Mechanical Vibrations

Lecture 12

CHAPTER 3

Harmonically Excited Vibration

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$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2}\right)\cos\omega_n t + \left(\frac{\dot{x}_0}{\omega_n}\right)\sin\omega_n t + \left(\frac{F_0}{k - m\omega^2}\right)\cos\omega t$$

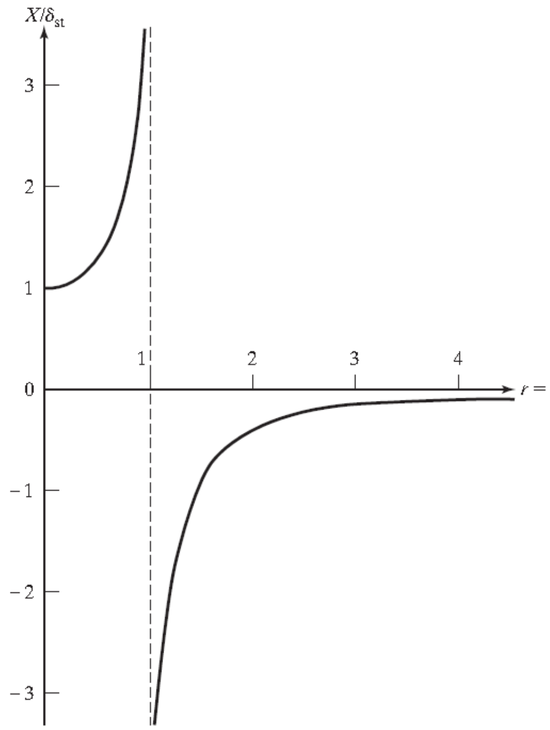
$$X = \frac{F_0}{k - m\omega^2} = \frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

The maximum amplitude X in Eq. (3.6) can be expressed as

$$\frac{X}{\delta_{st}} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

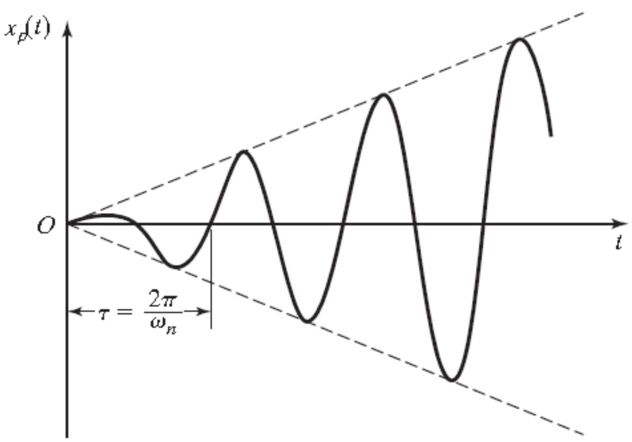
$$r = \omega / \omega_n$$

Case 1. When $0 < \omega/\omega_n < 1$



Case 2. When $\omega/\omega_n > 1$

Case 3. When $\omega/\omega_n = 1$,



$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \delta_{st} \left[\frac{\cos \omega t - \cos \omega_n t}{1 - \left(\frac{\omega}{\omega_n} \right)^2} \right]$$

$\omega / \omega_n = 1.$

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \frac{\delta_{st} \omega_n t}{2} \sin \omega_n t$$

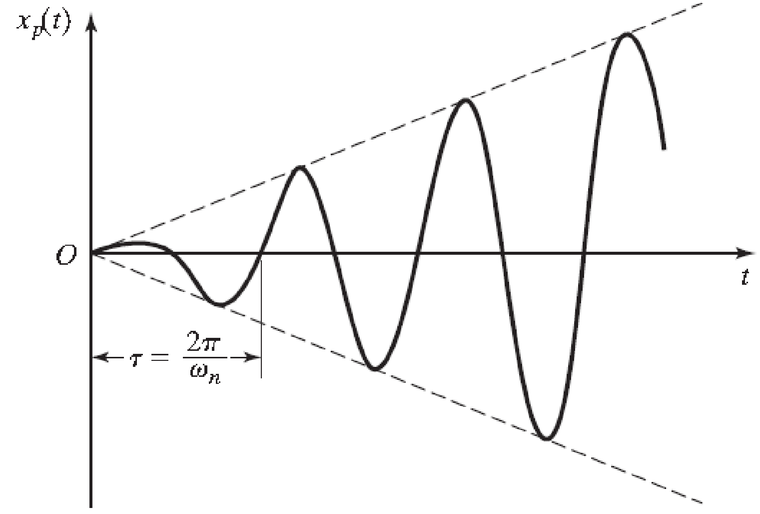


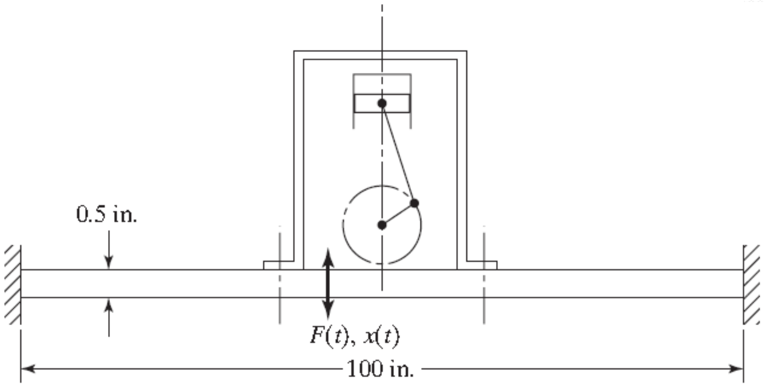
FIGURE 3.6 Response when $\omega / \omega_n = 1$.

$$x(t) = A \cos(\omega_n t - \phi) + \frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n} \right)^2} \cos \omega t; \quad \text{for } \frac{\omega}{\omega_n} < 1$$

$$x(t) = A \cos(\omega_n t - \phi) - \frac{\delta_{st}}{-1 + \left(\frac{\omega}{\omega_n} \right)^2} \cos \omega t; \quad \text{for } \frac{\omega}{\omega_n} > 1$$

EXAMPLE 3.1 Plate Supporting a Pump

A reciprocating pump, weighing 150 lb, is mounted at the middle of a steel plate of thickness 0.5 in., width 20 in., and length 100 in., clamped along two edges as shown in Fig. 3.9. During operation of the pump, the plate is subjected to a harmonic force, $F(t) = 50 \cos 62.832t$ lb. Find the amplitude of vibration of the plate.



Solution: The plate can be modeled as a fixed-fixed beam having Young's modulus $(E) = 30 \times 10^6$ psi, length $(l) = 100$ in., and area moment of inertia $(I) = \frac{1}{12}(20)(0.5)^3 = 0.2083$ in^4. The bending stiffness of the beam is given by

$$k = \frac{192EI}{l^3} = \frac{192(30 \times 10^6)(0.2083)}{(100)^3} = 1200.0 \text{ lb/in.} \tag{E.1}$$

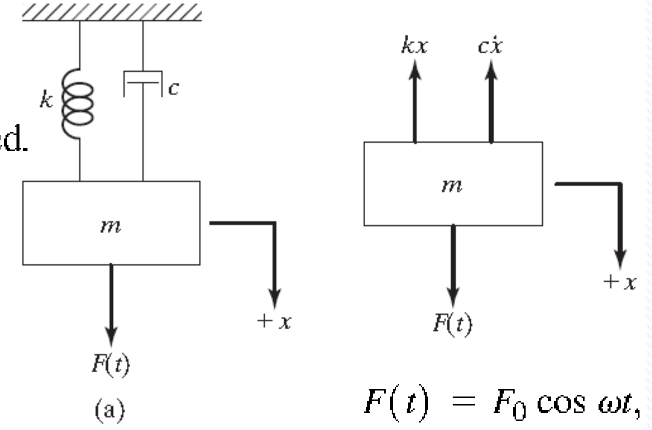
The amplitude of harmonic response is given by Eq. (3.6) with $F_0 = 50$ lb, $m = 150/386.4$ lb-sec^2/in. (neglecting the weight of the steel plate), $k = 1200.0$ lb/in., and $\omega = 62.832$ rad/s. Thus Eq. (3.6) gives

$$X = \frac{F_0}{k - m\omega^2} = \frac{50}{1200.0 - (150/386.4)(62.832)^2} = -0.1504 \text{ in.} \tag{E.2}$$

The negative sign indicates that the response $x(t)$ of the plate is out of phase with the excitation $F(t)$.

3.4 Response of a Damped System Under Harmonic Force

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t$$



$$F(t) = F_0 \cos \omega t,$$

$$x_p(t) = X \cos(\omega t - \phi)$$

where X and ϕ are constants to be determined.

$$X[(k - m\omega^2)\cos(\omega t - \phi) - c\omega \sin(\omega t - \phi)] = F_0 \cos \omega t$$

$$\cos(\omega t - \phi) = \cos \omega t \cos \phi + \sin \omega t \sin \phi$$

$$\sin(\omega t - \phi) = \sin \omega t \cos \phi - \cos \omega t \sin \phi$$

$$X[(k - m\omega^2)\cos \phi + c\omega \sin \phi] = F_0$$

$$X[(k - m\omega^2)\sin \phi - c\omega \cos \phi] = 0$$

$$X = \frac{F_0}{[(k - m\omega^2)^2 + c^2\omega^2]^{1/2}}$$

$$\phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right)$$

$$\omega_n = \sqrt{\frac{k}{m}} = \text{undamped natural frequency,}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{mk}}; \qquad \frac{c}{m} = 2\zeta\omega_n,$$

$$\delta_{st} = \frac{F_0}{k} = \text{deflection under the static force } F_0, \text{ and}$$

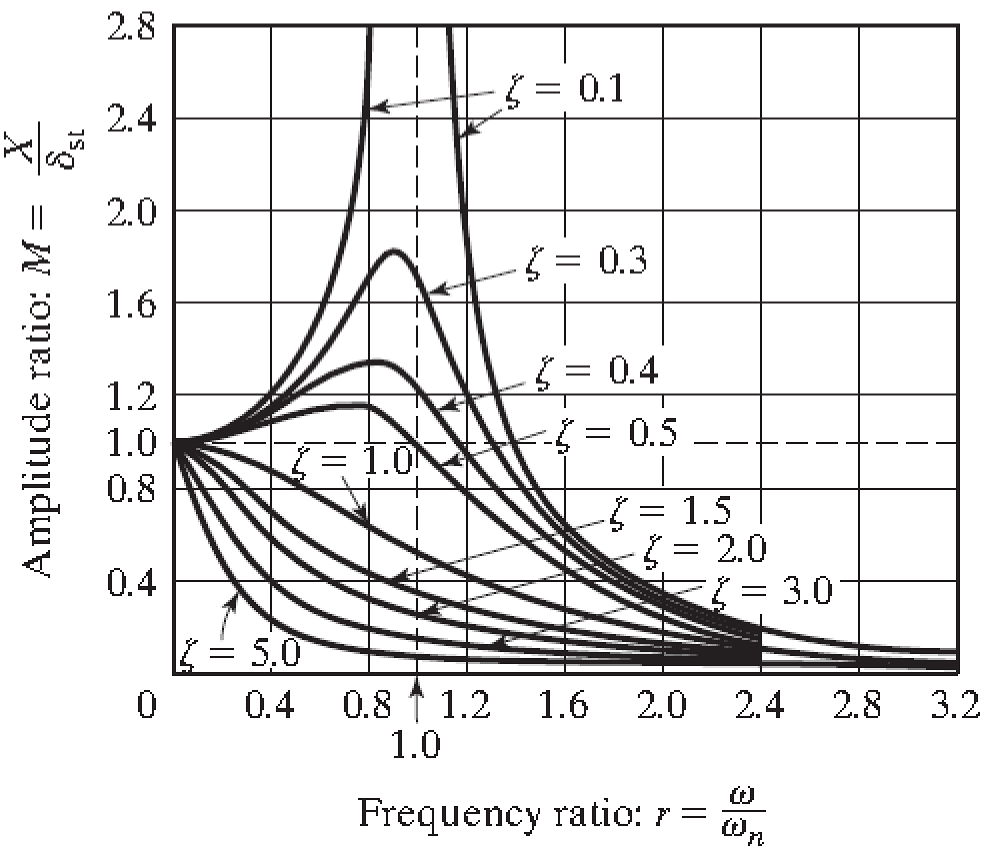
$$r = \frac{\omega}{\omega_n} = \text{frequency ratio}$$

$$\frac{X}{\delta_{st}} = \frac{1}{\left\{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2\right\}^{1/2}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$X = \frac{F_0}{[(k - m\omega^2)^2 + c^2\omega^2]^{1/2}}$$

$$\phi = \tan^{-1}\left\{\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right\} = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)$$

$$\phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right)$$



the maximum value of *M* occurs when $r = \sqrt{1 - 2\zeta^2}$ or $\omega = \omega_n\sqrt{1 - 2\zeta^2}$

The maximum value of *X* $\left(\frac{X}{\delta_{st}}\right)_{max} = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}$

and the value of *X* at $\omega = \omega_n$ by $\left(\frac{X}{\delta_{st}}\right)_{\omega=\omega_n} = \frac{1}{2\zeta}$

Total Response

$$x(t) = x_h(t) + x_p(t)$$

$$x(t) = X_0 e^{-\zeta\omega_n t}\cos(\omega_d t - \phi_0) + X\cos(\omega t - \phi)$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n$$

X and ϕ	X_0 and ϕ_0

$$x_0 = X_0\cos\phi_0 + X\cos\phi$$

$$\dot{x}_0 = -\zeta\omega_n X_0\cos\phi_0 + \omega_d X_0\sin\phi_0 + \omega X\sin\phi$$

$$\left.\begin{aligned} X_0 &= \left[(x_0 - X\cos\phi)^2 + \frac{1}{\omega_d^2}(\zeta\omega_n x_0 + \dot{x}_0 - \zeta\omega_n X\cos\phi - \omega X\sin\phi)^2\right]^{\frac{1}{2}} \\ \tan\phi_0 &= \frac{\zeta\omega_n x_0 + \dot{x}_0 - \zeta\omega_n X\cos\phi - \omega X\sin\phi}{\omega_d(x_0 - X\cos\phi)} \end{aligned}\right\} \quad (3.37)$$

EXAMPLE 3.3

Total Response of a System

Find the total response of a single-degree-of-freedom system with $m = 10$ kg, $c = 20$ N-s/m, $k = 4000$ N/m, $x_0 = 0.01$ m, and $\dot{x}_0 = 0$ under the following conditions:

a. An external force $F(t) = F_0 \cos \omega t$ acts on the system with $F_0 = 100$ N and $\omega = 10$ rad/s.
b. Free vibration with $F(t) = 0$.

Solution:

a. From the given data, we obtain

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{10}} = 20 \text{ rad/s}$$

$$\delta_{st} = \frac{F_0}{k} = \frac{100}{4000} = 0.025 \text{ m}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{20}{2\sqrt{(4000)(10)}} = 0.05$$

$$\omega_d = \sqrt{1 - \zeta^2}\,\omega_n = \sqrt{1 - (0.05)^2}(20) = 19.974984 \text{ rad/s}$$

$$r = \frac{\omega}{\omega_n} = \frac{10}{20} = 0.5$$

$$X = \frac{\delta_{st}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} = \frac{0.025}{\left[(1 - 0.05^2)^2 + (2 \cdot 0.5 \cdot 0.5)^2\right]^{1/2}} = 0.03326 \text{ m} \qquad \text{(E.1)}$$

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right) = \tan^{-1}\left(\frac{2 \cdot 0.05 \cdot 0.5}{1 - 0.5^2}\right) = 3.814075° \qquad \text{(E.2)}$$

Using the initial conditions, $x_0 = 0.01$ and $\dot{x}_0 = 0$, Eq. (3.36) yields:

$$0.01 = X_0 \cos\phi_0 + (0.03326)(0.997785)$$

$$X_0 \cos\phi_0 = -0.023186 \quad \text{(E.3)}$$

$$0 = -(0.05)(20)\, X_0 \cos\phi_0 + X_0 (19.974984) \sin\phi_0 + (0.03326)(10) \sin(3.814075°) \quad \text{(E.4)}$$

Substituting the value of $X_0 \cos\phi_0$ from Eq. (E.3) into (E.4), we obtain

$$X_0 \sin\phi_0 = -0.002268 \quad \text{(E.5)}$$

Solution of Eqs. (E.3) and (E.5) yields

$$X_0 = \left[(X_0 \cos\phi_0)^2 + (X_0 \sin\phi_0)^2 \right]^{1/2} = 0.023297 \quad \text{(E.6)}$$

and

$$\tan\phi_0 = \frac{X_0 \sin\phi_0}{X_0 \cos\phi_0} = 0.0978176$$

or

$$\phi_0 = 5.586765° \quad \text{(E.7)}$$

b. For free vibration, the total response is given by

$$x(t) = X_0 e^{-\zeta \omega_n t} \cos(\omega_d t - \phi_0) \tag{E.8}$$

Using the initial conditions $x(0) = x_0 = 0.01$ and $\dot{x}(0) = \dot{x}_0 = 0$, X_0 and ϕ_0 of Eq. (E.8) can be determined as (see Eqs. 2.73 and 2.75):

$$X_0 = \left[x_0^2 + \left(\frac{\zeta \omega_n x_0}{\omega_d} \right)^2 \right]^{1/2} = \left[0.01^2 + \left(\frac{0.05 \cdot 20 \cdot 0.01}{19.974984} \right)^2 \right]^{1/2} = 0.010012 \tag{E.9}$$

$$\phi_0 = \tan^{-1} \left(- \frac{\dot{x}_0 + \zeta \omega_n x_0}{\omega_d \, x_0} \right) = \tan^{-1} \left(- \frac{0.05 \cdot 20}{19.974984} \right) = -2.865984° \tag{E.10}$$

Note that the constants X_0 and ϕ_0 in cases (a) and (b) are very different.