Abstract Algebra (I)

Chapter 11: Fundamental Theorem of Finite Abelian Groups

- The Fundamental Theorem
- The Isomorphism Classes of Abelian Groups

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First Semester 2019-2020
Theorem (Fundamental Theorem of Finite Abelian Groups)

Every finite Abelian group is a direct product of cyclic groups of prime-power order. Moreover, the number of terms in the product and the orders of the cyclic groups are uniquely determined by the group. That is:

\[ G \cong \mathbb{Z}_{p_1^{n_1}} \oplus \mathbb{Z}_{p_2^{n_2}} \oplus \ldots \oplus \mathbb{Z}_{p_k^{n_k}} \]
The Isomorphism Classes of Abelian Groups

Let’s look at groups whose orders have the form $p^k$, where $p$ is prime and $k \leq 4$. In general, there is one group of order $p^k$ for each set of positive integers whose sum is $k$.

**Definition**

(Partition)
A set of positive integers $n_1, n_2, ..., n_t$ is called a partition of $k$ if $k = n_1 + n_2 + ... + n_t$.

In this case

$$\mathbb{Z}_{p^{n_1}} \oplus \mathbb{Z}_{p^{n_2}} \cdots \oplus \mathbb{Z}_{p^{n_t}}$$

is an Abelian group of order $p^k$. 
**Partition**

<table>
<thead>
<tr>
<th>Order of $G$</th>
<th>Partitions of $k$</th>
<th>Possible direct products for $G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>1</td>
<td>$\mathbb{Z}_p$</td>
</tr>
<tr>
<td>$p^2$</td>
<td>2</td>
<td>$\mathbb{Z}_{p^2}$</td>
</tr>
<tr>
<td></td>
<td>$1+1$</td>
<td>$\mathbb{Z}_p \oplus \mathbb{Z}_p$</td>
</tr>
<tr>
<td>$p^3$</td>
<td>3</td>
<td>$\mathbb{Z}_{p^3}$</td>
</tr>
<tr>
<td></td>
<td>$2+1$</td>
<td>$\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$</td>
</tr>
<tr>
<td></td>
<td>$1+1+1$</td>
<td>$\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$</td>
</tr>
<tr>
<td>$p^4$</td>
<td>4</td>
<td>$\mathbb{Z}_{p^4}$</td>
</tr>
<tr>
<td></td>
<td>$3+1$</td>
<td>$\mathbb{Z}_{p^3} \oplus \mathbb{Z}_p$</td>
</tr>
<tr>
<td></td>
<td>$2+2$</td>
<td>$\mathbb{Z}<em>{p^2} \oplus \mathbb{Z}</em>{p^2}$</td>
</tr>
<tr>
<td></td>
<td>$2+1+1$</td>
<td>$\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$</td>
</tr>
<tr>
<td></td>
<td>$1+1+1+1$</td>
<td>$\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$</td>
</tr>
</tbody>
</table>

**Figure:** Isomorphism Classes
Now, we know how to construct all the Abelian groups of prime power order, $p^k$.

we move to the problem of constructing all Abelian groups of a certain order $n$, where $n$ has two or more distinct prime divisors.

We begin by writing $n$ in prime-power decomposition form $n = p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k}$.

Next, we individually form all Abelian groups of order $p_1^{n_1}$, then $p_2^{n_2}$, and so on.
Example

Find all non isomorphic Abelian groups of order $n = 1176$. 
Example

Find all non isomorphic Abelian groups of order $n = 1176$.

- Note that $1176 = 2^3 \cdot 3 \cdot 7^2$. 
Example
Let $G = \{1, 8, 12, 14, 18, 21, 27, 31, 34, 38, 44, 47, 51, 53, 57, 64\}$ under multiplication modulo 65. Write $G$ as an external product of cyclic groups.

Proof.
Note that $G = 16 = 2^4$.
Partition of 4 is: $4, 3 + 1, 2 + 2, 2 + 1 + 1, 1 + 1 + 1 + 1$.
So we have 5 classes $\mathbb{Z}_{16} \mathbb{Z}_8 \oplus \mathbb{Z}_2 \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
$G$ is isomorphic to only one of these 5 groups, which one?
Example

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- So we have 5 classes

$$
\mathbb{Z}_{16}
$$

$$
\mathbb{Z}_8 \oplus \mathbb{Z}_2
$$

$$
\mathbb{Z}_4 \oplus \mathbb{Z}_4
$$

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- So we have 5 classes

\[
\mathbb{Z}_{16} \ 
\mathbb{Z}_8 \oplus \mathbb{Z}_2 \\
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\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2
\]

- $G$ is isomorphic to only one of these 5 groups, which one?
Table of orders of elements of $G$

<table>
<thead>
<tr>
<th>Proof.</th>
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Proof.
Table of orders of elements of $G$

Proof.

To decide which one, we calculate the orders of the elements of $G$.

<table>
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<tr>
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<tr>
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Why? It remains $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ and $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. We rule out $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ since it has more than 3 elements of order 2 but $G$ has only 3 elements of order 3. Therefore $G$ is isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_4$. 
**Table of orders of elements of $G$**

**Proof.**

- To decide which one, we calculate the orders of the elements of $G$.

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- From the table of orders, we rule out $\mathbb{Z}_{16}$, $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Why?
Table of orders of elements of $G$

Proof.

- To decide which one, we calculate the orders of the elements of $G$.

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- From the table of orders, we rule out $\mathbb{Z}_{16}$, $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Why?

- It remains $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ and $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$
Table of orders of elements of $G$

Proof.

- To decide which one, we calculate the orders of the elements of $G$.

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- From the table of orders, we rule out $\mathbb{Z}_{16}$, $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Why?

- It remains $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ and $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

- We rule out $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ since it has more than 3 elements of order 2 but $G$ has only 3 elements of order 3.
Table of orders of elements of $G$

Proof.

To decide which one, we calculate the orders of the elements of $G$.

From the table of orders, we rule out $\mathbb{Z}_{16}$, $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Why?

It remains $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ and $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

We rule out $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ since it has more than 3 elements of order 2 but $G$ has only 3 elements of order 2 or order 3.

Therefore $G$ is isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_4$. 
Example

Let $G = \{1, 8, 17, 19, 26, 28, 37, 44, 46, 53, 62, 64, 71, 73, 82, 89, 91, 98, 107, 109, 116, 118, 127, 134\}$ under multiplication modulo 135. Since $G$ has order 24, it is isomorphic to one of the following:

$$\mathbb{Z}_8 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_{24},$$

$$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_{12} \oplus \mathbb{Z}_3,$$

$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_{6} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2,$$

decide which one.
Proof.

- Since $|8| = 12$ we rule out the third one,
  \[ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_6 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, \]

- Since $|109| = |134| = 2$, $G$ has at least 2 elements of order 2 and $\mathbb{Z}_8 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_{24}$, has only one element of order 2, Why? we rule out the first one.

- Therefore $G \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_{12} \oplus \mathbb{Z}_3$. 
Corollary

If $m$ divides the order of a finite Abelian group $G$, then $G$ has a subgroup of order $m$.

Example

Let $G$ be an Abelian group of order 72, we show that $G$ always has a subgroup of order 12.

Proof.

Since $|G| = 72 = 2^3 \cdot 3^2$, then $G$ is isomorphic to one of the following:

\[
\begin{align*}
Z_8 \oplus Z_9, & \quad Z_8 \oplus Z_3 \oplus Z_3, \\
Z_4 \oplus Z_2 \oplus Z_9, & \quad Z_4 \oplus Z_2 \oplus Z_3 \oplus Z_3, \\
Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_9, & \quad Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_3 \oplus Z_3.
\end{align*}
\]

Therefore, we show each of these groups has a subgroup of order 12.
Exercises of Ch 11

1-10
12
15
19
24-26