Mechanical Vibrations

Lecture 9

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$$x(t) = e^{-\zeta\omega_n t}\left\{C_1' \cos\sqrt{1-\zeta^2}\,\omega_n t + C_2' \sin\sqrt{1-\zeta^2}\,\omega_n t\right\}$$

$$C_1' = x_0 \quad \text{and} \quad C_2' = \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n}$$

$$X = X_0 = \sqrt{(C_1')^2 + (C_2')^2} = \frac{\sqrt{x_0^2\omega_n^2 + \dot{x}_0^2 + 2x_0\dot{x}_0\zeta\omega_n}}{\sqrt{1-\zeta^2}\,\omega_n}$$

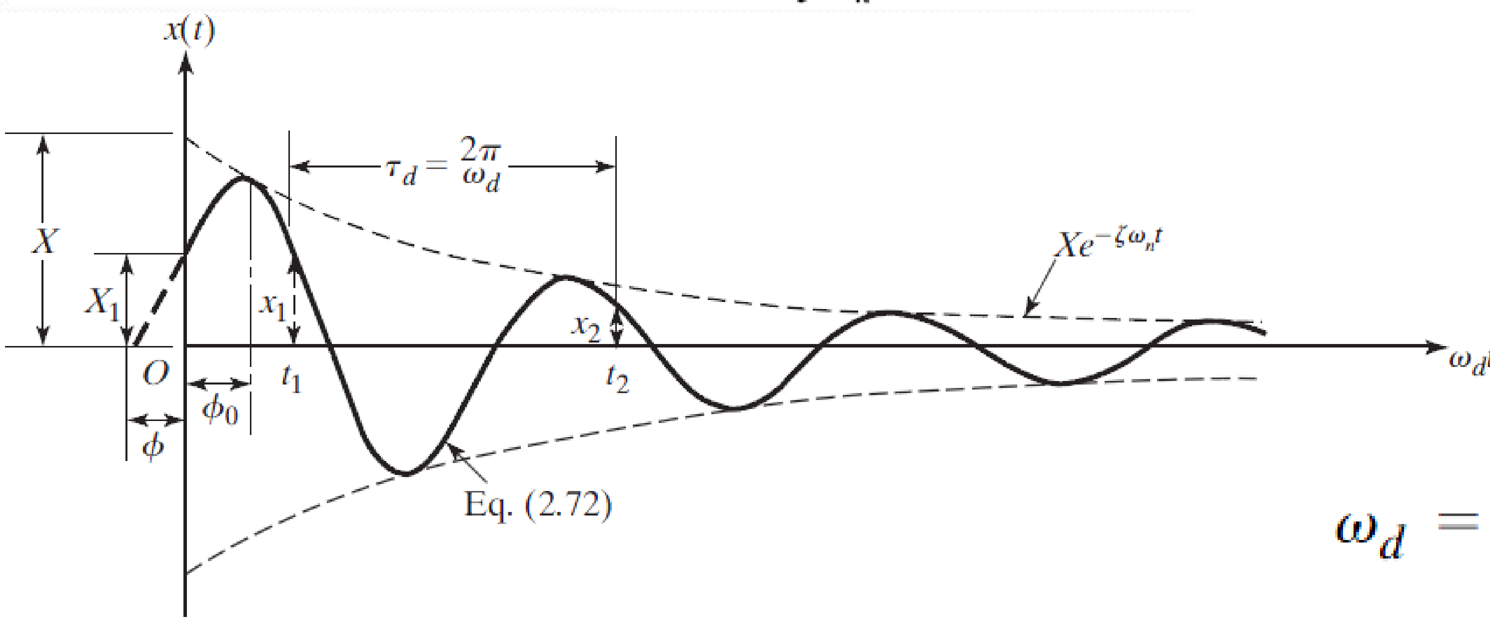
$$x(t) = X_0 e^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi_0\right)$$

$$\phi_0 = \tan^{-1}\left(\frac{C_1'}{C_2'}\right) = \tan^{-1}\left(\frac{x_0\omega_n\sqrt{1-\zeta^2}}{\dot{x}_0 + \zeta\omega_n x_0}\right)$$

$$x(t) = X e^{-\zeta\omega_n t}\cos\left(\sqrt{1-\zeta^2}\,\omega_n t - \phi\right)$$

$$\phi = \tan^{-1}\left(\frac{C_2'}{C_1'}\right) = \tan^{-1}\left(\frac{\dot{x}_0 + \zeta\omega_n x_0}{x_0\omega_n\sqrt{1-\zeta^2}}\right)$$

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0\cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n}\sin\sqrt{1-\zeta^2}\,\omega_n t\right\}$$

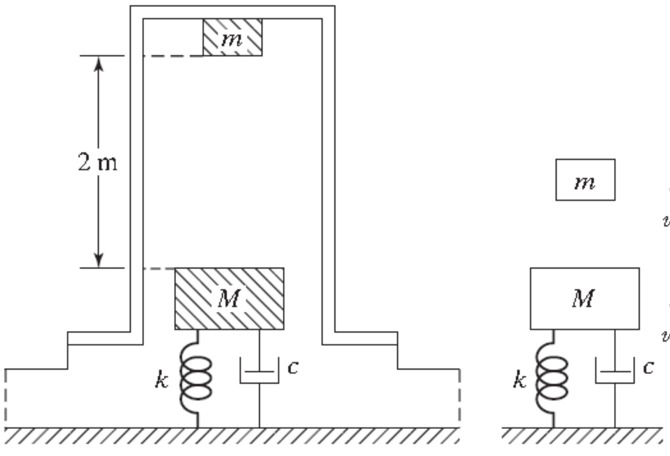


$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n$$

EXAMPLE 2.10

Response of Anvil of a Forging Hammer

The anvil of a forging hammer weighs 5,000 N and is mounted on a foundation that has a stiffness of 5×10^6 N/m and a viscous damping constant of 10,000 N-s/m. During a particular forging operation, the tup (i.e., the falling weight or the hammer), weighing 1,000 N, is made to fall from a height of 2 m onto the anvil (Fig. 2.29(a)). If the anvil is at rest before impact by the tup, determine the response of the anvil after the impact. Assume that the coefficient of restitution between the anvil and the tup is 0.4.



$$v_{t1} = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 2} = 6.26099 \text{ m/s}$$

$$M(v_{a2} - v_{a1}) = m(v_{t1} - v_{t2})$$

$$\frac{5000}{9.81}(v_{a2} - 0) = \frac{1000}{9.81}(6.26099 - v_{t2})$$

$$510.204082\, v_{a2} = 638.87653 - 102.040813\, v_{t2}$$

$$r = -\left(\frac{v_{a2} - v_{t2}}{v_{a1} - v_{t1}}\right) \qquad 0.4 = -\left(\frac{v_{a2} - v_{t2}}{0 - 6.26099}\right)$$

$$v_{a2} = v_{t2} + 2.504396$$

$$v_{a2} = 1.460898 \text{ m/s}; \qquad v_{t2} = -1.043498 \text{ m/s}$$

Thus the initial conditions of the anvil are given by

$$x_0 = 0; \qquad \dot{x}_0 = 1.460898 \text{ m/s}$$

$$\zeta = c/c_c \qquad c_c = 2m\sqrt{\frac{k}{m}} = 2\sqrt{km} = 2m\omega_n$$

$$\zeta = \frac{c}{2\sqrt{kM}} = \frac{1000}{2\sqrt{(5 \times 10^6)\left(\frac{5000}{9.81}\right)}} = 0.0989949$$

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{5 \times 10^6}{\left(\frac{5000}{9.81}\right)}} = 98.994949 \text{ rad/s}$$

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} = 98.994949\sqrt{1 - 0.0989949^2} = 98.024799 \text{ rad/s}$$

$$x(t) = X_0 e^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi_0\right) \qquad \phi_0 = \tan^{-1}\left(\frac{x_0\omega_n\sqrt{1-\zeta^2}}{\dot{x}_0 + \zeta\omega_n x_0}\right)$$

$$x(t) = e^{-\zeta\omega_n t}\left\{\frac{\dot{x}_0}{\omega_d}\sin\omega_d t\right\}$$

$$= e^{-9.799995\,t}\{0.01490335 \sin 98.024799\,t\}\text{ m}$$

Case 2. *Critically damped system* ($\zeta = 1$ or $c = c_c$ or $c/2m = \sqrt{k/m}$).

$$x(t) = (C_1 + C_2 t)e^{-\omega_n t}$$

$$C_1 = x_0$$

$$C_2 = \dot{x}_0 + \omega_n x_0$$

$$x(t) = [x_0 + (\dot{x}_0 + \omega_n x_0)t]e^{-\omega_n t}$$

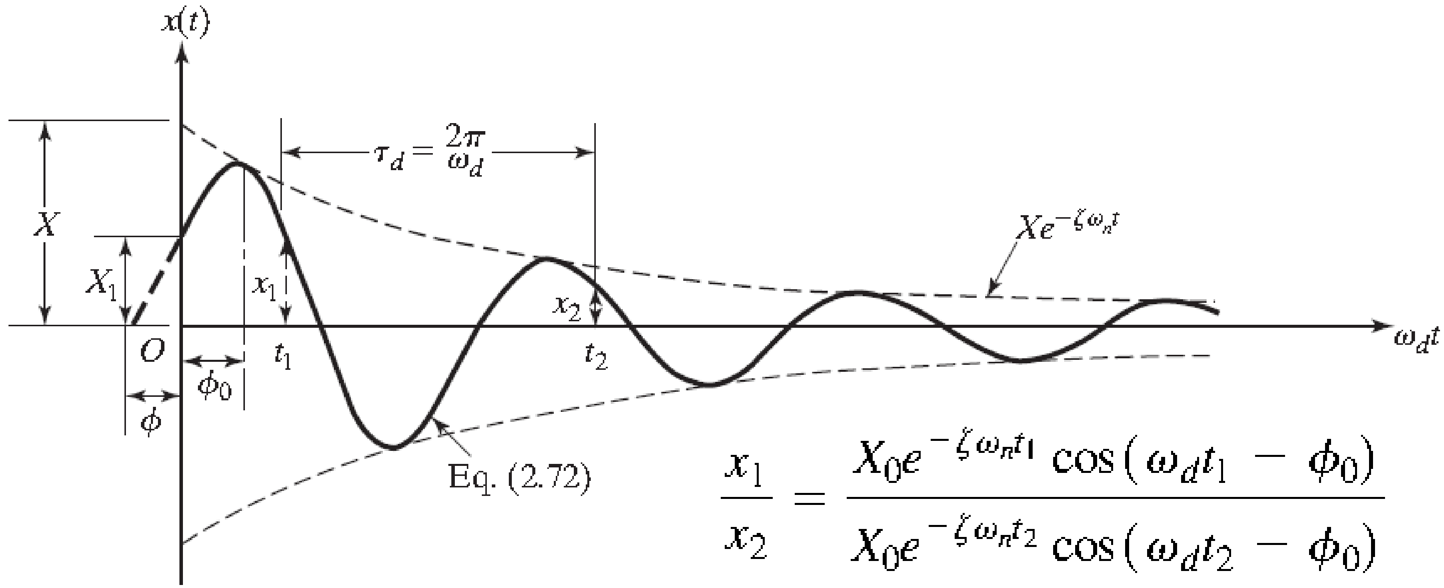
Case 3. *Overdamped system* ($\zeta > 1$ or $c > c_c$ or $c/2m > \sqrt{k/m}$). As $\sqrt{\zeta^2 - 1} > 0$,

$$x(t) = C_1 e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + C_2 e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t}$$

$$C_1 = \frac{x_0 \omega_n (\zeta + \sqrt{\zeta^2 - 1}) + \dot{x}_0}{2\omega_n \sqrt{\zeta^2 - 1}}$$

$$C_2 = \frac{-x_0 \omega_n (\zeta - \sqrt{\zeta^2 - 1}) - \dot{x}_0}{2\omega_n \sqrt{\zeta^2 - 1}}$$

Logarithmic Decrement



$$\frac{x_1}{x_2} = \frac{X_0 e^{-\zeta\omega_n t_1} \cos(\omega_d t_1 - \phi_0)}{X_0 e^{-\zeta\omega_n t_2} \cos(\omega_d t_2 - \phi_0)}$$

But $t_2 = t_1 + \tau_d$, where $\tau_d = 2\pi/\omega_d$ is the period of damped vibration. Hence $\cos(\omega_d t_2 - \phi_0) = \cos(2\pi + \omega_d t_1 - \phi_0) = \cos(\omega_d t_1 - \phi_0)$,

$$\frac{x_1}{x_2} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n (t_1 + \tau_d)}} = e^{\zeta\omega_n \tau_d}$$

$$\delta = \ln\frac{x_1}{x_2} = \zeta\omega_n\tau_d = \zeta\omega_n \frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi}{\omega_d}\cdot\frac{c}{2m}$$

For small damping

$$\delta \simeq 2\pi\zeta \qquad \text{if} \qquad \zeta \ll 1$$

The logarithmic decrement is dimensionless and is actually another form of the dimensionless damping ratio ζ. Once δ is known, ζ can be

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}$$

If the damping in the given system is not known, we can determine it experimentally by measuring any two consecutive displacements x_1 and x_2. By taking the natural logarithm of the ratio of x_1 and x_2, we obtain δ.

$$\frac{x_1}{x_{m+1}} = \frac{x_1}{x_2}\frac{x_2}{x_3}\frac{x_3}{x_4}\cdots\frac{x_m}{x_{m+1}}$$

$$\frac{x_j}{x_{j+1}} = e^{\zeta\omega_n\tau_d}$$

$$\frac{x_1}{x_{m+1}} = \left(e^{\zeta\omega_n\tau_d}\right)^m = e^{m\zeta\omega_n\tau_d}$$

$$\delta = \frac{1}{m}\ln\left(\frac{x_1}{x_{m+1}}\right)$$

* if ζ is known → δ ?

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

* if δ is known → ζ ?

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2+\delta^2}}$$

if ζ is not known → δ ?

$$\delta = \ln \frac{X_{n-1}}{X_n}$$

لـ amplitude معينة والتي بعدها

$$\delta = \frac{1}{n} \ln \frac{X_1}{X_{n+1}}$$

بين ال amplitude الأولى والأخيرة

EXAMPLE 2.11

Shock Absorber for a Motorcycle

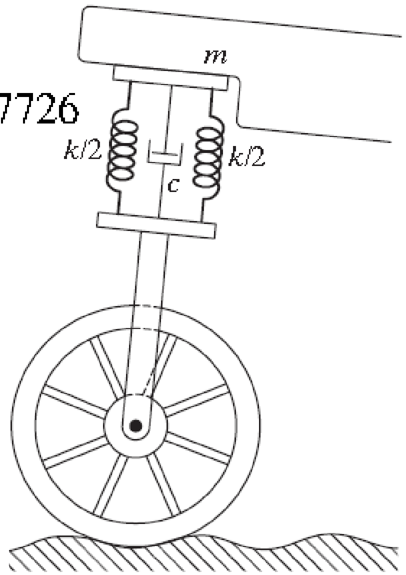
An underdamped shock absorber is to be designed for a motorcycle of mass 200 kg (Fig. 2.30(a)). When the shock absorber is subjected to an initial vertical velocity due to a road bump, the resulting displacement-time curve is to be as indicated in Fig. 2.30(b). Find the necessary stiffness and damping constants of the shock absorber if the damped period of vibration is to be 2 s and the amplitude x_1 is to be reduced to one-fourth in one half cycle (i.e., $x_{1.5} = x_1/4$). Also find the minimum initial velocity that leads to a maximum displacement of 250 mm.

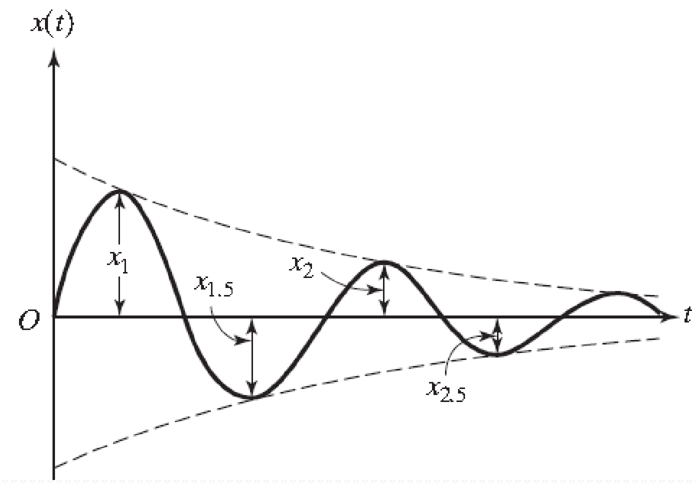
$$\delta = \ln\frac{x_1}{x_2} \qquad \delta = \ln\left(\frac{x_1}{x_2}\right) = \ln(16) = 2.7726$$

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = 0.4037$$

$$\tau_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1 - \zeta^2}}$$

$$\omega_n = \frac{2\pi}{2\sqrt{1 - (0.4037)^2}} = 3.4338 \text{ rad/s}$$





$$c_c = 2m\sqrt{\frac{k}{m}} = 2\sqrt{km} = 2m\omega_n \qquad c_c = 2m\omega_n = 2(200)(3.4338) = 1373.54 \text{ N-s/m}$$

$$c = \zeta c_c = (0.4037)(1373.54) = 554.4981 \text{ N-s/m} \qquad k = m\omega_n^2 = (200)(3.4338)^2 = 2358.2652 \text{ N/m}$$