Chapter 9
Normal Subgroups and Factor Groups

- Normal Subgroups
- Factor Groups
- Applications of Factor Groups
- Internal Direct Products
Normal Subgroups

As we saw in Chapter 7, if $G$ is a group and $H$ is a subgroup of $G$, it is not always true that $aH = Ha$ for all $a$ in $G$. In this chapter, we study a type of subgroups $H$ for which $aH = Ha$ for all $a$ in $G$. Such subgroups are called Normal subgroups.

**Definition (Normal Subgroup)**

A subgroup $H$ of a group $G$ is called a normal subgroup of $G$ if $aH = Ha$ for all $a$ in $G$. We denote this by $H 	riangleleft G$.

- Note that if $aH = Ha$, it is not true that $ah = ha$ for all $a \in G$ and $h \in H$. But
- $ah = h_1 a$ and $ha = ah_2$ for some $h_1, h_2 \in H$. 

Ahmed EL-Mabhouh

Abstract Algebra I
Theorem (1, Normal Subgroup Test)

A subgroup $H$ of $G$ is normal in $G$ if and only if $xHx^{-1} \subseteq H$

Proof.

If $H \triangleleft G$, then $xH = Hx$ for any $x \in G$. Therefore, $H = xHx^{-1}$, and so $xHx^{-1} \subseteq H$ for all $x$ in $G$.

Conversely, assume $xHx^{-1} \subseteq H$ for all $x$ in $G$. We want to show $aH = Ha$ for all $a \in G$.

Let $a = x$, we have $aHa^{-1} \subseteq H$ and so $aH \subseteq Ha$ (we multiply on the right by $a$.)

Let $a^{-1} = x$, we have $a^{-1}H(a^{-1})^{-1} = a^{-1}Ha \subseteq H$ and so $Ha \subseteq aH$ (we multiply on the left by $a$.)

Therefore, $aH = Ha$ for all $a \in G$ and so $H \triangleleft G$. 
Example (1)

Every subgroup of an Abelian group is normal.

Example (2)

The center $Z(G)$ is a normal subgroup of $G$, WHY?

Example (3)

The group $SL(2, \mathbb{R})$ of $2 \times 2$ matrices with determinant 1 is a normal subgroup of $GL(2, \mathbb{R})$.

Proof.

We show that $A \begin{bmatrix} SL(2, \mathbb{R}) \end{bmatrix} A^{-1} \subseteq SL(2, \mathbb{R})$. For any $B \in SL(2, \mathbb{R})$, we have

$\det(ABA^{-1}) = \det(A) \det(B) \det(A^{-1}) = \det(B) = 1$.

Therefore, $ABA^{-1} \in SL(2, \mathbb{R})$. That is $A \begin{bmatrix} SL(2, \mathbb{R}) \end{bmatrix} A^{-1} \subseteq SL(2, \mathbb{R})$. Hence, $SL(2, \mathbb{R}) \triangleleft GL(2, \mathbb{R})$. \qed
Example (4)
The alternating group $A_n$ of even permutations is a normal subgroup of $S_n$. Note that $(12)(123) \neq (123)(12)$ but $(12)(123) = (132)(12)$. $\alpha A_n = A_n\alpha$ for any $\alpha \in S_n$.

Example (5)
The subgroup of rotations $R$ in $D_n$ is normal in $D_n$.

Proof.
For any rotation $r$ and any reflection $f$, we have $fr = r^{-1}f$, whereas for any rotations $r$ and $r'$, we have $rr' = r'r$. That means, for any $x \in D_n$, we have $xR = Rx$. 

Ahmed EL-Mabhouh
Abstract Algebra I
Recall that the multiplication table for $A_4$ is given in the following diagram:

\[
\begin{array}{cccccccccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 & \alpha_9 & \alpha_{10} & \alpha_{11} & \alpha_{12} \\
(1) = \alpha_1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
(12)(34) = \alpha_2 & 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 & 10 & 9 & 12 & 11 \\
(13)(24) = \alpha_3 & 3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 & 11 & 12 & 9 & 10 \\
(14)(23) = \alpha_4 & 4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 & 12 & 11 & 10 & 9 \\
(123) = \alpha_5 & 5 & 8 & 6 & 7 & 9 & 12 & 10 & 11 & 1 & 4 & 2 & 3 \\
(243) = \alpha_6 & 6 & 7 & 5 & 8 & 10 & 11 & 9 & 12 & 2 & 3 & 1 & 4 \\
(142) = \alpha_7 & 7 & 6 & 8 & 5 & 11 & 10 & 12 & 9 & 3 & 2 & 4 & 1 \\
(134) = \alpha_8 & 8 & 5 & 7 & 6 & 12 & 9 & 11 & 10 & 4 & 1 & 3 & 2 \\
(132) = \alpha_9 & 9 & 11 & 12 & 10 & 1 & 3 & 4 & 2 & 5 & 7 & 8 & 6 \\
(143) = \alpha_{10} & 10 & 12 & 11 & 9 & 2 & 4 & 3 & 1 & 6 & 8 & 7 & 5 \\
(234) = \alpha_{11} & 11 & 9 & 10 & 12 & 3 & 1 & 2 & 4 & 7 & 5 & 6 & 8 \\
(124) = \alpha_{12} & 12 & 10 & 9 & 11 & 4 & 2 & 1 & 3 & 8 & 6 & 5 & 7 \\
\end{array}
\]

$H = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $K = \{\alpha_1, \alpha_5, \alpha_9\}$ are subgroups of $A_4$. Why $H \triangleleft A_4$ but $H \not\triangleleft A_4$?

Note that the only subgroup of order 4 is $H$ since any element not in $H$ has order 3. Also $\beta H \beta^{-1}$ is a subgroup of order 4, and so $\beta H \beta^{-1} = H$. While $\alpha_2 \alpha_5 \alpha_2^{-1} = \alpha_7 \notin K$. Therefore, $\alpha_2 K \alpha_2^{-1} \notin K$
Theorem (2, Factor Group)

Let $G$ be a group and let $H$ be a normal subgroup of $G$. The set

$$G/H = \{aH | a \in G\}$$

is a group under the operation $(aH)(bH) = abH$

Proof.

- First, we show that multiplication is well defined.
- Let $aH = a'H$ and $bH = b'H$, we show $aHbH = a'Hb'H$. That is, $abH = a'b'H$
- Since $aH = a'H$ and $bH = b'H$, then $a' = ah_1$ and $b' = bh_2$ for some $h_1, h_2$ in $H$
- Therefore,

$$a'b'H = ah_1bh_2H = ah_1bH = ah_1Hb^2 = aHb = abH$$

(Since $H \triangleleft G$, we have $bH = Hb$)
Proof.

- For associativity, 
  \[ aH(bHcH) = aH(bcH) = a(bc)H = (ab)cH = (aHbH)cH. \]

- The identity is \( eH = H \) since for each \( aH \) we have 
  \( eHaH = eaH = aH. \)

- The inverse of \( aH \) is \( (aH)^{-1} = a^{-1}H \) since 
  \[ aHa^{-1}H = aa^{-1}H = eH = H. \]

- Therefore, \( G/H \) is a group.

The group \( G/H \) is called the factor group of \( G \) by \( H \) or
the quotient group of \( G \) by \( H \).
Example (6)

Let $G = \mathbb{Z}$ and $H = 4\mathbb{Z} = \{0, \pm 4, \pm 8, \ldots\}$. Since $\mathbb{Z}$ is an Abelian group, then $4\mathbb{Z} \triangleleft \mathbb{Z}$. So

$$\mathbb{Z}/4\mathbb{Z} = \{0 + 4\mathbb{Z}, 1 + 4\mathbb{Z}, 2 + 4\mathbb{Z}, 3 + 4\mathbb{Z}\}$$

is a group where the multiplication table is given by:

<table>
<thead>
<tr>
<th></th>
<th>$0 + 4\mathbb{Z}$</th>
<th>$1 + 4\mathbb{Z}$</th>
<th>$2 + 4\mathbb{Z}$</th>
<th>$3 + 4\mathbb{Z}$</th>
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<td>$0 + 4\mathbb{Z}$</td>
<td>$0 + 4\mathbb{Z}$</td>
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<td>$2 + 4\mathbb{Z}$</td>
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<td>$3 + 4\mathbb{Z}$</td>
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<td>$0 + 4\mathbb{Z}$</td>
<td>$1 + 4\mathbb{Z}$</td>
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</tr>
</tbody>
</table>

Note that $|\mathbb{Z}/4\mathbb{Z}| = 4$ and $|1 + 4\mathbb{Z}| = 4$, so $\mathbb{Z}/4\mathbb{Z}$ is a cyclic group of order 4 and so $\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}_4$.

Notice the similarity of multiplication tables of both $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}_4$. 

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Example (7)

Let $G = \mathbb{Z}_{18}$ and let $H = \langle 6 \rangle = \{0, 6, 12\}$
List down all distinct elements of $G/H$
How addition is performed?
Is $G/H$ cyclic?

Proof.
Example (8)

Let \( \mathcal{K} = \{R_0, R_{180}\} \), then the factor group is \( D_4/\mathcal{K} = \{\mathcal{K}, R_{90}\mathcal{K}, H\mathcal{K}, D\mathcal{K}\} \) where

\[
R_0\mathcal{K} = \mathcal{K},
R_{90}\mathcal{K} = \{R_{90}, R_{270}\},
V\mathcal{K} = \{V, H\},
D\mathcal{K} = \{D, D'\},
\]

Since \( |D_4/\mathcal{K}| = 4 \), then \( D_4/\mathcal{K} \cong \mathbb{Z}_4 \) or \( D_4/\mathcal{K} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \).

Decide which one.
the following table shows how a factor group of $G$ is related to $G$ itself.

<table>
<thead>
<tr>
<th></th>
<th>$R_0$</th>
<th>$R_{180}$</th>
<th>$R_{90}$</th>
<th>$R_{270}$</th>
<th>$H$</th>
<th>$V$</th>
<th>$D$</th>
<th>$D'$</th>
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</thead>
<tbody>
<tr>
<td>$R_0$</td>
<td>$R_0$</td>
<td>$R_{180}$</td>
<td>$R_{90}$</td>
<td>$R_{270}$</td>
<td>$H$</td>
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<tr>
<td>$R_{180}$</td>
<td>$R_{180}$</td>
<td>$R_0$</td>
<td>$R_{270}$</td>
<td>$R_{90}$</td>
<td>$V$</td>
<td>$H$</td>
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<tr>
<td>$R_{90}$</td>
<td>$R_{90}$</td>
<td>$R_{270}$</td>
<td>$R_{180}$</td>
<td>$R_0$</td>
<td>$D'$</td>
<td>$D$</td>
<td>$H$</td>
<td>$V$</td>
</tr>
<tr>
<td>$R_{270}$</td>
<td>$R_{270}$</td>
<td>$R_{90}$</td>
<td>$R_{180}$</td>
<td>$R_0$</td>
<td>$D$</td>
<td>$D'$</td>
<td>$V$</td>
<td>$H$</td>
</tr>
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<td>$H$</td>
<td>$H$</td>
<td>$V$</td>
<td>$D$</td>
<td>$D'$</td>
<td>$R_0$</td>
<td>$R_{180}$</td>
<td>$R_{90}$</td>
<td>$R_{270}$</td>
</tr>
<tr>
<td>$V$</td>
<td>$V$</td>
<td>$H$</td>
<td>$D'$</td>
<td>$D$</td>
<td>$R_{180}$</td>
<td>$R_0$</td>
<td>$R_{270}$</td>
<td>$R_{90}$</td>
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<tr>
<td>$D$</td>
<td>$D$</td>
<td>$D'$</td>
<td>$V$</td>
<td>$H$</td>
<td>$R_{270}$</td>
<td>$R_{90}$</td>
<td>$R_{0}$</td>
<td>$R_{180}$</td>
</tr>
<tr>
<td>$D'$</td>
<td>$D'$</td>
<td>$D$</td>
<td>$H$</td>
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<td>$R_{180}$</td>
<td>$R_{0}$</td>
<td>$R_{0}$</td>
<td>$R_{180}$</td>
</tr>
</tbody>
</table>
Let $H = \{1, 2, 3, 4\}$ be the normal subgroup of $A_4$. (Here $i$ denotes the permutation $\alpha_i$). Then $A_4/H = \{H, 5H, 9H\}$ where $H = \{1, 2, 3, 4\}$, $5H = \{5, 6, 7, 8\}$, and $9H = \{9, 10, 11, 12\}$.
Why normal is so important?
Let \( H = \{1, 5, 9\} \) which is not a normal subgroup of \( A_4 \).
(Here \( i \) denotes the permutation \( \alpha_i \)). Then the left cosets are
\( 1H = H = \{1, 5, 9\} \), \( 2H = \{2, 6, 10\} \), \( 3H = \{3, 7, 11\} \) and
\( 4H = \{4, 8, 12\} \). Look at the multiplication table

\[
\begin{array}{cccc|cccc|cccc}
 & 1 & 5 & 9 & 2 & 6 & 10 & 3 & 7 & 11 & 4 & 8 & 12 \\
1 & 1 & 5 & 9 & 2 & 6 & 10 & 3 & 7 & 11 & 4 & 8 & 12 \\
5 & 5 & 9 & 1 & 8 & 12 & 4 & 6 & 10 & 2 & 7 & 11 & 3 \\
9 & 9 & 1 & 5 & 11 & 3 & 7 & 12 & 4 & 8 & 10 & 2 & 6 \\
\end{array}
\]
Example (9)

Let 
\[ G = U(32) = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31\} \]
and 
\[ H = U_{16}(32) = \{1, 17\}. \]

Then \( G/H \) as an Abelian group of order \( |G|/|H| = 16/2 = 8 \).

This group is isomorphic to one of the following groups:
\( \mathbb{Z}_8 \), \( \mathbb{Z}_4 \oplus \mathbb{Z}_2 \), or \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \). Decide which one.

Proof.

- The distinct elements of \( G/H \) are:
  
  \[ 1H = \{1, 17\}, \ 3H = \{3, 19\}, \ 5H = \{5, 21\}, \ 7H = \{7, 23\} \]
  
  \[ 9H = \{9, 25\}, \ 11H = \{11, 27\}, \ 13H = \{13, 29\}, \ 15H = \{15, 31\} \]

- Since \( |3H| = 4 \), and \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) has no elements of order 4, then \( G/H \not\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \).
Proof.

Since $|7H| = |9H| = 2$ then $G/H$ has at least two elements of order 2 but $\mathbb{Z}_8$ has only one element of order 2, so $G/H \not\cong \mathbb{Z}_8$.

Therefore $G/H \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$.

Example (10)

Let $G = U(32)$ and $K = \{1, 15\}$. Then $|G/K| = |G|/|K| = 16/2 = 8$. This group is isomorphic to one of the following groups: $\mathbb{Z}_8$, $\mathbb{Z}_4 \oplus \mathbb{Z}_2$, or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Decide which one.

Note that $|3K| = 8$. 

Applications of Factor Groups

In the following example, we give another proof to show that $A_4$ Has No Subgroup of Order 6 using properties of factor groups.

**Example (11)**

Show that $A_4$ Has No Subgroup of Order 6.

**Proof.**

Assume $A_4$ has a subgroup $H$ of order 6, then $|A_4 : H| = 2$, and so $H \triangleleft A_4$. Hence, $|A_4/H| = 2$ and so by Lagrange's theorem, for each $\alpha \in A_4$, we have $\alpha^2 H = (\alpha H)^2 = H$. Hence, $\alpha^2 \in H$ for all $\alpha$ in $A_4$. Back to the multiplication table of $A_4$, we see that we have 9 distinct elements of the form $\alpha^2$ all of them are in $H$, which is a contradiction.
Theorem

Let $G$ be a group and let $Z(G)$ be the center of $G$. If $G/Z(G)$ is cyclic, then $G$ is Abelian.

Proof.

Assume $G/Z(G)$ is cyclic, so $G/Z(G) = \langle gZ(G) \rangle$ for some $g \in G$. Let $a, b \in G$. We show that $ab = ba$.

Since $a, b \in G$, then $aZ(G)$ and $bZ(G)$ are in $G/Z(G)$. Therefore, for some integers $i, j$ we have

$$aZ(G) = (gZ(G))^i = g^iZ(G), \quad bZ(G) = (gZ(G))^j = g^jZ(G)$$

Thus, $a = g^i x$ for some $x$ in $Z(G)$ and $b = g^j y$ for some $y$ in $Z(G)$, hence

$$ab = (g^i x) (g^j y) = g^i (xg^j) y = g^i (g^j x) y$$

$$= (g^i g^j) (xy) = (g^j g^i) (yx) = (g^j y) (g^i x) = ba$$
Remarks

1. If $H \leq Z(G)$ and $G/H$ is cyclic, then $G$ is Abelian.

2. The contrapositive of theorem:
   If $G$ is non-Abelian, then $G/Z(G)$ is not cyclic.

Example (12)

Let $G$ be a non Abelian group with $|G| = pq$ where $p$ and $q$ are primes, then $Z(G) = \{e\}$.

Proof.
Theorem

For any group $G$, $G/Z(G)$ is isomorphic to $\text{Inn}(G)$.

Proof.

Let $T: G/Z(G) \to \text{Inn}(G)$ given by: $T(gZ(G)) = \phi_g$ where $\phi_g$ is the inner automorphism given by: $\phi_g(x) = gxg^{-1}$ for all $x$ in $G$.

We show that $T$ is an isomorphism.

- **$T$ is well defined:** Let $gZ(G) = hZ(G)$ and we verify that $T(gZ(G)) = T(hZ(G))$ that is $\phi_g = \phi_h$.
  
  From $gZ(G) = hZ(G)$, we have that $h^{-1}g$ belongs to $Z(G)$. So, for all $x$ in $G$, $h^{-1}gx = xh^{-1}g$. Therefore, $gxg^{-1} = hxh^{-1}$ for all $x$ in $G$, and, therefore, $\phi_g = \phi_h$.

- Reversing the argument in part 1, shows that $T$ is one-to-one, as well.

- **$T$ is Onto**

- **$T$ is operation-preserving,** note that $\phi_g \phi_h = \phi_{gh}$ for all $g$ and $h$ in $G$
Example (13)

Show that $\text{Inn}(D_6)$ is isomorphic to $D_3$

Proof.
Theorem (Cauchy’s Theorem for Abelian Groups)

Let $G$ be a finite Abelian group and let $p$ be a prime that divides the order of $G$. Then $G$ has an element of order $p$.

Proof.

By the Second Principle of Mathematical Induction on $|G|$. If $|G| = 2$, it is true that $G$ has an element of order 2. Assume the result is true for all Abelian groups with order $< |G|$. Note that $G$ has an element of prime order, since for $x \in G$ and $x \neq e$ where $|x| = m$, let $q$ be a prime factor of $m$. That is $|x| = qn$, then $|x^n| = q$.

Therefore, let $x$ be an element of $G$ where $|x| = q$ for some prime $q$. If $p = q$, we are finished; so assume that $q \neq p$.

since $G$ is Abelian, then $H = \langle x \rangle \triangleleft G$, and so, we have the factor group $\overline{G} = G/H$ and $|\overline{G}| = |G|/q < |G|$.

since $p \neq q$, then $p \parallel |\overline{G}|$. By induction hypothesis, $\overline{G}$ has an element say, $y \in \langle x \rangle$ with $|y \in \langle x \rangle| = p$.

By exercise 65 of this chapter, $G$ has an element of order $p$. 

Ahmed EL-Mabhouh
Abstract Algebra I
Internal Direct Products

**Definition**

We say that $G$ is the internal direct product of $H$ and $K$ and write $G = H \times K$ if

1. $H$ and $K$ are normal subgroups of $G$
2. $G = HK = \{hk : h \in H, k \in K\}$
3. $H \cap K = \{e\}$

Note that $HK \cong H \oplus K$.

**Figure:** For the internal direct product, $H$ and $K$ must be subgroups of the same group.
Example (14)

Let $D_6 = H \times K$, where $H = \{ R_0, R_{120}, R_{240}, F, R_{120}F, R_{240}F \}$ and $K = \{ R_0, R_{180} \}$.

Example (15)

Let $G = S_3$, $H = \langle (123) \rangle$, and $K = \langle (12) \rangle$. Is $S_3 = H \times K$? Why?

Note that $H, K$ are both cyclic and $|H|, |K|$ are relatively prime, so $H \oplus K$ is cyclic but $S_3$ is not cyclic.

Therefore, $H \oplus K \nless S_3$ although $S_3 = HK$.

This is because $K$ is not normal.
Internal Direct Products

Definition

Let $H_1, H_2, \ldots, H_n$ be a finite collection of subgroups of $G$. We say that $G$ is the internal direct product of $H_1, H_2, \ldots, H_n$ and write $G = H_1 \times H_2 \times \cdots \times H_n$, if

1. $H_i \triangleleft G$ for all $i = 1, 2, \ldots, n$,
2. $G = H_1 H_2 \cdots H_n = \{h_1 h_2 \cdots h_n | h_i \in H_i\}$
3. $(H_1 H_2 \cdots H_i) \cap H_{i+1} = \{e\}$ for $i = 1, 2, \ldots, n - 1$

Theorem

If a group $G$ is the internal direct product of a finite number of subgroups $H_1, H_2, \ldots, H_n$, then $G$ is isomorphic to the external direct product of $H_1, H_2, \ldots, H_n$. That is:

$$H_1 \times H_2 \times \cdots \times H_n \cong H_1 \oplus H_2 \oplus \cdots \oplus H_n$$
Classification of Groups of Order \( p^2 \)

**Theorem**

Every group of order \( p^2 \), where \( p \) is a prime, is isomorphic to \( \mathbb{Z}_{p^2} \) or \( \mathbb{Z}_p \oplus \mathbb{Z}_p \)

**Corollary**

If \( G \) is a group of order \( p^2 \), where \( p \) is a prime, then \( G \) is Abelian.
If \( m = n_1 n_2 \cdots n_k \), where \( \gcd(n_i, n_j) = 1 \) for \( i \neq j \), then

\[
U(m) = U_{m/n_1}(m) \times U_{m/n_2}(m) \times \cdots \times U_{m/n_k}(m)
\]

\[
\approx U(n_1) \oplus U(n_2) \oplus \cdots \oplus U(n_k)
\]

**Example (16)**

\[
U(105) = U(15 \cdot 7) = U_{15}(105) \times U_7(105)
\]

\[
= \{1, 16, 31, 46, 61, 76\} \times \{1, 8, 22, 29, 43, 64, 71, 92\}
\]

\[
\approx U(7) \oplus U(15)
\]

\[
U(105) = U(3 \cdot 5 \cdot 7) = U_{35}(105) \times U_{21}(105) \times U_{15}(105)
\]

\[
= \{1, 71\} \times \{1, 22, 43, 64\} \times \{1, 16, 31, 46, 61, 76\}
\]

\[
\approx U(3) \oplus U(5) \oplus U(7)
\]