Mechanical Vibrations

Lecture 8

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Free Vibration of an Undamped Torsional System

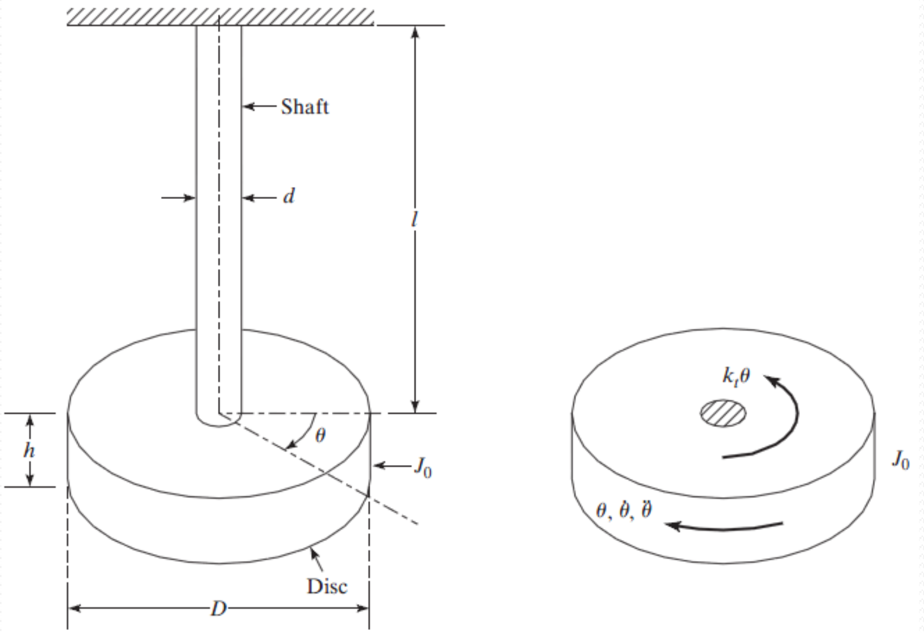
$$J_0\ddot{\theta} + k_t\theta = 0$$

$$\omega_n = \left(\frac{k_t}{J_0}\right)^{1/2}$$

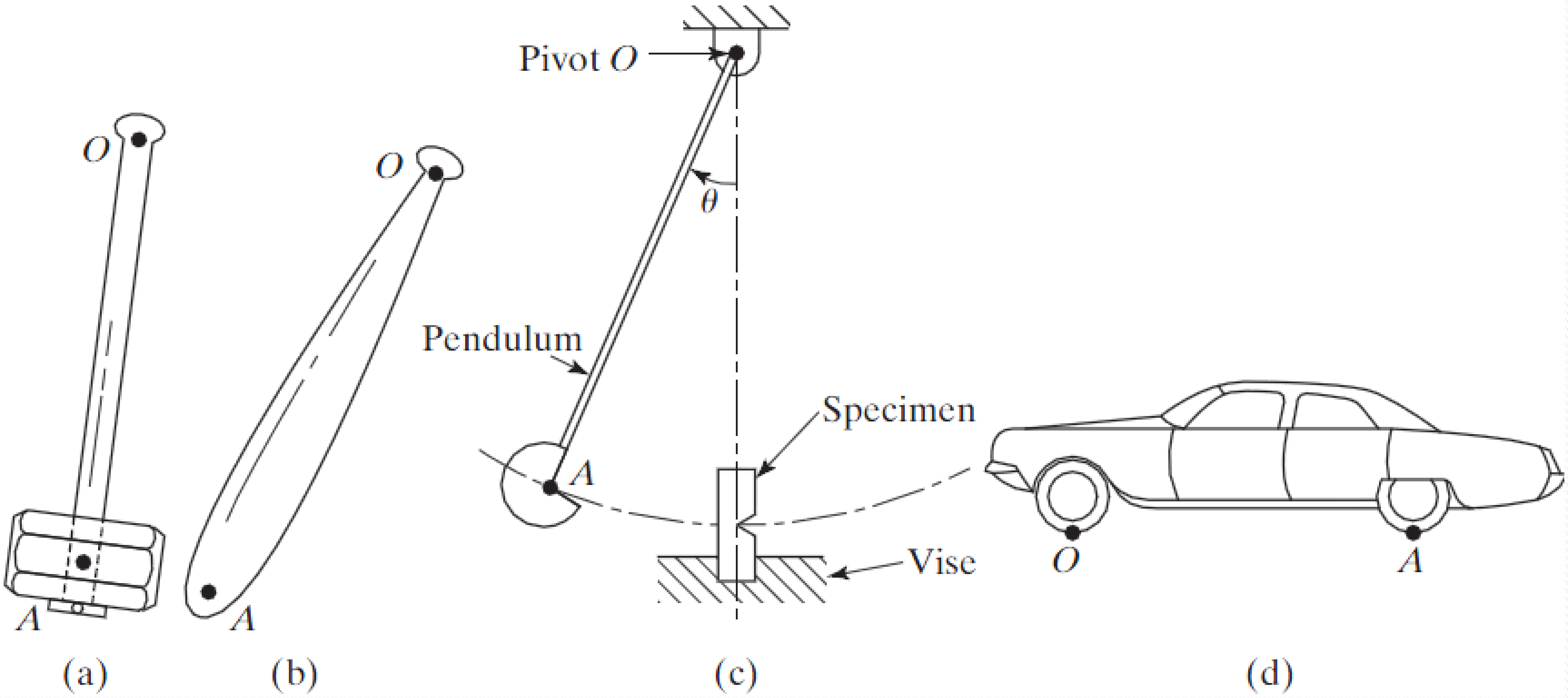
$$\theta(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t$$

$$A_1 = \theta_0$$

$$A_2 = \dot{\theta}_0 / \omega_n$$



Center of Percussion



Inertia and mass moment of Inertia

Effect of Mass of a Spring on Natural Frequency ω

Effect of Mass of Column on Natural Frequency of Water Tank

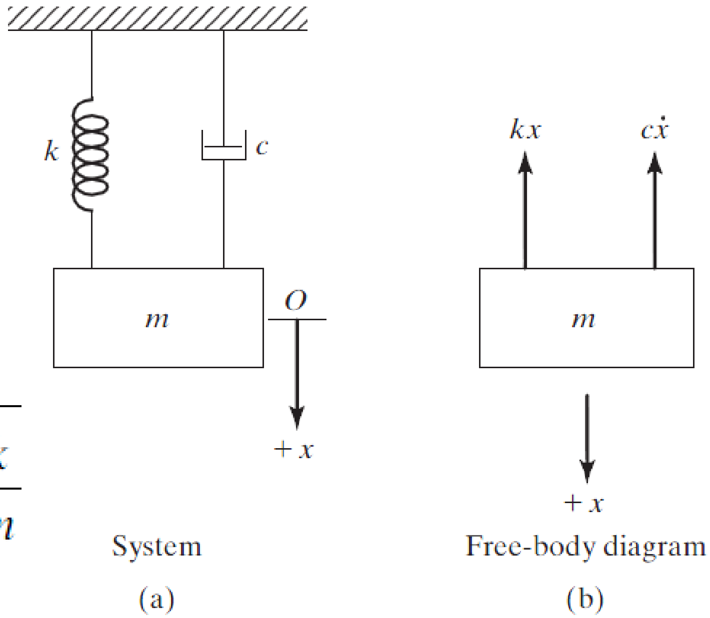
Free Vibration with Viscous Damping

$$m\ddot{x} = -c\dot{x} - kx$$

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$ms^2 + cs + k = 0$$

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$



$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

$$x(t) = C_1 e^{\left\{-\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right\}t} + C_2 e^{\left\{-\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right\}t}$$

Critical Damping Constant and the Damping Ratio

$$x(t) = C_1 e^{\left\{-\frac{c}{2m}+\sqrt{\left(\frac{c}{2m}\right)^2-\frac{k}{m}}\right\}t} + C_2 e^{\left\{-\frac{c}{2m}-\sqrt{\left(\frac{c}{2m}\right)^2-\frac{k}{m}}\right\}t}$$

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0$$

$$c_c = 2m\sqrt{\frac{k}{m}} = 2\sqrt{km} = 2m\omega_n$$

For any damped system, the damping ratio ζ is defined as the ratio of the damping constant to the critical damping constant:

$$\zeta = c/c_c \qquad (2.66)$$

$$\frac{c}{2m} = \frac{c}{c_c}\cdot\frac{c_c}{2m} = \zeta\omega_n$$

$$s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n$$

$$x(t) = C_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + C_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}$$

$$x(t) = C_1 e^{(-\zeta+\sqrt{\zeta^2-1})\omega_n t} + C_2 e^{(-\zeta-\sqrt{\zeta^2-1})\omega_n t}$$

Case 1. *Underdamped system* ($\zeta < 1$ or $c < c_c$ or $c/2m < \sqrt{k/m}$). For this condition, $(\zeta^2 - 1)$ is negative and the roots s_1 and s_2 can be expressed as

$$s_1 = (-\zeta + i\sqrt{1-\zeta^2})\omega_n$$

$$s_2 = (-\zeta - i\sqrt{1-\zeta^2})\omega_n$$

$$x(t) = C_1 e^{(-\zeta+i\sqrt{1-\zeta^2})\omega_n t} + C_2 e^{(-\zeta-i\sqrt{1-\zeta^2})\omega_n t}$$

$$= e^{-\zeta\omega_n t}\left\{C_1 e^{i\sqrt{1-\zeta^2}\omega_n t} + C_2 e^{-i\sqrt{1-\zeta^2}\omega_n t}\right\}$$

$$= e^{-\zeta\omega_n t}\left\{(C_1 + C_2)\cos\sqrt{1-\zeta^2}\omega_n t + i(C_1 - C_2)\sin\sqrt{1-\zeta^2}\omega_n t\right\}$$

$$= e^{-\zeta\omega_n t}\left\{C_1'\cos\sqrt{1-\zeta^2}\omega_n t + C_2'\sin\sqrt{1-\zeta^2}\omega_n t\right\}$$

$$x(t) = e^{-\zeta\omega_n t}\left\{C_1' \cos\sqrt{1-\zeta^2}\,\omega_n t + C_2' \sin\sqrt{1-\zeta^2}\,\omega_n t\right\}$$

$$x(t) = X_0 e^{-\zeta\omega_n t} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi_0\right)$$

$$x(t) = X e^{-\zeta\omega_n t} \cos\left(\sqrt{1-\zeta^2}\,\omega_n t - \phi\right)$$

where (C_1', C_2'), (X, ϕ), and (X_0, ϕ_0) are arbitrary constants to be determined from the initial conditions.

For the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$, C_1' and C_2' can be found:

$$C_1' = x_0 \quad \text{and} \quad C_2' = \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n} \tag{2.71}$$

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n} \sin\sqrt{1-\zeta^2}\,\omega_n t\right\}$$

The constants (X, ϕ) and (X_0, ϕ_0) can be expressed as

$$X = X_0 = \sqrt{(C_1')^2 + (C_2')^2} = \frac{\sqrt{x_0^2 \omega_n^2 + \dot{x}_0^2 + 2 x_0 \dot{x}_0 \zeta \omega_n}}{\sqrt{1 - \zeta^2}\, \omega_n}$$

$$\phi_0 = \tan^{-1}\left(\frac{C_1'}{C_2'}\right) = \tan^{-1}\left(\frac{x_0 \omega_n \sqrt{1 - \zeta^2}}{\dot{x}_0 + \zeta \omega_n x_0}\right)$$

$$\phi = \tan^{-1}\left(\frac{C_2'}{C_1'}\right) = \tan^{-1}\left(\frac{\dot{x}_0 + \zeta \omega_n x_0}{x_0 \omega_n \sqrt{1 - \zeta^2}}\right)$$

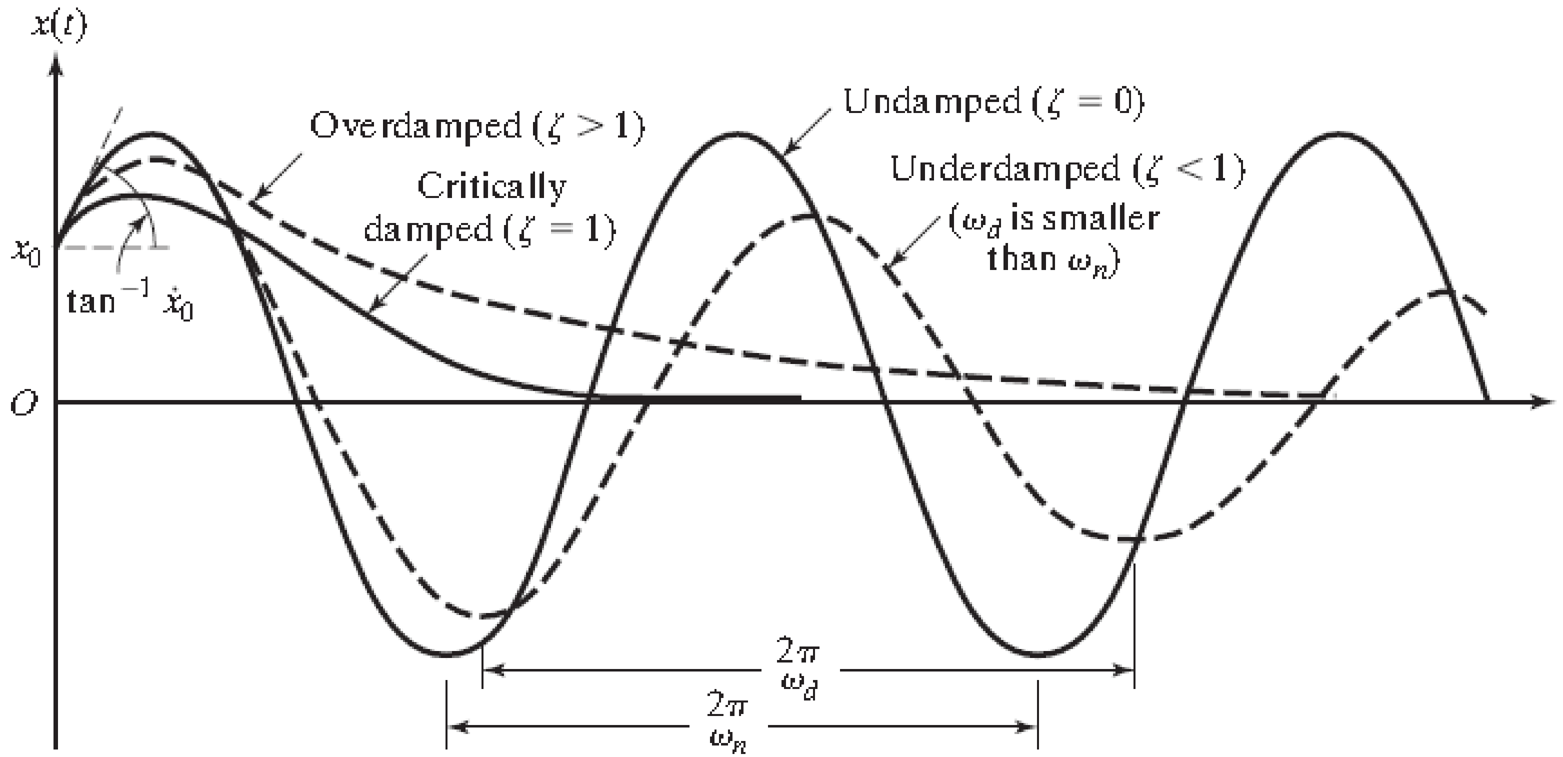


FIGURE 2.24 Comparison of motions with different types of damping.