Mechanical Vibrations

Lecture 13

C H A P T E R 3

Harmonically Damped Excited Vibration

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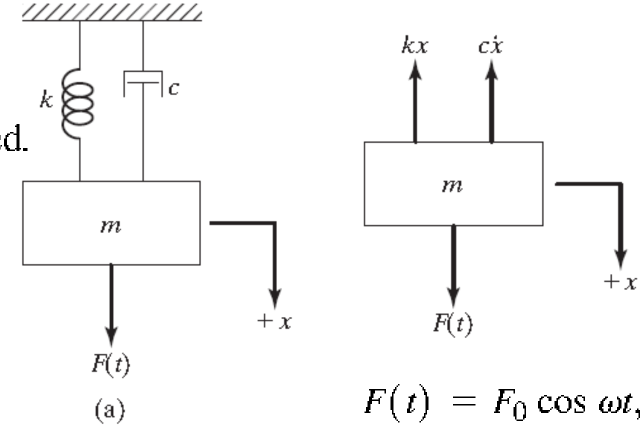
3.4 Response of a Damped System Under Harmonic Force

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t$$

$$x_p(t) = X \cos(\omega t - \phi)$$

where X and ϕ are constants to be determined.



$$F(t) = F_0 \cos \omega t,$$

$$X[(k - m\omega^2) \cos(\omega t - \phi) - c\omega \sin(\omega t - \phi)] = F_0 \cos \omega t$$

$$\cos(\omega t - \phi) = \cos \omega t \cos \phi + \sin \omega t \sin \phi$$

$$\sin(\omega t - \phi) = \sin \omega t \cos \phi - \cos \omega t \sin \phi$$

$$X[(k - m\omega^2) \cos \phi + c\omega \sin \phi] = F_0$$

$$X[(k - m\omega^2) \sin \phi - c\omega \cos \phi] = 0$$

$$X = \frac{F_0}{[(k - m\omega^2)^2 + c^2\omega^2]^{1/2}}$$

$$\phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right)$$

$$\omega_n = \sqrt{\frac{k}{m}} = \text{undamped natural frequency,}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{mk}}; \qquad \frac{c}{m} = 2\zeta\omega_n,$$

$$\delta_{st} = \frac{F_0}{k} = \text{deflection under the static force } F_0, \text{ and}$$

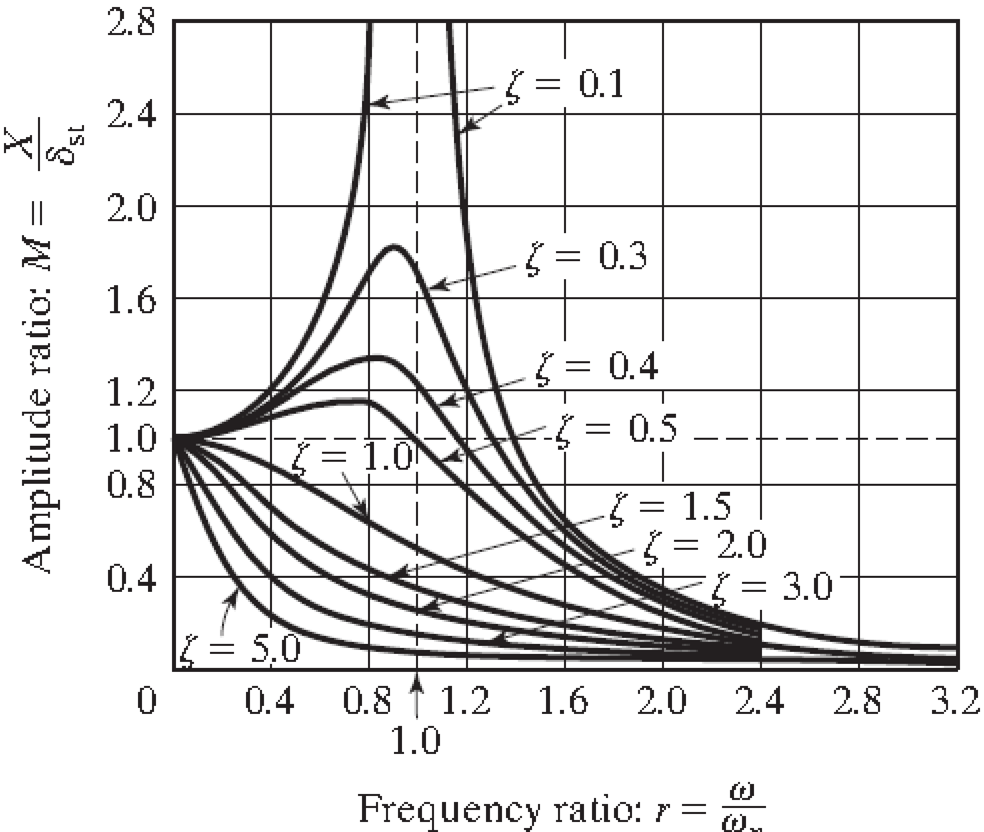
$$r = \frac{\omega}{\omega_n} = \text{frequency ratio}$$

$$X = \frac{F_0}{[(k - m\omega^2)^2 + c^2\omega^2]^{1/2}}$$

$$\phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right)$$

$$\frac{X}{\delta_{st}} = \frac{1}{\left\{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2\right\}^{1/2}} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

$$\phi = \tan^{-1}\left\{\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right\} = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)$$



the maximum value of *M* occurs when $r = \sqrt{1 - 2\zeta^2}$ or $\omega = \omega_n\sqrt{1 - 2\zeta^2}$

The maximum value of *X* $\left(\frac{X}{\delta_{st}}\right)_{max} = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}$

and the value of *X* at $\omega = \omega_n$ by $\left(\frac{X}{\delta_{st}}\right)_{\omega=\omega_n} = \frac{1}{2\zeta}$

Total Response

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t$$

$$x(t) = x_h(t) + x_p(t)$$

$$x(t) = X_0 e^{-\zeta \omega_n t} \cos(\omega_d t - \phi_0) + X \cos(\omega t - \phi)$$

$$\omega_d = \sqrt{1 - \zeta^2}\, \omega_n$$

X and ϕ

$$X = \frac{F_0}{[(k - m\omega^2)^2 + c^2\omega^2]^{1/2}} \qquad \phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right)$$

X_0 and ϕ_0

$$x_0 = X_0 \cos \phi_0 + X \cos \phi$$

$$\dot{x}_0 = -\zeta \omega_n X_0 \cos \phi_0 + \omega_d X_0 \sin \phi_0 + \omega X \sin \phi$$

$$X_0 = \left[(x_0 - X \cos \phi)^2 + \frac{1}{\omega_d^2} (\zeta \omega_n x_0 + \dot{x}_0 - \zeta \omega_n X \cos \phi - \omega X \sin \phi)^2 \right]^{\frac{1}{2}} \tag{3.37}$$

$$\tan \phi_0 = \frac{\zeta \omega_n x_0 + \dot{x}_0 - \zeta \omega_n X \cos \phi - \omega X \sin \phi}{\omega_d (x_0 - X \cos \phi)}$$

EXAMPLE 3.3

Total Response of a System

Find the total response of a single-degree-of-freedom system with $m = 10$ kg, $c = 20$ N-s/m, $k = 4000$ N/m, $x_0 = 0.01$ m, and $\dot{x}_0 = 0$ under the following conditions:

a. An external force $F(t) = F_0 \cos \omega t$ acts on the system with $F_0 = 100$ N and $\omega = 10$ rad/s.
b. Free vibration with $F(t) = 0$.

Solution:

a. From the given data, we obtain

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{10}} = 20 \text{ rad/s}$$

$$\delta_{st} = \frac{F_0}{k} = \frac{100}{4000} = 0.025 \text{ m}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{20}{2\sqrt{(4000)(10)}} = 0.05$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = \sqrt{1-(0.05)^2}(20) = 19.974984 \text{ rad/s}$$

$$r = \frac{\omega}{\omega_n} = \frac{10}{20} = 0.5$$

$$X = \frac{\delta_{st}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{0.025}{\left[(1-0.05^2)^2 + (2 \cdot 0.5 \cdot 0.5)^2\right]^{1/2}} = 0.03326 \text{ m} \qquad \text{(E.1)}$$

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{2 \cdot 0.05 \cdot 0.5}{1-0.5^2}\right) = 3.814075° \qquad \text{(E.2)}$$

Using the initial conditions, $x_0 = 0.01$ and $\dot{x}_0 = 0$, Eq. (3.36) yields:

$$0.01 = X_0 \cos \phi_0 + (0.03326)(0.997785)$$

$$X_0 \cos \phi_0 = -0.023186 \qquad \text{(E.3)}$$

$$0 = -(0.05)(20)\, X_0 \cos \phi_0 + X_0 (19.974984) \sin \phi_0 + (0.03326)(10) \sin(3.814075°) \qquad \text{(E.4)}$$

Substituting the value of $X_0 \cos \phi_0$ from Eq. (E.3) into (E.4), we obtain

$$X_0 \sin \phi_0 = -0.002268 \qquad \text{(E.5)}$$

Solution of Eqs. (E.3) and (E.5) yields

$$X_0 = \left[(X_0 \cos \phi_0)^2 + (X_0 \sin \phi_0)^2\right]^{1/2} = 0.023297 \qquad \text{(E.6)}$$

and

$$\tan \phi_0 = \frac{X_0 \sin \phi_0}{X_0 \cos \phi_0} = 0.0978176$$

or

$$\phi_0 = 5.586765° \qquad \text{(E.7)}$$

b. For free vibration, the total response is given by

$$x(t) = X_0 e^{-\zeta \omega_n t} \cos(\omega_d t - \phi_0) \tag{E.8}$$

Using the initial conditions $x(0) = x_0 = 0.01$ and $\dot{x}(0) = \dot{x}_0 = 0$, X_0 and ϕ_0 of Eq. (E.8) can be determined as (see Eqs. 2.73 and 2.75):

$$X_0 = \left[x_0^2 + \left(\frac{\zeta \omega_n x_0}{\omega_d}\right)^2\right]^{1/2} = \left[0.01^2 + \left(\frac{0.05 \cdot 20 \cdot 0.01}{19.974984}\right)^2\right]^{1/2} = 0.010012 \tag{E.9}$$

$$\phi_0 = \tan^{-1}\left(-\frac{\dot{x}_0 + \zeta \omega_n x_0}{\omega_d x_0}\right) = \tan^{-1}\left(-\frac{0.05 \cdot 20}{19.974984}\right) = -2.865984° \tag{E.10}$$

Note that the constants X_0 and ϕ_0 in cases (a) and (b) are very different.