Fuzzy expert systems

Main points:

Fuzzy logic is determined as a set of mathematical principles for knowledge representation based on degrees of membership rather than on crisp membership of classical binary logic.

Fuzzy sets

The basic idea of the fuzzy set theory is that an element belongs to a fuzzy set with a certain degree of membership as a real number in the interval [0,1]

\[
\mu_A(x) = \begin{cases} 1 & \text{if } x \text{ is totally in } A \\ 0 & \text{if } x \text{ is not in } A \\ 0 < \mu_A(x) < 1 & \text{if } x \text{ is partly in } A 
\end{cases}
\]

horizontal axis represents the universe of discourse: the range of all possible values applicable to a chosen variable.

\[\mu_A(x) : X \rightarrow [0,1]\] membership function of set A

where:

Linguistic variables:

Linguistic variables: speed

Linguistic values: slow, medium, fast
Hedges:

hedges terms that modify the shape of fuzzy sets. They include adverbs such as very, somewhat, more or less and slightly.

<table>
<thead>
<tr>
<th>Hedge</th>
<th>Mathematical expression</th>
<th>Graphical representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>A little</td>
<td>$[\mu_A(x)]^{1.3}$</td>
<td><img src="image1.png" alt="Graph" /></td>
</tr>
<tr>
<td>Slightly</td>
<td>$[\mu_A(x)]^{1.7}$</td>
<td><img src="image2.png" alt="Graph" /></td>
</tr>
<tr>
<td>Very</td>
<td>$[\mu_A(x)]^2$</td>
<td><img src="image3.png" alt="Graph" /></td>
</tr>
<tr>
<td>Extremely</td>
<td>$[\mu_A(x)]^3$</td>
<td><img src="image4.png" alt="Graph" /></td>
</tr>
<tr>
<td>Very very</td>
<td>$[\mu_A(x)]^4$</td>
<td><img src="image5.png" alt="Graph" /></td>
</tr>
<tr>
<td>More or less</td>
<td>$\sqrt{\mu_A(x)}$</td>
<td><img src="image6.png" alt="Graph" /></td>
</tr>
<tr>
<td>Somewhat</td>
<td>$\sqrt{\mu_A(x)}$</td>
<td><img src="image7.png" alt="Graph" /></td>
</tr>
</tbody>
</table>
| Indeed      | $2[\mu_A(x)]^2$ if $0 \leq \mu_A \leq 0.5$
               | $1 - 2[1 - \mu_A(x)]^2$ if $0.5 < \mu_A \leq 1$ | ![Graph](image8.png)     |

*Representation of hedges in fuzzy logic*
Operations of fuzzy sets:

<table>
<thead>
<tr>
<th>Operation</th>
<th>Formula</th>
</tr>
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<tbody>
<tr>
<td>Complement</td>
<td>$\mu_{\bar{A}}(x) = 1 - \mu_A(x)$</td>
</tr>
<tr>
<td>Containment</td>
<td>a set can contain other sets</td>
</tr>
<tr>
<td>Intersection</td>
<td>$\mu_{A \cap B}(x) = \min[\mu_A(x), \mu_B(x)] = \mu_A(x) \cap \mu_B(x)$ where $x \in X$</td>
</tr>
<tr>
<td>Union</td>
<td>$\mu_{A \cup B}(x) = \max[\mu_A(x), \mu_B(x)] = \mu_A(x) \cup \mu_B(x)$ where $x \in X$</td>
</tr>
</tbody>
</table>

Fuzzy rules:

**If** Linguistic variables is Linguistic value

**Then** result

Evaluating “degree of membership” the rule antecedent (IF part) THEN apply the result to the consequent (THEN part of the rule).

* If the antecedent is true to some degree of membership, then the consequent is also true to that same degree.

* If the consequent of a fuzzy rule have multiple parts, in this case, all parts of the consequent are affected equally by the antecedent.

Fuzzy inference:

It's a process of mapping from a given input to an output, we need to obtain a single crisp solution for the output variable using one of the following methods.

inference methods:

1. Mamdani-style:
2. Sugeno-style:

Mamdani

The Mamdani-style fuzzy inference process is performed in four steps:

1) Fuzzification of the input variables
2) Rule evaluation
3) Aggregation of the rule outputs
4) Defuzzification.
**Fuzzification:**

From the crisp inputs (by the expert), $x_1$ and $y_1$ we determine the degree to which these inputs belong to each of the appropriate fuzzy sets.

*خلالها بحدد درجة انتماء عناصر المجموعة عن طريق تقاطع الخط العمودي مع الدالة الانتهاء*.

**Rule evaluation:**

If there multiple antecedents we need to obtain a single number that (the truth value) represents the result of the antecedent evaluation, then applied to the consequent membership function.

**Aggregation of the rule outputs:**

we take the membership functions of all rule consequents previously clipped and combine them into a single fuzzy set.

- If there a multiple output variable then the output is **one fuzzy set for each output variable**.

**Defuzzification:**

Find the final output of a fuzzy system that will be a crisp number. **We Use Centroid technique to** finds the point where a vertical line would slice the aggregate set into two equal masses.

Mathematically this centre of gravity (COG):

$$COG = \frac{\sum_{x=a}^{b} \mu_A(x) x}{\sum_{x=a}^{b} \mu_A(x)}$$

**Sugeno-style inference:**

it's the same mamdani process but it changed only a rule consequent. Instead of a fuzzy set, he used a mathematical function of the input variable that rule consequents are singletons that is unity at a single particular $k$ point on the universe of discourse and zero everywhere else

$$IF \quad x \text{ is } A$$
$$AND \quad y \text{ is } B$$
$$THEN \quad z \text{ is } k$$

To find the weighted average (WA) of these singletons

$$WA = \frac{\mu(k1) \times k1 + \mu(k2) \times k2 + \mu(k3) \times k3}{\mu(k1) + \mu(k2) + \mu(k3)}$$
Questions:

Q 1) Determine the truth value of the following propositions $P_1$ and $P_2$.

$P_1 = \text{“P is very true”}$,  $P_2 = \text{“P is false”}$,

where $P = \text{“30 is high”}$, the truth value of $P$ is 0.3,

Solution:

$$\mu_{\text{very true}} = (\mu_{\text{true}})^2$$

$P_1=0.09$  \hspace{1cm} $P_2=0.7$

Q 2) For a given two fuzzy sets:  $A = ( 0/0 , 0.5/20 , 0.65/40 , 0.85/60 , 1.0 /80 , 1.0 /100 )$

$B = ( 0/0 , 0.45/20 , 0.6/40 , 0.8/60 , 0.95/80 , 1.0/100 )$

Find the following membership functions using standard fuzzy operations:

a)  $\mu_{A \cup B} (x)$

b)  $\mu_{A \cap B} (x)$

c)  $\mu_{\overline{A}} (x)$

d)  $\mu_{\overline{B}} (x)$

e)  $\mu_{\overline{A} \cup A} (x)$

f)  $\mu_{\overline{A} \cap A} (x)$

g)  $\mu_{\overline{A} \cap \overline{B}} (x)$

Solution:

a)  $\mu_{A \cup B} (x)= (0/0 , 0.5/20 , 0.65/40 , 0.85/60 , 1/80 , 1/100 )$

b)  $\mu_{A \cap B} (x)= (0/0 , 0.45/20 , 0.6/40 , 0.8/60 , 0.95/80 , 1/100 )$

c)  $\mu_{\overline{A}} (x)= (1/0 , 0.95/20 , 0.35/40 , 0.15/60 , 0/80 , 0/100 )$

d)  $\mu_{\overline{B}} (x)= (1/0 , 0.55/20 , 0.4/40 , 0.2/60 , 0.05/80 , 0/100 )$

e)  $\mu_{\overline{A} \cup A} (x)= (1/0 , 0.5/20 , 0.65/40 , 0.85/60 , 1/80 , 1/100 )$

f)  $\mu_{\overline{A} \cap A} (x)= (0/0 , 0.5/20 , 0.35/40 , 0.15/60 , 0/80 , 0/100 )$

g)  $\mu_{\overline{A} \cap \overline{B}} (x)= (1.0/0 , 0.55/20 , 0.4/40 , 0.2/60 , 0.5/80 , 1.0/100 )$
Q 3) Suppose we define membership for linguistic variables “heavy” and “light” as follows:

Heavy = (0.2/5, 0.4/8, 0.6/12, 0.8/20, 1/30)
Light = (1/5, 0.8/8, 0.5/12, 0.1/20, 0/30)

Develop membership functions for the following linguistic phrases:

a) Very heavy.
b) A little heavy
c) Not very Light
d) Not very heavy and not very light

Solution:

Very heavy = (0.04/5, 0.16/8, 0.36/12, 0.64/20, 1/30)
A little heavy = (0.12/5, 0.3/8, 0.51/12, 0.75/20, 1/30)
Very Light = (1/5, 0.64/8, 0.25/12, 0.01/20, 0/30)
Not very Light = (0/5, 0.36/8, 0.75/12, 0.99/20, 1/30)
Not very heavy and not very light = [1 - μ heavy (x)] ∩ [1 - μ light (x)]
= (0/5, 0.36/8, 0.64/12, 0.36/20, 0/30)
Q 4) Consider the following fuzzy expert system for weather forecast:

R1: IF arrow position is down
    THEN clouds (cf = 0.8)

R2: IF arrow position is in the middle and moving down
    THEN clouds (cf = 0.6)

R3: IF arrow position is in the middle and moving up
    THEN sunny (cf = 0.6)

R4: IF arrow position is up
    THEN sunny (cf = 0.8)

The following two plots represent the membership functions of two fuzzy variables describing the position of the arrow of barometer (left) and the direction of its movement (right):

The air pressure is measured in millibars, and the speed of its change in millibars per hour. When a given input pressure is 1020 millibars and the pressure changes –2 millibars every hour. Answer the following questions.

a) How much is the arrow position Down, Up or in the Middle?

\[ \mu_{\text{up}}(p) = 0.5 \quad , \quad \mu_{\text{down}}(p) = 0.0 \quad , \quad \mu_{\text{middle}}(p) = 0.25 \]
b) How much is the arrow moving Down or Up?

\[ \mu_{\text{up}}(m) = 0.0, \quad \mu_{\text{down}}(m) = 0.75 \]

c) Using the membership values found above and confidences of the rules in the table calculate the degree of confidence in that the sky is clear or cloudy?

 Degrees of membership of the arrow position and movement (Fuzzification)
\[
\begin{align*}
\mu (P=\text{arrow position is down}) &= 0 \\
\mu (P=\text{arrow position is in the middle}) &= 0.25 \\
\mu (P=\text{arrow position is up}) &= 0.5 \\
\mu (M=\text{arrow is moving down}) &= 0.75 \\
\mu (M=\text{arrow is moving up}) &= 0
\end{align*}
\]

First, we find the combined membership values of condition parts of the rules (Rule evaluation):

**Rule 1**: IF arrow is down
\[ \mu (P=\text{arrow is down}) = \]

**Rule 2**: IF arrow is in the middle and moving down
\[ \mu (P=\text{arrow is in the middle} \text{ and } M=\text{moving down}) = \min[0.25, 0.75] = 0.25 \]

**Rule 3**: IF arrow is in the middle and moving up
\[ \mu (P=\text{arrow is in the middle} \text{ and } M=\text{moving up}) = \min[0.25, 0] = 0, \]

**Rule 4**: IF arrow is up
\[ \mu (P=\text{arrow is up}) = 0.5 \]
To calculate the degree of confidence in conclusion of each rule, we need to multiply the membership values of their conditions by the degrees of confidence (cf) of the rules itself:

\[
\mu(\text{result}) = \mu(\text{condition}) \times \text{cf(rule)}.
\]

Using the cf values for rules we can calculate:
- \(\mu_1(\text{w}=\text{clouds}) = 0 \times 0.8 = 0\)
- \(\mu_2(\text{w}=\text{clouds}) = 0.25 \times 0.6 = 0.15\)
- \(\mu_1(\text{w}=\text{sunny}) = 0 \times 0.6 = 0\)
- \(\mu_2(\text{w}=\text{sunny}) = 0.5 \times 0.8 = 0.4\)

Because we have two conclusions about each type of weather, we need to use the following formula for combining memberships of two conclusions:

\[
\mu(x) = \mu_1(x) + \mu_2(x) - \mu_1(x) \times \mu_2(x) \quad \text{where } \mu_1(x) \text{ and } \mu_2(x) \geq 0.
\]

So, the results are:
- \(\mu(\text{W}=\text{clouds}) = 0 + 0.15 - 0 \times 0.15 = 0.15\)
- \(\mu(\text{W}=\text{sunny}) = 0 + 0.4 - 0 \times 0.4 = 0.4\)

**d)** The following plot describes sunny and clouds sets for the output variable weather forecast, find the final crisp output?

\[
\text{COG} = \frac{(0+10+20+30+40) \times 0.15 + (50 \times 0.35) + (60+70+80+90+100) \times 0.4}{5 \times 0.15 + 5 \times 0.4 + 0.35} = 71.3
\]

The crisp output is 71.3. It means, for instance, that the weather forecast is 71.3% according to crisp inputs.