_{r0} = E_{t0}$

$H_i(0) + H_r(0) = H_t(0) \rightarrow \frac{1}{\eta_1}(E_{i0} - E_{r0}) = \frac{E_{t0}}{\eta_2}$

17

Reflection of a plane wave at normal incidence

From the last two equations:

$$E_{r0} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E_{i0}$$

$\rightarrow$ Reflection Coefficient $\boxed{\Gamma = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}}$, or $E_{r0} = \Gamma E_{i0}$

and $E_{t0} = \frac{2\eta_2}{\eta_2 + \eta_1} E_{i0}$

$\rightarrow$ Transmission Coefficient $\boxed{\tau = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2}{\eta_2 + \eta_1}}$, or $E_{t0} = \tau E_{i0}$

Note that:

1. $1 + \Gamma = \tau$
2. Both Γ and τ are dimensionless and may be complex.
   (Γ and τ are real for lossless media, and complex for lossy media)
3. $0 \le |\Gamma| \le 1$

18

Reflection of a plane wave at normal incidence

When medium 1 is a perfect dielectric (lossless , σ1=0), and medium 2 is a perfect conductor (σ2=∞):

For conductor $\eta = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^0$

Medium 1	Medium 2
a perfect dielectric (lossless , σ1=0)	a perfect conductor (σ2=∞)

➢ $\eta_2 = 0 \rightarrow \Gamma = -1 \rightarrow \tau = 0$

➢ The wave is totally reflected and there is no transmitted wave ($E_2 = 0$).

➢ The totally reflected wave combines with the incident wave to form a ***standing wave***.

➢ ***A standing wave** "stands" and does not* travel; it consists of two travelling waves (E_i and E_r) of equal amplitudes but in opposite directions.

19

Reflection of a plane wave at normal incidence

The standing wave in medium 1 is:

Medium 1	Medium 2
a perfect dielectric (lossless , σ1=0)	a perfect conductor (σ2=∞)

$E_{1s} = E_{is} + E_{rs} = \left(E_{i0} e^{-\gamma_1 z} + E_{r0} e^{\gamma_1 z}\right) a_x$

But $\Gamma = \frac{E_{r0}}{E_{i0}} = -1$, $\sigma_1 = 0$, $\alpha_1 = 0$, $\gamma_1 = j\beta_1$

$E_{1s} = -E_{i0}\left(e^{j\beta_1 z} - e^{-j\beta_1 z}\right) a_x$

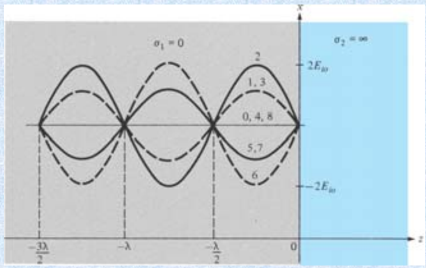
or $E_{1s} = -2jE_{i0} \sin\beta_1 z \; a_x$ (since $\sin A = \frac{e^{jA} - e^{-jA}}{2j}$)

Thus $E_1 = \mathrm{Re}\left(E_{1s} e^{j\omega t}\right)$, *or* $\boxed{E_1 = 2E_{i0} \sin\beta_1 z \sin\omega t \; a_x}$

Similarly, it can be shown that: $\boxed{H_1 = \frac{2E_{i0}}{\eta_1} \cos\beta_1 z \cos\omega t \; a_y}$

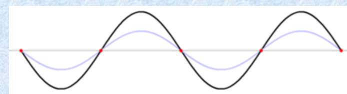
20

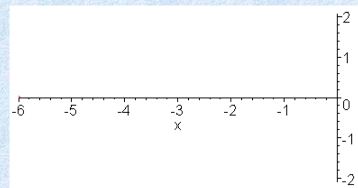
Reflection of a plane wave at normal incidence



Standing waves $E = 2E_{io} \sin \beta_1 z \sin \omega t \, \mathbf{a}_x$. The curves 0, 1, 2, 3, 4, . . ., are, respectively, at times $t = 0, T/8, T/4, 3T/8, T/2, \ldots$; $\lambda = 2\pi/\beta_1$.

Standing Waves Examples





Standing wave on a string

http://www.walter-fendt.de/ph14e/stwaverefl.htm

Reflection of a plane wave at normal incidence

Medium 1 : perfect dielectric σ_1=0

Medium 2: perfect dielectric σ_2=0

- η_1 and η_2 are real and so are Γ and τ.
- There is a standing wave in medium 1 but there is also a transmitted wave in medium 2. (*incident wave is partly reflected and partly transmitted*).
- However, the incident and reflected waves have amplitudes that are not equal in magnitude.
- Two cases:

  case 1: when $\eta_2 > \eta_1$

  case 2: when $\eta_2 < \eta_1$

CASE 1

Medium 1 : perfect dielectric σ_1=0, Medium 2: perfect dielectric σ_2=0

If $\eta_2 > \eta_1$, $\Gamma = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$, $\Gamma > 0$,

$\Gamma = |\Gamma| e^{j0} = |\Gamma| \angle 0^o$

τ and Γ are real

$$\mathrm{E}_{1s} = \mathrm{E}_{is} + \mathrm{E}_{rs} = E_{oi}(e^{-j\beta_1 z} + \Gamma e^{+j\beta_1 z})$$
$$= E_{oi} e^{-j\beta_1 z}(1 + |\Gamma| e^{+2j\beta_1 z})$$
$$\rightarrow |\mathrm{E}_{1s}| = E_{oi} \left|1 + |\Gamma| e^{+2j\beta_1 z}\right|$$

$|\mathrm{E}_1|$ is maximum when $e^{+2j\beta_1 z} = 1 \rightarrow |\mathrm{E}_1|_{\max} = E_{oi}(1 + |\Gamma|)$

$$-2\beta_1 z_{\max} = 0, 2\pi, 4\pi, 6\pi \ldots$$
$$\text{or } -\beta_1 z_{\max} = 0, \pi, 2\pi, 3\pi, \ldots \quad \rightarrow z_{\max} = -\frac{n\pi}{\beta_1} = -\frac{n\lambda_1}{2} \qquad n = 0,1,2,3$$

$|\mathrm{E}_1|$ is minimum when $e^{+2j\beta_1 z} = -1 \rightarrow |\mathrm{E}_1|_{\min} = E_{oi}(1 - |\Gamma|)$

$$-2\beta_1 z_{\min} = \pi, 3\pi, 5\pi \ldots \quad \text{or} \quad -\beta_1 z_{\min} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots$$

$$z_{\min} = -\frac{(2n+1)\pi}{2\beta_1} = -\frac{(2n+1)\lambda_1}{4} \qquad n = 0,1,2,3$$

CASE 2

Medium 1 : perfect dielectric σ_1=0, Medium 2: perfect dielectric σ_2=0

If $\eta_2 < \eta_1$, $\Gamma = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$, $\Gamma < 0$,

$\Gamma = |\Gamma| e^{j\pi} = |\Gamma| \angle 180^o$

τ and Γ are real

$$E_{1s} = E_{is} + E_{rs} = E_{oi}(e^{-j\beta_1 z} + \Gamma e^{+j\beta_1 z})$$
$$= E_{oi} e^{-j\beta_1 z}(1 - |\Gamma| e^{+2j\beta_1 z})$$
$$\rightarrow |E_{1s}| = E_{oi} \left|1 - |\Gamma| e^{+2j\beta_1 z}\right|$$

$|E_1|$ is maximum when $e^{+2j\beta_1 z} = -1 \rightarrow |E_1|_{max} = E_{oi}(1 + |\Gamma|)$

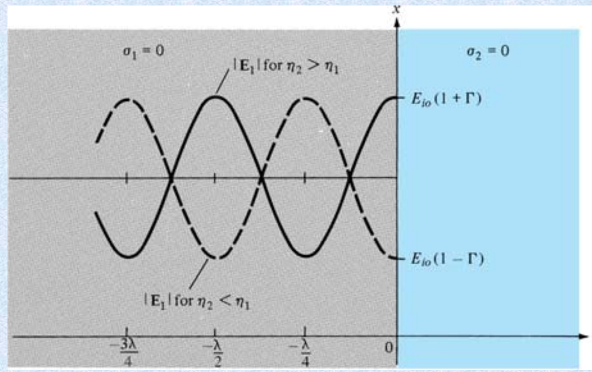
$-2\beta_1 z_{max} = \pi, 3\pi, 5\pi ...$ or $-\beta_1 z_{max} = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2} ...$

$$z_{max} = -\frac{(2n+1)\pi}{2\beta_1} = -\frac{(2n+1)\lambda_1}{4} \qquad n = 0,1,2,3$$

$|E_1|$ is minimum when $e^{+2j\beta_1 z} = 1 \rightarrow |E_1|_{min} = E_{oi}(1 - |\Gamma|)$

$-2\beta_1 z_{min} = 0, 2\pi, 4\pi, 6\pi ...$

or $-\beta_1 z_{min} = 0, \pi, 2\pi, 3\pi, ...$ $\rightarrow z_{min} = -\dfrac{n\pi}{\beta_1} = -\dfrac{n\lambda_1}{2}$ $\quad n = 0,1,2,3$



Standing waves due to reflection at an interface between two lossless media; $\lambda = 2\pi/\beta_1$.

Standing Wave Ratio, SWR

- Measures the amount of reflections, the more reflections, the larger the standing wave that is formed.
- The ratio of $|E_1|_{max}$ to $|E_1|_{min}$

$$s = \frac{|E_1|_{max}}{|E_1|_{min}} = \frac{|H_1|_{max}}{|H_1|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

or

$$|\Gamma| = \frac{s-1}{s+1}$$

- Since $0 \le |\Gamma| \le 1$, it follows that $1 \le s \le \infty$.
  - * When Γ=0, s=1, no reflection, total transmission.
  - * When $|\Gamma|$=1, $s=\infty$, no transmission, total reflection.
- s is dimensionless, expressed in decibels (dB) as: s dB=20log$_{10}$ s

27

Example 10.8

In freespace ($z \le 0$),a plane wave with

$$\mathbf{H}_i = 10\cos(10^8 t - \beta z)\mathbf{a}_x \text{ mA/m}$$

is incident normally on a lossless medium ($\varepsilon=2\varepsilon_o, \mu=8\mu_o$) in region $z \ge 0$.
Determine the reflected wave $\mathbf{H}_r$, $\mathbf{E}_r$ and the transmitted wave $\mathbf{H}_t$, $\mathbf{E}_t$

Solution :

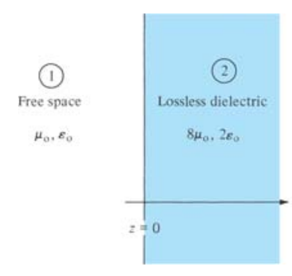
$$\beta_1 = \frac{\omega}{c} = \frac{10^8}{3\times10^8} = \frac{1}{3}$$

$$\eta_1 = \eta_o = 120\pi$$

$$\beta_2 = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_o\varepsilon_o}\sqrt{\mu_r\varepsilon_r}$$

$$\beta_2 = \frac{\omega}{c}.(4) = 4\beta_1 = \frac{4}{3}$$

$$\eta_2 = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_o}{\varepsilon_o}}\sqrt{\frac{\mu_r}{\varepsilon_r}} = 2\eta_o$$



28

Example 10.8 – solution continued

Given that $\mathbf{H}_i = 10\cos(10^8 t - \beta_1 z)\mathbf{a}_x$ we expect that

$$\mathbf{E}_i = \mathbf{E}_{io}\cos(10^8 t - \beta_1 z)\mathbf{a}_{Ei}$$

where $\mathbf{a}_{Ei} = \mathbf{a}_{Hi} \times \mathbf{a}_{ki} = \mathbf{a}_x \times \mathbf{a}_z = -\mathbf{a}_y$ and $E_{io} = \eta_1 H_{io} = 10\eta_o$

Hence, $\mathbf{E}_i = -10\eta_o \cos(10^8 t - \beta_1 z)\mathbf{a}_y$ mV/m

Now $\dfrac{E_{ro}}{E_{io}} = \Gamma = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \dfrac{2\eta_o - \eta_o}{2\eta_o + \eta_o} = \dfrac{1}{3}$, $\quad E_{ro} = \dfrac{1}{3}E_{io}$

Thus $\mathbf{E}_r = -\dfrac{10}{3}\eta_o \cos\left(10^8 t + \dfrac{1}{3}z\right)\mathbf{a}_y$ mV/m

from which we easily obtain $\mathbf{H}_r$ as

$$\mathbf{H}_r = -\frac{10}{3}\cos\left(10^8 t + \frac{1}{3}z\right)\mathbf{a}_x \text{ mA/m}$$

29

Example 10.8 – solution continued

Similarly,

$$\frac{E_{to}}{E_{io}} = \tau = 1 + \Gamma = \frac{4}{3} \quad \text{or} \quad E_{to} = \frac{4}{3}E_{io}$$

Thus

$$\mathbf{E}_t = E_{to}\cos\left(10^8 t + \beta_2 z\right)\mathbf{a}_{E_t}$$

where $\mathbf{a}_{Et} = \mathbf{a}_{Ei} = -\mathbf{a}_y$ .Hence,

$$\mathbf{E}_t = -\frac{40}{3}\eta_o \cos\left(10^8 t - \frac{4}{3}z\right)\mathbf{a}_y \text{ mV/m}$$

from which we obtain

$$\mathbf{H}_t = \frac{20}{3}\cos\left(10^8 t - \frac{4}{3}z\right)\mathbf{a}_x \text{ mA/m}$$

30

Example 10.9

Given a uniform plane wave in air as

$$\mathbf{E}_i = 40\cos(\omega t - \beta z)\mathbf{a}_x + 30\sin(\omega t - \beta z)\mathbf{a}_y \text{ V/m}$$

(a) Find $\mathbf{H}_i$

(b) If the wave encounters a perfectly conducting plate normal to the z axis at z = 0, find the reflected wave $\mathbf{E}_r$ and $\mathbf{H}_r$.

(c) What are the total **E** and **H** fields for $z \leq 0$?

(d) Calculate the time-average Poynting vectors for $z \leq 0$ and $z \geq 0$.

31

Example 10.9 -solution

Solution

(a) This is similar to the problem in Example 10.3.

We may treat the wave as consisting of two waves $\mathbf{E}_{i1}$ and $\mathbf{E}_{i2}$ where

$$\mathbf{E}_i = 40\cos(\omega t - \beta z)\mathbf{a}_x \text{ , } \mathbf{E}_{i2} = 30\sin(\omega t - \beta z)\mathbf{a}_y$$

At atmospheric pressure, air has $\varepsilon_r = 1.0006 \cong 1$.

Thus air may be regarded as free space.

Let $\mathbf{H}_i = \mathbf{H}_{i1} + \mathbf{H}_{i2}$

$$\mathbf{H}_{i1} = H_{i1o}\cos(\omega t - \beta z)\mathbf{a}_{H_1}$$

$$H_{i1o} = \frac{E_{i1o}}{\eta_o} = \frac{40}{120\pi} = \frac{1}{3\pi}$$

$$\mathbf{a}_{H_1} = \mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y$$

Hence $\mathbf{H}_{i1} = \dfrac{1}{3\pi}\cos(\omega t - \beta z)\mathbf{a}_y$

32

Example 10.9 -solution

Similarly,

$$\mathbf{H}_{i2} = H_{i2o}\ \sin(\omega t - \beta z)\mathbf{a}_{H_2}$$

where

$$H_{i2o} = \frac{E_{i2o}}{\eta_o} = \frac{30}{120\pi} = \frac{1}{4\pi}$$

$$\mathbf{a}_{H_2} = \mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_z \times \mathbf{a}_y = -\mathbf{a}_x$$

Hence

$$\mathbf{H}_{i2} = -\frac{1}{4\pi}\ \sin(\omega t - \beta z)\mathbf{a}_x$$

and

$$\mathbf{H}_i = \mathbf{H}_{i1} + \mathbf{H}_{i2} = -\frac{1}{4\pi}\ \sin(\omega t - \beta z)\mathbf{a}_x + \frac{1}{3\pi}\ \cos(\omega t - \beta z)\mathbf{a}_y \quad \text{mA/m}$$

This problem can also be solved using Method 2 of Example 10.3.

33

Example 10.9 -solution

(b) Since medium 2 is perfectly conducting,

$$\frac{\sigma_2}{\omega\varepsilon_2} >> 1 \ \rightarrow \ \eta_2 << \eta_1$$

that is $\Gamma \approx -1$, $\tau = 0$

showing that the incident **E** and **H** fields are totally reflected.

$$E_{ro} = \Gamma E_{io} = -E_{io}$$

Hence, $\mathbf{E}_r = -40\ \cos(\omega t + \beta z)\mathbf{a}_x \ - 30\ \sin(\omega t + \beta z)\mathbf{a}_y$ V/m

$$H_r = \frac{1}{3\pi}\ \cos(\omega t + \beta z)\mathbf{a}_y \ - \frac{1}{4\pi}\ \sin(\omega t + \beta z)\mathbf{a}_x \quad \text{A/m}$$

(c) The total fields in air

$$\mathbf{E}_1 = \mathbf{E}_i + \mathbf{E}_r \ \text{ and } \ \mathbf{H}_1 = \mathbf{H}_i + \mathbf{H}_r$$

can be shown to be standing wave.

The total fields in the conductor are $\mathbf{E}_2 = \mathbf{E}_t = 0$, $\mathbf{H}_2 = \mathbf{H}_t = 0$.

34

Example 10.9 -solution

(d) For $z \le 0$,

$$\wp_{1ave} = \frac{|E_{1s}|^2}{2\eta_1}\mathbf{a}_k = \frac{1}{2\eta_o}[E^2_{io}\mathbf{a}_z - E^2_{ro}\mathbf{a}_z]$$

$$= \frac{1}{240\pi}\left[\left(40^2 + 30^2\right)\mathbf{a}_z - \left(40^2 + 30^2\right)\mathbf{a}_z\right] = 0$$

For $z \ge 0$,

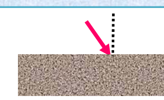
$$\wp_{2ave} = \frac{|E_{2s}|^2}{2\eta_2}\mathbf{a}_k = \frac{E^2_{to}}{2\eta_2}\mathbf{a}_z = 0$$

because the whole incident power is reflected.

35

Oblique incidence

- Wave arrives at an angle.
- Assume lossless media.
- Uniform plane wave in <u>general form</u>



$$E(\mathbf{r},t) = E_o \cos(\mathbf{k}\cdot\mathbf{r} - \omega t) = \text{Re}[E_o e^{j(\mathbf{k}\cdot\mathbf{r} - \omega t)}]$$

$\mathbf{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$ position vector

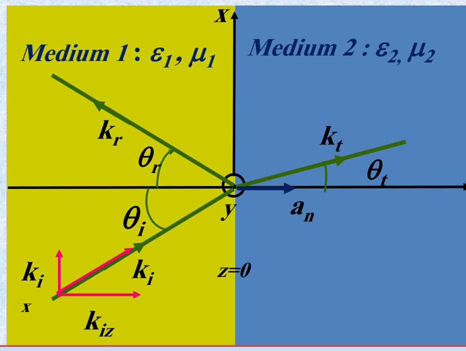
$\mathbf{k} = k_x\hat{a}_x + k_y\hat{a}_y + k_z\hat{a}_z$ wave number or propagation vector

$$k^2 = k_x^2 + k_y^2 + k_z^2 = \omega^2\mu\varepsilon$$

- For <u>lossless</u> <u>unbounded</u> media, $k = \beta$

36

Oblique incidence

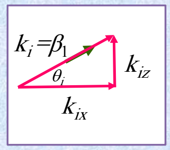


- θ_i is ***angle of incidence***.
- The plane defined by propagation vector **k** and a unit normal vector $\mathbf{a}_n$ to the boundary is called ***plane of incidence***.

37

Oblique incidence

$$E_i = E_{io}\cos(k_{ix}x + k_{iy}y + k_{iz}z - \omega t)$$
$$E_r = E_{ro}\cos(k_{rx}x + k_{ry}y + k_{rz}z - \omega t)$$
$$E_t = E_{to}\cos(k_{tx}x + k_{ty}y + k_{tz}z - \omega t)$$



where $k_i = k_r = \beta_1 = \omega\sqrt{\mu_1\varepsilon_1}$

$k_t = \beta_2 = \omega\sqrt{\mu_2\varepsilon_2}$

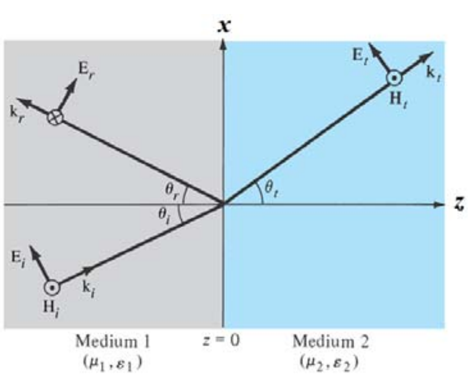
$k_{ix} = \beta_1\cos\theta_i$

$k_{iz} = \beta_1\sin\theta_i$

38

Parallel Polarization

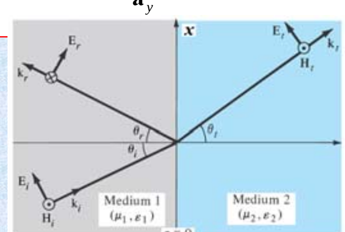
It's defined as **E** is || to incidence plane (**E**-field lies in the xz-plane)



39

Parallel Polarization

$$E_{is} = E_{io}(\cos\theta_i\ \mathbf{a}_x - \sin\theta_i\ \mathbf{a}_z)\ e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$
$$H_{is} = \frac{E_{io}}{\eta_1}e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}\ \mathbf{a}_y$$
$$E_{rs} = E_{ro}(\cos\theta_r\ \mathbf{a}_x + \sin\theta_r\ \mathbf{a}_z)\ e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$
$$H_{rs} = -\frac{E_{ro}}{\eta_1}e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}\ \mathbf{a}_y$$

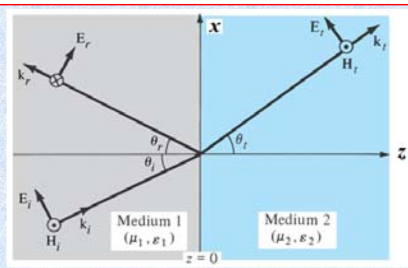


40

Parallel Polarization

$$E_{ts} = E_{to}(\cos\theta_t\, \mathbf{a}_x - \sin\theta_t\, \mathbf{a}_z)e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$H_{ts} = \frac{E_{to}}{\eta_2} e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}\, \mathbf{a}_y$$



Parallel Polarization

Tangential components of E and H should be continuous at the boundary z=0,

$$E_{io}(\cos\theta_i)e^{-j\beta_1 x\sin\theta_i} + E_{ro}(\cos\theta_r)e^{-j\beta_1 x\sin\theta_r} = E_{to}(\cos\theta_t)e^{-j\beta_2 x\sin\theta_t}$$

$$\frac{E_{io}}{\eta_1}e^{-j\beta_1 x\sin\theta_i} - \frac{E_{ro}}{\eta_1}e^{-j\beta_1 x\sin\theta_i} = \frac{E_{to}}{\eta_2}e^{-j\beta_2 x\sin\theta_t}$$

The exponential terms must be equal for the previous equations to be valid: $\rightarrow \beta_1\sin\theta_i = \beta_1\sin\theta_r = \beta_2\sin\theta_t$

or $\boxed{\theta_i = \theta_r}$ (Incidence angle = reflection angle)

$\rightarrow$ $\boxed{\frac{\beta_1}{\beta_2} = \frac{\sin\theta_t}{\sin\theta_i} = \sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}} = \frac{n_1}{n_2}}$ (snell's law)

Parallel Polarization

Hence,

$E_{io}\cos\theta_i + E_{ro}\cos\theta_r = E_{to}\cos\theta_t$ (x-components of E)

$\frac{E_{io}}{\eta_1} - \frac{E_{ro}}{\eta_1} = \frac{E_{to}}{\eta_2}$ (y-component of H)

Reflection coefficient $\boxed{\Gamma_{\parallel} = \frac{E_{ro}}{E_{io}} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}}$, $E_{ro} = \Gamma_{\parallel}E_{io}$

Transmission coefficient $\boxed{\tau_{\parallel} = \frac{E_{to}}{E_{io}} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}}$, $E_{to} = \tau_{\parallel}E_{io}$

where $\tau_{\parallel} = (1 + \Gamma_{\parallel})\frac{\cos\theta_i}{\cos\theta_t}$

Parallel Polarization - Brewster angle, θ_B

- defined as the incidence angle at which the reflection coefficient is 0 (**all transmission**).

By setting $\theta_i = \theta_B$:−

$$\Gamma_{\parallel} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_{B\parallel}}{\eta_2\cos\theta_t + \eta_1\cos\theta_{B\parallel}} = 0$$

$\rightarrow \eta_2\cos\theta_t = \eta_1\cos\theta_{B\parallel}$

or

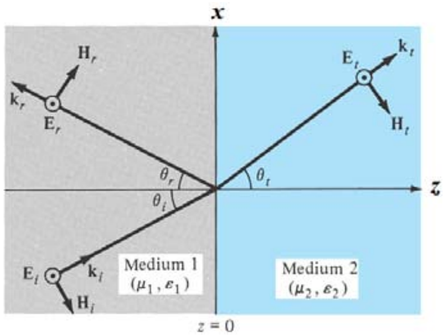
$$\eta_2^2\left(1 - \sin^2\theta_t\right) = \eta_1^2\left(1 - \sin^2\theta_{B\parallel}\right)$$

Since $\frac{\sin\theta_t}{\sin\theta_i} = \sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}}$, and $\theta_i = \theta_{B\parallel}$

$\rightarrow$ $\boxed{\sin^2\theta_{B\parallel} = \frac{1 - (\varepsilon_1\mu_2/\varepsilon_2\mu_1)}{1 - (\varepsilon_1/\varepsilon_2)^2}}$

Perpendicular Polarization

In this case, the E field is **perpendicular** to the plane of incidence (the xz-plane)



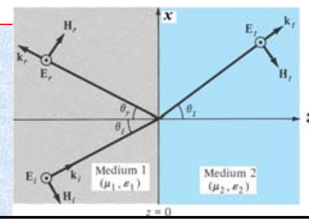
Perpendicular Polarization

$$E_{is} = E_{io}\ e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}\ \mathbf{a}_y$$

$$H_{is} = \frac{E_{io}}{\eta_1}(-\cos\theta_i\ \mathbf{a}_x + \sin\theta_i\ \mathbf{a}_z)\ e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$E_{rs} = E_{ro}e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}\ \mathbf{a}_y$$

$$H_{rs} = \frac{E_{ro}}{\eta_1}(\cos\theta_r\ \mathbf{a}_x + \sin\theta_r\ \mathbf{a}_z)\ e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

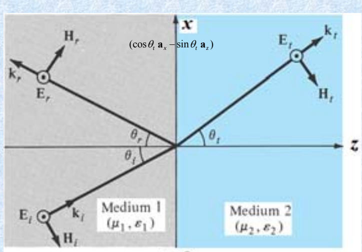


46

Perpendicular Polarization

$$E_{ts} = E_{to}\ e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}\ \mathbf{a}_y$$

$$H_{ts} = \frac{E_{to}}{\eta_2}(-\cos\theta_t\ \mathbf{a}_x + \sin\theta_t\ \mathbf{a}_z)e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$



47

Perpendicular Polarization

Tangential components of E and H should be continuous at the boundary z=0, and by setting $\theta_i = \theta_r$:–

$E_{io} + E_{ro} = E_{to}$ (y-component of E)

$\left(\frac{E_{io}}{\eta_1} - \frac{E_{ro}}{\eta_1}\right)\cos\theta_i = \frac{E_{to}}{\eta_2}\cos\theta_t$ (x-component of H)

Reflection coefficient $\Gamma_\perp = \frac{E_{ro}}{E_{io}} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$, $E_{ro} = \Gamma_\perp E_{io}$

Transmission coefficient $\tau_\perp = \frac{E_{to}}{E_{io}} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$, $E_{to} = \tau_\perp E_{io}$

where $1 + \Gamma_\perp = \tau_\perp$

48

Perpendicular Polarization - Brewster angle

- For no reflection (total transmission):

By setting $\theta_i = \theta_{B\perp}$:-

$$\Gamma_\perp = \frac{\eta_2 \cos\theta_{B\perp} - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_{B\perp} + \eta_1 \cos\theta_t} = 0$$

$$\rightarrow \eta_2 \cos\theta_{B\perp} = \eta_1 \cos\theta_t$$

or

$$\eta_2^2 \left(1 - \sin^2\theta_{B\perp}\right) = \eta_1^2 \left(1 - \sin^2\theta_t\right)$$

$$\textit{Since}\ \ \frac{\sin\theta_t}{\sin\theta_i} = \sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}},\ \text{and}\ \theta_i = \theta_{B\perp}$$

$$\rightarrow \boxed{\sin^2\theta_{B\perp} = \frac{1 - (\varepsilon_2\mu_1/\varepsilon_1\mu_2)}{1 - (\mu_1/\mu_2)^2}}$$

Summary

Property	Normal Incidence	Perpendicular	Parallel
Reflection coefficient	$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$	$\Gamma_\perp = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$	$\Gamma_\parallel = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$
Transmission coefficient	$\tau = \frac{2\eta_2}{\eta_2 + \eta_1}$	$\tau_\perp = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$	$\tau_\parallel = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$
Relation	$\tau = 1 + \Gamma$	$\tau_\perp = 1 + \Gamma_\perp$	$\tau_\parallel = \left(1 + \Gamma_\parallel\right)\frac{\cos\theta_i}{\cos\theta_t}$

Example 10.10

An EM wave travels in free space with the electric field component

$$\mathbf{E}_s = 100 e^{j(0.866y + 0.5z)}\ \mathbf{a}_x\ \text{V/m}$$

Determine

(a) ω and λ

(b) The magnetic field component

(c) The time average power in the wave

Example 10.10 -solution

(a) Comparing the given E with

$$\mathbf{E}_s = \mathbf{E}_o\ e^{j\mathbf{k}\cdot\mathbf{r}} = E_o\ e^{j(k_x x + k_y y + k_z z)}\mathbf{a}_x$$

it is clear that

$$k_x = 0\ ,\ k_y = 0.866\ \ ,\ k_z = 0.5$$

$$k = \sqrt{k_x{}^2 + k_y{}^2 + k_z{}^2} = \sqrt{(0.866)^2 + (0.5)^2} = 1$$

But in free space,

$$k = \beta = \omega\sqrt{\mu_o\varepsilon_o} = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

Hence, $\omega = kc = 3\times10^8$ rad/s

$$\lambda = \frac{2\pi}{k} = 2\pi = 6.283\ \text{m}$$

Example 10.10 -solution

(b) the corresponding magnetic field is given by $\mathbf{H}_s = \dfrac{a_k \times \mathbf{E}_s}{\eta}$

$$a_k = \frac{0.866\mathrm{a}_y + 0.5\mathrm{a}_z}{\sqrt{0.866^2 + 0.5^2}} = 0.866\mathbf{a}_y + 0.5\mathbf{a}_z$$

$$\mathbf{H}_s = \frac{0.866\mathbf{a}_y + 0.5\mathbf{a}_z}{\eta} \times 100\mathbf{a}_x e^{j(0.866y+0.5z)} \; ; \; \eta = (120\pi)$$

$$\mathbf{H}_s = (0.132\mathbf{a}_y - 0.23\mathbf{a}_z)\, e^{j(0.866y+0.5z)} \quad \text{A/m}$$

(c) The time average power is

$$\wp_{avg} = \frac{1}{2}\mathrm{Re}\left(\mathbf{E}_s \times \mathbf{H}_s^*\right) = \frac{E_o^{\;2}}{2\eta}\mathbf{a}_k = \frac{(100)^2}{2(120\pi)}(0.866\mathbf{a}_y + 0.5\mathbf{a}_z)$$

$$= 11.49\mathbf{a}_y + 6.631\mathbf{a}_z \quad \text{W/m2}$$

Example 10.11

A uniform plane wave in air with

$$\mathbf{E} = 8\cos(\omega t - 4x - 3z)\,\mathbf{a}_y \quad \text{V/m}$$

is incident on a dielectric slab $(z \geq 0)$ with $\mu_r = 1$, $\varepsilon_r = 2.5$, σ=0. Find

(a) The polarization of the wave

(b) The angle of incidence

(c) The reflected **E** field

(d) The transmitted **H** field

Example 10.11 - solution

(a) From the incident **E** field, it is evident that the propagation vector is

$$\mathbf{k}_i = 4\mathbf{a}_x + 3\mathbf{a}_z \rightarrow k_i = 5 = \omega\sqrt{\mu_o \varepsilon_o} = \frac{\omega}{c}$$

Hence, ω=5c=15×10^8 rad/s

A unit vector normal to the interface (z = 0) is $\mathbf{a}_z$.
The plane containing **k** and $\mathbf{a}_z$ is y = constant, which is the xz-plane, the plane of incidence. Since $\mathbf{E}_i$ is normal to this plane, we have perpendicular polarization (similar to Figure 10.17).

Example 10.11 - solution

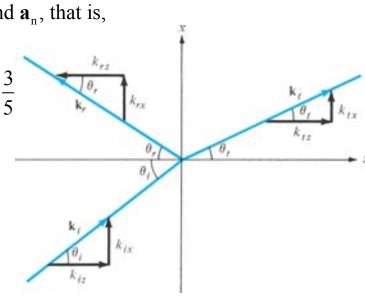
(b) from the figure, $\tan\theta_i = \dfrac{\mathbf{k}_{ix}}{\mathbf{k}_{iz}} = \dfrac{4}{3} \rightarrow \theta_i = 53.13^o$

Alternatively, we can obtain θ_i from the fact that θ_i is the angle between **k** and $\mathbf{a}_n$, that is,

$$\cos\theta_i = \mathbf{a}_k . \mathbf{a}_n$$

$$= \left(\frac{4\mathbf{a}_x + 3\mathbf{a}_z}{5}\right).\mathbf{a}_z = \frac{3}{5}$$

or $\theta_i = 53.13^o$



Example 10.11 - solution

(c) Let $\mathbf{E}_r = E_{ro}\cos(\omega t - \mathbf{k}_r.\mathbf{r})\mathbf{a}_y$

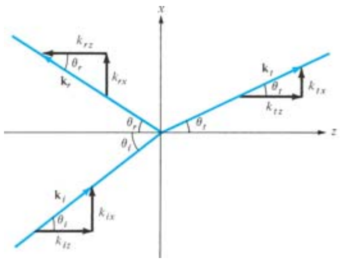
which is similar to form to the given $\mathbf{E}_i$. The unit vector $\mathbf{a}_y$ is chosen in view of the fact that the tangential component of **E** must be continuous at the interface. From the Figure:

$\mathbf{k}_r = \mathbf{k}_{rx}\mathbf{a}_x - \mathbf{k}_{rz}\mathbf{a}_z$

$k_{rx} = k_r \sin\theta_r$, $k_{rz} = k_r \cos\theta_r$

But $\theta_r = \theta_i$ and $k_r = k_i = 5$ because both k_r and k_i are in the same medium. Hence

$k_r = 4\mathbf{a}_x - 3\mathbf{a}_z$



Example 10.11 - solution

To find E_{r0}, we need θ_t. From Snell's law

$$\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i = \frac{c\sqrt{\mu_1\varepsilon_1}}{c\sqrt{\mu_2\varepsilon_2}}\sin\theta_i = \frac{\sin 53.13^o}{\sqrt{2.5}}$$

or $\theta_t = 30.39^o$

$$\Gamma_\perp = \frac{E_{ro}}{E_{io}} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$

where $\eta_1 = \eta_0 = 377$, $\eta_2 = \sqrt{\dfrac{\mu_0\mu_{r2}}{\varepsilon_o\varepsilon_{r2}}} = \dfrac{377}{\sqrt{2.5}} = 238.4$

$$\Gamma_\perp = \frac{E_{ro}}{E_{io}} = \frac{238.4\cos 53.13^o - 377\cos 30.39^o}{238.4\cos 53.13^o + 377\cos 30.39^o} = -0.389$$

Hence, $E_{ro} = \Gamma_\perp E_{io} = -0.389(8) = -3.112$

$\mathbf{E}_r = -3.112\cos(15\times10^8 t - 4x + 3z)\mathbf{a}_y$ V/m

Example 10.11 - solution

(d) Similarly, let the transmitted electric field be

$\mathbf{E}_t = E_{to}\cos(\omega t - \mathbf{k}_t.\mathbf{r})\mathbf{a}_y$

where $k_t = \beta_2 = \omega\sqrt{\mu_2\varepsilon_2} = \dfrac{\omega}{c}\sqrt{\mu_{r2}\varepsilon_{r2}} = \dfrac{15\times10^8}{3\times10^8}\sqrt{1\times2.5} = 7.906$

From the Figure ,

$k_{tx} = k_t \sin\theta_t = 4$

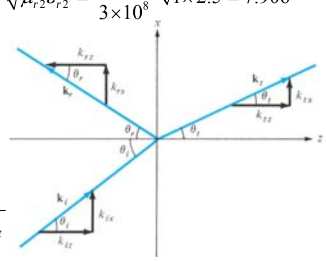
$k_{tz} = k_t \cos\theta_t = 6.819$

$\mathbf{k}_t = 4\mathbf{a}_x + 6.819\mathbf{a}_z$

Notice that $k_{ix} = k_{rx} = k_{tx}$

$$\tau_\perp = \frac{E_{to}}{E_{io}} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$

$$= \frac{2\times238.4\cos 53.13^o}{238.4\cos 53.13^o + 377\cos 30.39^o} = 0.611$$



Example 10.11 - solution

The same result could be obtained from the relation $\tau = 1 + \Gamma$. Hence,

$E_{to} = \tau_\perp E_{io} = 0.611\times8 = 4.888$

$\mathbf{E}_t = 4.88\cos(15\times10^8 t - 4x - 6.819z)\mathbf{a}_y$

From $\mathbf{E}_t$, $\mathbf{H}_t$ is easily obtained as

$$\mathbf{H}_t = \frac{\mathbf{a}_{k_t}\times\mathbf{E}_t}{\eta_2} = \frac{4\mathbf{a}_x + 6.819\mathbf{a}_z}{7.906(238.4)}\times4.888\mathbf{a}_y\cos(\omega t - \mathbf{k}_r.\mathbf{r})$$

$\mathbf{H}_t = (-17.69\mathbf{a}_x + 10.37\mathbf{a}_z)\cos(15\times10^8 t - 4x - 6.819z)$ mA/m