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Between Scott and Alexandroff spaces

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Abstract

In this paper, a new type of posets is considered. More precise, the class of posets which are \( g - ACC \) is strictly contains the class of posets satisfying \( ACC \). This class is important, and we will see that a poset being \( g - ACC \) is necessary and sufficient condition for equality of the two topologies; the Scott topology
and the Alexandroff topology, induced from the poset. A class of $T_0$–Alexandroff spaces whose corresponding posets is $g–ACC$ is called $g$-Artinian $T_0$–Alexandroff spaces. This class is more general than the class of Artinian spaces. We give alternative of characterizations of a space to being $g$-Artinian space.

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### 1 Introduction

Let $(P, \leq)$ be a poset. The set of all maximal (resp. minimal) elements is denoted by $M$ (resp $m$). If $A$ is a subset of $P$, then the order of $P$ induces an order in $A$. In this case, we define $M(A)$ (resp. $m(A)$) to be the set of all maximal (resp. minimal) elements of $A$ under the induced order. If $A$ is a subset of $P$ with supremum, we write the supremum element as $\text{sup} A$ or $\lor A$.

We say that $P$ satisfies the *ascending chain condition* (briefly, $ACC$), if each increasing chain $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots$ in $P$ is finally constant. That is, there exists $k \in \mathbb{N}$ such that $x_k = x_{k+1} = \cdots$. We say that $P$ satisfies the *descending chain condition* (briefly, $DCC$), if each decreasing chain $x_1 \geq x_2 \geq \cdots \geq x_n \geq \cdots$ in $P$ is finally constant. And we say that $P$ is of *finite chain condition* (briefly, $FCC$), if it satisfies both $ACC$ and $DCC$. It should be noted that, in the case when $P$ satisfies $ACC$ (resp. $DCC$), the set $M$ (resp. $m$) is nonempty set.

Alexandroff spaces were first studied in 1937 by P. Alexandroff [1]. It is a topological space in which arbitrary intersection of open sets is open. Equivalently, each singleton has a minimal neighborhood base. So, any discrete space is Alexandroff, and any finite space is also Alexandroff.

For each $T_0$–Alexandroff space $(X, \tau)$, there is a corresponding poset $(X, \leq_{\tau})$, in one to one and onto way, where each one of them is completely determined by the other. Given a poset $(P, \leq)$, then $\mathbb{B} = \{ \uparrow x : x \in X \}$ is a base for a topology on $P$. This topology, denoted by $\tau _{<}$, is $T_0$–Alexandroff topology. On the other hand, if $(X, \tau)$ is Alexandroff space, we define the a pre-order $\leq_{\tau}$, called *(Alexandroff) specialization pre-order*, by $a \leq_{\tau} b$ if and only if $a \in \{ b \}$ or equivalently $b \in U$ whenever $U$
is a neighborhood of \( a \). This pre-order is a partial order if and only if \( X \) is \( T_0 \). Moreover, if \((X, \leq)\) is a poset and \( \tau(\leq) \) its induced \( T_0 \)-Alexandroff topology, then the specialization order of \( \tau(\leq) \) is the order \( \leq \) itself; that is, \( \leq_{\tau(\leq)} = \leq \). If \((X, \tau)\) is a \( T_0 \)-Alexandroff space with specialization order \( \leq_{\tau} \), then the induced topology by the specialization order is the topology \( \tau \) itself; that is, \( \tau(\leq_{\tau}) = \tau \).

For a \( T_0 \)-Alexandroff space \((X, \tau)\) with corresponding poset \((X, \leq)\), and for \( x \in X \), the collection consisting of one set \( \{\uparrow x\} \) is the minimal neighborhood base for \( x \) and sometimes denoted by \( \{V(x)\} \). Moreover, \( \{x\} = \downarrow x \), and \( A \) is open (closed) set in \( X \) iff \( A \) is up (down) set in the corresponding poset.

Mathematically, Alexandroff spaces have important role in the study of the structure of the lattice of topologies. Alexandroff spaces are used in domain theory in computer science, applied mathematics, and as a consequence of the very important role of finite spaces in digital topology.

In [4], we introduce a class of Alexandroff spaces called Artinian \( T_0 \)-Alexandroff spaces. A space \( X \) belongs to this class if its corresponding poset satisfies the ACC. Some of basic concepts and ideas was studied, We got strong results for Artinian \( T_0 \)-Alexandroff spaces.

In this parer, we introduce a new class of Alexandroff spaces called \( g \)-Artinian \( T_0 \)-Alexandroff spaces. This class of spaces strictly contains the class of Artinian \( T_0 \)-Alexandroff space. We prove that a space \( X \) is \( g \)-Artinian \( T_0 \)-Alexandroff space iff the Scott topology \( \sigma(X) \) induced from the corresponding poset coincides with the original Alexandroff topology on \( X \).

Throughout this paper, \((X, \tau(\leq))\) denotes a \( T_0 \)-Alexandroff space with corresponding poset \((X, \leq)\). For each element \( x \in X \), \( \uparrow x \) or \( V(x) \) denotes the minimal neighborhood. The notation := means equal by definition.

2 Preliminaries and Definitions

Definition 2.1. [4] Let \((X, \tau(\leq))\) be a \( T_0 \)-Alexandroff Space. If the corresponding poset \((X, \leq)\) satisfies

(i) ACC, then \( X \) is called Artinian \( T_0 \)-Alexandroff space.
(ii) DCC, then $X$ is called Noetherian $T_0$—Alexandroff space.

(iii) FCC, then $X$ is called generalized locally finite space ($g$-locally finite space).

Recall that a topological space $(X, \tau)$ is called locally finite if each element $x$ of $X$ is contained in a finite open set and a finite closed set. Note that for a space $X$, the following implications hold:

$$T_0 \text{— finite} \Rightarrow T_0 \text{— locally finite} \Rightarrow g\text{-locally finite} \Rightarrow \text{both Artinian and Noetherian } T_0\text{— Alexandroff spaces},$$

and the converse is not always true.

A subset $U$ of a poset $P$ is Scott open\cite{5} if $U$ is an up set, and for any directed set $S$ with supremum, if $\bigvee S \in U$, then there exists $s_o \in S$ such that $s_o \in U$; that is, $S \cap U \neq \emptyset$.

The collection $\sigma(P)$ of all Scott open sets forms a topology on $P$ called Scott topology. A subset $F$ of a poset $P$ is called Scott closed if its complement is Scott open. So $F$ is Scott closed if $F = \downarrow F$ ($F$ is a down set), and if $U$ is a directed set contained in $F$ and $\bigvee U$ exists, then $\bigvee U \in F$.

The Scott topology on a poset is $T_0$. Moreover a subspace of a Scott space is a Scott space.

Needless to say that, on any poset $P$, every Scott open is Alexandroff open. So the Scott topology is coarser than the Alexandroff topology; $\sigma(P) \subseteq \tau(\leq)$. The converse is not always true. For example, the right ray topology on $\mathbb{R}$ is the Scott topology induced by the usual order. For $x \in \mathbb{R}$ the set $U = [x, \infty) = \uparrow x$ is Alexandroff open which is not Scott open.

\textbf{Lemma 2.2.} \cite{7} If $P$ is a poset, then the set $U_x = \{z \in P : z \nleq x\} = P - \downarrow x$ is a Scott open set. Equivalently, $\forall x \in P$, $\downarrow x$ is Scott closed.

\section{New Class of posets}

Let $(P, \leq)$ be a poset. Define $P^{dir}$ to be the collection of all directed subsets of $P$, $P^{\text{di}}$ the collection of all directed subsets of $P$ with supremum, and $P^{d}$ the collection of all subsets of $P$ with maximum element.
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Since each set in $P$ with maximum is directed, we have the following implications: $P^d \subseteq P^{di} \subseteq P^{dir}$. The converse is not true. In fact, one of the following cases holds for a given poset:

**Case 1.** $P^{dir} \neq P^{di}$ and $P^{di} \neq P^d$. For example, in the real numbers $\mathbb{R}$ with usual order, the set $A = (0, \infty) \in P^{dir}$ which is not in $P^{di}$. And the set $B = (0, 1) \in P^{di}$ which is not in $P^d$.

**Case 2.** $P^{dir} = P^{di}$ while $P^{di} \neq P^d$. For example, in the set $[0, 1]$ with usual order, any subset of $[0, 1]$ is bounded by 1, so $P^{dir} = P^{di}$. But the set $B = (0, 1) \in P^{di}$ which is not in $P^d$.

**Case 3.** $P^{dir} \neq P^{di}$ while $P^{di} = P^d$. For example, in the set of natural numbers $\mathbb{N}$ with its usual order, the set of odd numbers is directed set without supremum. If $A$ is a set with supremum $\bigvee A$, then $\bigvee A \in A$, so $P^{di} = P^d$.

**Case 4.** $P^{dir} = P^{di} = P^d$. For example, take any finite poset.

**Definition 3.1.** [6] Let $(P, \leq)$ be a poset. We say that $P$ is

1. directed-complete (briefly dcpo) if $P^{dir} = P^{di}$; that is, every directed subset of $P$ has a supremum.

2. complete partial order (briefly cpo) if $P$ is a dcpo with a least element.

**Definition 3.2.** A poset $(P, \leq)$ is called generalized ascending chain condition (briefly g-ACC) if $P^{di} = P^d$; that is, each directed set with supremum has a maximum.

**Proposition 3.3.** A poset $(P, \leq)$ satisfies ACC if $P$ is g−ACC and dcpo.

**Proof.** ($\Rightarrow$) Let $U$ be a directed subset of $P$. Since $P$ satisfies ACC, $M(U) \neq \emptyset$. If $x, y \in M(U) \subseteq U$, then by directed property of $U$, $\exists z \in U$ such that $x \leq z$ and $y \leq z$. By maximality of $x, y$, we get that $x = y = z$, and hence $|M(U)| = 1$. Thus the supremum of $U$ exists in $U$, and this implies that $P^{dir} = P^{di} = P^d$.

($\Leftarrow$) Let $x_1 \leq x_2 \leq x_3 \leq \cdots$ be an increasing chain in $P$. Set $U = \{x_1, x_2, x_3, \cdots\}$, so $U$ is directed set, and so $\bigvee U$ exists in $U$; say $x_k = \bigvee U$. Now if $j \geq k$, we get that $x_k \leq x_j \leq \bigvee U = x_k$. Thus $x_k = x_{k+1} = x_{k+2} = \cdots$. $\square$
Definition 3.4. Let \((P, \leq)\) be a poset. Then

1. an element \(a \in P\) is called compact, if for any \(U \in P^d\), \(a \leq \bigvee U\) implies that there exists \(u \in U\) such that \(a \leq u\). That is, every directed set with join above \(a\) has a member above \(a\). So \(\uparrow a \cap U \neq \emptyset\). The set of all compact elements of \(P\) will be denoted by \(K_P\).

2. Let \(x, y \in P\). We say \(x\) approximates \(y\) [5], and we write \(x \ll y\), if for any \(U \in P^d\), \(y \leq \bigvee U\) implies that there exists \(u \in U\) such that \(x \leq u\). That is, every directed set with join above \(y\) has a member above \(x\). Some authors (see [2]) prefer the term "is way-below" rather than "approximates".

3. \(P\) is called algebraic, if \(\forall x \in P\), the set \(\downarrow_K x := \{ a \in K_P : a \leq x \}\) is a directed set with supremum and \(x = \bigvee(\downarrow_K x)\). Algebraic dcpo is referred to be domain[3].

Proposition 3.5. [6] Let \(P\) be a poset, and let \(K_P\) denote the set of all compact elements in \(P\). Then, for any \(a \in K_P\), \(\uparrow a = \{ x : a \leq x \}\) is Scott open.

We define the set \(\uparrow x := \{ y : x \ll y \}\) and \(\downarrow x := \{ y : y \ll x \}\). It is well known that if \(x \ll y\), then \(x \leq y\), and the converse is not always true. The next proposition provides necessary and sufficient conditions where the converse is true.

Proposition 3.6. Let \((P, \leq)\) be a poset. Then the following are equivalent:

a. \(\forall x, y \in P\), \(x \ll y\) iff \(x \leq y\) (\(\uparrow x = \uparrow x\)).

b. \(P\) is \(g - ACC\).

c. \(K_P = P\).

Proof. (a \(\Rightarrow\) b) Suppose that \(U\) is a directed set with supremum \(\bigvee U\) exists in \(P\). Since \(\forall x \in P\), \(x \leq x\) implies \(x \ll x\), we get \(\bigvee U \ll \bigvee U\). So \(\exists u \in U\) such that \(\bigvee U \leq u\). Therefore \(\bigvee U \in U\).

(b \(\Rightarrow\) c) Let \(a \in P\), and suppose that \(U \in P^d\) such that \(a \leq \bigvee U\). Since \(\bigvee U \in U\), we have that \(\uparrow a \cap U \neq \emptyset\). Hence \(a\) is compact.
(c ⇒ a) Suppose that \( x \leq y \) in \( P \), and let \( U \) be a directed set with supremum \( \bigvee U \) such that \( y \leq \bigvee U \). Now \( x \) is compact and \( x \leq \bigvee U \), so there exists \( u \in U \) such that \( x \leq u \). Thus \( x \ll y \).

**Corollary 3.7.** If a poset \( P \) is \( g - \text{ACC} \), then \( P \) is algebraic.

**Proof.** Direct, since \( (\downarrow_n x) = \downarrow x \). □

4 Generalized Artinian \( T_0 \)-Alexandroff Spaces

If \( P \) is a finite poset, then \( \sigma(P) = \tau(\leq) \). In general, this is not always true. In the next theorem, we give a necessary and sufficient condition where the equality between the two topologies holds. Then we introduce a new class of Alexandroff spaces containing the class of Artinian \( T_0 \)-Alexandroff spaces. We call it the class of \( g \)-Artinian \( T_0 \)-Alexandroff spaces.

**Theorem 4.1.** Let \( (P, \leq) \) be a poset. Then \( \sigma(P) = \tau(\leq) \) iff \( P \) is \( g - \text{ACC} \).

**Proof.** (⇒) Let \( U \) be a directed set with supremum \( \bigvee U \), and let \( O = \uparrow (\bigvee U) \). \( O \) is Alexandroff open set, so it is Scott open. Moreover \( \bigvee U \in O \), so \( U \cap O \neq \emptyset \). This implies that \( U \cap O = \{ \bigvee U \} \). Hence \( \bigvee U \in U \).

(⇐) Obvious, since each directed set \( U \in P^\times \) with supremum \( \bigvee U \), we have \( \bigvee U \in U \). □

**Definition 4.2.** A \( T_0 \)-Alexandroff space \( (X, \tau(\leq)) \) is called \( g \)-Artinian \( T_0 \)-Alexandroff space if the corresponding poset \( (X, \leq) \) is \( g - \text{ACC} \).

By the above theorem, \( X \) is \( g \)-Artinian \( T_0 \)-Alexandroff space iff the Scott topology \( \sigma(X) \) coincides with the Alexandroff topology \( \tau(\leq) \).

**Corollary 4.3.** If \( X \) is an Artinian \( T_0 \)-Alexandroff space, then \( X \) is a \( g \)-Artinian \( T_0 \)-Alexandroff space.

The following theorem is a consequence of Proposition 3.6, Theorem 4.1, and Definition 4.2.

**Main Theorem 4.4.** Let \( (P, \leq) \) be a poset, \( \tau(\leq) \) the Alexandroff topology, and \( \sigma(P) \) the Scott topology on \( P \). Then all the following are equivalent:
1. The poset $P$ is $g - ACC$.
2. $\forall x \in P$, $\uparrow x = \uparrow x$.
4. $\tau(\leq) = \sigma(P)$.
5. $(P, \tau(\leq))$ is a $g$-Artinian $T_0$-Alexandroff space.

**Proposition 4.5.** Let $(X, \tau(\leq))$ be a $T_0$-Alexandroff space with corresponding poset $(X, \leq)$, $U \subseteq X$, and for $x \in X$, $V(x)$ the minimal neighborhood of $x$. Then

1. $U \in X^{dir}$ in the sense of poset iff in the sense of topology, and $\forall x, y \in U$, $\exists z \in U$ such that $\{x, y\} \subseteq z$.
2. $U \in X^{di}$ iff $U$ is directed and $\exists z \in X$ such that $V(z) = \bigcap_{x \in U} V(x)$.
3. $U \in X^{d}$ iff $\exists z \in U$ such that $U \subseteq z$.

**Proof.** We will use the fact that $x \leq z$ iff $x \in z$ iff $z \in V(x)$ iff $V(z) \subseteq V(x)$. 1) $(\Leftrightarrow)$ $U$ is directed iff $\forall x, y \in U$, $\exists z \in U$ such that $x \leq z$ and $y \leq z$ iff $\exists z \in U$ such that $\{x, y\} \subseteq z$.

2) $(\Rightarrow)$ Suppose that $U$ is a directed set with supremum, say $z = \bigvee U$. Then for any $x \in U$, $x \leq z$, and hence $V(z) \subseteq V(x)$. Thus $V(z) \subseteq \bigcap_{x \in U} V(x)$. On the other hand, if $y \in \bigcap_{x \in U} V(x)$, then $x \leq y \forall x \in U$. Thus $z \leq y$, and hence $y \in V(z)$.

$(\Leftarrow)$ If $V(z) = \bigcap_{x \in U} V(x)$, then $z$ is an upper bound for $U$. Moreover, if $y$ is an upper bound of $U$, then $y \in V(x), \forall x \in U$. So $y \in \bigcap_{x \in U} V(x) = V(z)$. Thus $z \leq y$.

3) $(\Rightarrow)$ Suppose $z = \max U$, so $\forall x \in U$, $x \leq z$. Therefore $x \in z$.

$(\Leftarrow)$ If $\exists z \in U$ such that $x \in z \forall x \in U$, then $z$ is an upper bound of $U$. Now $U$ contains one of its upper bound, so $\max U$ exists and $\max U = z$. $\Box$
The following proposition is a direct way to test whether a \( T_0 \)-Alexandroff space being a g-Artinian.

**Proposition 4.6.** Let \( (X, \tau(\leq)) \) be a \( T_0 \)-Alexandroff space. Then \( X \) is g-Artinian \( T_0 \)-Alexandroff space iff whenever a subset \( U \) of \( X \) satisfies the two conditions:

1. \( \forall x, y \in U, \exists w \in U \text{ such that } \{x, y\} \subseteq w \), and
2. \( \exists z \in X \text{ such that } V(z) = \bigcap_{x \in U} V(x) \),

then \( z \in U \).

**Proof.** (\( \Rightarrow \)) The corresponding poset \( (X, \leq) \) is \( g-ACC \), so \( X^{d_{\uparrow}} = X^{d_{\downarrow}} \).

Now if \( U \) satisfies the two conditions, then by Proposition 4.5(2), \( U \in X^{d_{\uparrow}} \) is directed with supremum \( \bigvee U = z \). Hence \( U \in X^{d_{\downarrow}} \) and max \( U = \bigvee U = z \).

(\( \Leftarrow \)) Let \( U \in X^{d_{\downarrow}} \), so by Proposition 4.5(2), \( z = \bigvee U \) satisfies \( V(z) = \bigcap_{x \in U} V(x) \). Thus \( U \) is a set in \( X \) satisfies the conditions (1) and (2) above, so by given \( z \in U \). Hence \( U \in X^{d_{\downarrow}} \).

\( \square \)

**References**


