Chapter 3
Relations and Functions

This chapter discusses ordered pairs and the Cartesian product of two sets. Then it defines a relation as a set of ordered pairs. The intimate connection between a partition and an equivalence relation on a set is closely examined. The concept of a function is introduced as a special kind of relation. Moreover, the important properties of functions are studied.

3.1 Cartesian Products of Two Sets

Definition. (Ordered pairs)

(a) Given any two objects $a$ and $b$, the object $(a, b)$ is called the ordered pair $a, b$.

(b) If $(a, b)$ is an ordered pair, then $a$ is called the first coordinate and $b$ is called the second coordinate.

(c) We say that two ordered pairs $(a, b)$ and $(c, d)$ are equal, and write $(a, b) = (c, d)$, if and only if $a = c$ and $b = d$.

Remark. The adjective “ordered” here emphasizes that the order in which the objects $a$ and $b$ appear in $(a, b)$ is essential.

Example 1. If $(1, 2)$ and $(2, 1)$ are two ordered pairs, then $(1, 2) \neq (2, 1)$.

Example 2. $(x, y) = (2, 3)$ if and only if $x = 2$ and $y = 3$.

Definition. (Cartesian products)

Let $A$ and $B$ be sets. The cartesian product of $A$ and $B$, denoted by $A \times B$, is the set of all ordered pairs $(x, y)$ with $x \in A$ and $y \in B$. In symbols

$$ A \times B = \{(x, y) : x \in A \land y \in B\}. $$

Example 1. Let $A = \{1, 2, 3\}$ and let $B = \{a, b\}$. Find $A \times B$ and $B \times A$.

Solution.
Example 2. Let $A$ be any set. Find $A \times \phi$ and $\phi \times A$.

Solution.

Theorem 3.1. Let $A, B,$ and $C$ be any three sets. Then

(a) $A \times (B \cap C) = (A \times B) \cap (A \times C),$

(b) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Proof.
Theorem 3.2. Let $A$, $B$, and $C$ be any three sets. Then $A \times (B - C) = (A \times B) - (A \times C)$.

Proof.

Example 1. Prove that if $C \neq \phi$ and $A \times C \subseteq B \times C$, then $A \subseteq B$.

Solution.
Example 2. Prove that $A \times B = \emptyset$ if and only if $A = \emptyset$ or $B = \emptyset$.

Solution.

Example 3. Prove or disprove: $\wp(A \times B) = \wp(A) \times \wp(B)$.

Solution.
3.2 Relations

Given sets $A$ and $B$, not necessarily distinct, when we say that an element $a$ of $A$ is related to another element $b$ of $B$ by a relation $R$ we are making a statement about the ordered pair $(a, b)$ in the Cartesian product $A \times B$. Therefore, a mathematical definition of a relation can be precisely given in term of ordered pairs in Cartesian product of sets.

**Definition.** (Relations)
A relation $R$ from a set $A$ to a set $B$ is a subset of the Cartesian product $A \times B$.

**Notation.** If $R$ is a relation and $(a, b) \in R$, then we write $a R b$ (read: $a$ is $R$–related to $b$).

**Example 1.** Let $A = \{a, b\}$ and let $B = \{1, -1\}$. Then $R = \{(a, 1), (b, 1)\}$ is a relation from $A$ to $B$.

**Example 2.** Let $A = \{1, 2, 3\}$ and let $R = \{(1, 1), (2, 2), (1, 3)\}$. Then $R$ is a relation from $A$ to $A$.

**Remark.** If $R$ is a relation from $A$ to $A$, then $R$ is called a relation on $A$.

**Definition.** (Inverse relation)
Let $A$ and $B$ be two sets, not necessarily distinct, and let $R$ be a relation from $A$ to $B$. The inverse of the relation $R$ is the relation $R^{-1}$ from $B$ to $A$ such that $b R^{-1} a$ if and only if $a R b$. That is $R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}$.

**Example 1.** Let $A = \{1, 2\}$, $B = \{-1, 0, -2\}$, and let $R$ be a relation from $A$ to $B$ given by $R = \{(1, -1), (2, -2)\}$. Find $R^{-1}$.

**Solution.**

**Example 2.** Find $R^{-1}$ if $R = \{(n, m) \in \mathbb{N} \times \mathbb{Z} : n$ divides $m\}$.

**Solution.**

**Example 3.** Let $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x > y\}$. Find $R^{-1}$.

**Solution.**
**Definition.** (Domain and image of a relation)
Let $R$ be a relation from a set $A$ to a set $B$.

(a) The domain of $R$ is the set $\text{Dom}(R) = \{a \in A : (a, b) \in R \text{ for some } b \in B\}$.

(b) The image of $R$ is the set $\text{Im}(R) = \{b \in B : (a, b) \in R \text{ for some } a \in A\}$.

**Remark.** If $R$ is a relation from a set $A$ to a set $B$, then $\text{Dom}(R) \subseteq A$ and $\text{Im}(R) \subseteq B$.

**Example 1.** Let $R = \{(1, 1), (1, -2), (2, 5)\}$. Find $\text{Dom}(R)$ and $\text{Im}(R)$.

**Solution.**

**Example 2.** Let $R = \{(n, m) \in \mathbb{N} \times \mathbb{Z} : n \text{ divides } m\}$. Find $\text{Dom}(R)$ and $\text{Im}(R)$.

**Solution.**

**Example 3.** Prove that for any relation $R$, $\text{Dom}(R) = \text{Im}(R^{-1})$.

**Solution.**

**Example 4.** Let $R$ be a relation from $A$ to $B$ and let $D \subseteq A$. By the restriction of $R$ to $D$ we mean the relation $R|D = \{(x, y) \in R : x \in D\}$ from $D$ to $B$. Prove that $R|D = R \cap (D \times \text{Im}(R))$.

**Solution.**
Example 5. Let $R$ be a relation from $A$ to $B$ and let $X \subseteq A$. Define

$$R(X) = \{ y \in B : (x,y) \in R \text{ for some } x \in X \}.$$ 

Prove that if $D \subseteq A$ and $E \subseteq A$, then

$$R(D \cap E) \subseteq (R(D) \cap R(E)).$$

Solution.

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**Equivalence Relations**

**Definition.** (Equivalence relations)
Let $R$ be a relation on a set $X$. Then

(a) $R$ is reflexive if and only if $x R x$ for all $x \in X$,

(b) $R$ is symmetric if and only if $x R y \Rightarrow y R x$,

(c) $R$ is transitive if and only if $x R y \land y R z \Rightarrow x R z$,

(d) $R$ is an equivalence relation if and only if $R$ is reflexive, symmetric, and transitive.

**Example 1.** Let $X = \{1,2\}$ and let $R$ be a relation on $X$ given by $R = \{(1,1),(1,2),(2,2)\}$. Show that $R$ is reflexive and transitive, but not symmetric.

Solution.
Example 2. Let $X = \{a, b, c\}$ and let $R$ be a relation on $X$ given by $R = \{(a, b), (b, a), (a, c), (c, a), (b, b)\}$. Show that $R$ is symmetric but neither transitive nor reflexive.

Solution.

Example 3. Let $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : xy > 0\}$. Determine whether $R$ is reflexive, symmetric, transitive or not.

Solution.
Example 4. Let $X = \{0, 1\}$ and let $R$ be a relation on $X$ given by $R = \{(1, 1), (0, 0), (0, 1), (1, 0)\}$. Show that $R$ is an equivalence relation on $X$.

Solution.

Remark. Given a nonempty set $X$, there always exist at least two equivalence relations on $X$; the diagonal (or identity) relation $\Delta_x = \{(x, x) : x \in X\}$, and the relation $R = X \times X$.

Example 1. Let $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x - y = 3k \text{ for some } k \in \mathbb{Z}\}$. Show that $R$ is an equivalence relation on $\mathbb{Z}$.

Solution.

Example 2. Let $m$ be an arbitrary fixed positive integer and let $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x - y = mk \text{ for some } k \in \mathbb{Z}\}$. Then $R$ is an equivalence relation on $\mathbb{Z}$ and it is called the congruence relation modulo $m$ on $\mathbb{Z}$. If $(x, y) \in R$, then we write $x \equiv y$ (modulo $m$).
Example 3. Let $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x + y = 3k \text{ for some } k \in \mathbb{Z}\}$. Determine whether $R$ is reflexive, symmetric, transitive or not.

Solution.

Example 4. Let $R$ be a relation on a set $X$. Prove that $R$ is reflexive if and only if $\Delta_x \subseteq R$.

Solution.

Exercise 3.2 (1-18)
3.3 Partitions and Equivalence Relations

Definition. (Partition)
Let $X$ be a nonempty set. A partition $P$ of $X$ is a set of nonempty subsets of $X$ such that

(1) if $A, B \in P$ and $A \neq B$, then $A \cap B = \emptyset$,

(2) $\bigcup_{A \in P} A = X$.

Remark. Intuitively, a partition of a set $X$ is a cutting up of $X$ into nonempty disjoint pieces.

Example 1. Let $X = \{1, 2, 3, 4, 5\}$ and let $P = \{\{1, 5\}, \{2, 3\}, \{4\}\}$. Is $P$ a partition of $X$? Explain.
Solution.

Example 2. Let $X = \mathbb{Z}$ and let $E = \{x \in \mathbb{Z} : x \text{ is even}\}$, $D = \{x \in \mathbb{Z} : x \text{ is odd}\}$. Then $P = \{E, D\}$ is a partition of $X$.

Example 3. Let $X = \mathbb{Z}$ and for $j = 0, 1, 2$ define $Z_j = \{x \in \mathbb{Z} : x - j = 3k \text{ for some } k \in \mathbb{Z}\}$. Then $P = \{Z_0, Z_1, Z_2\}$ is a partition of $\mathbb{Z}$.

Remark. There is a close connection between the partition of a nonempty set and an equivalence relation on the set.

Definition. (Equivalence classes)
Let $\mathcal{E}$ be an equivalence relation on a nonempty set $X$.

(1) For each $x \in X$, we define the equivalence class determined by $x$ to be the set $x/\mathcal{E} = \{y \in X : (x, y) \in \mathcal{E}\}$.

(2) The set of all equivalence classes in $X$ is called $X$ modulo $\mathcal{E}$ and it is denoted by $X/\mathcal{E}$. In symbols: $X/\mathcal{E} = \{x/\mathcal{E} : x \in X\}$.

Remark. If $\mathcal{E}$ is an equivalence relation on a nonempty set $X$, then every equivalence class $x/\mathcal{E}$ is a subset of $X$.

Example 1. Let $X = \{1, 2\}$ and let $\mathcal{E} = \{(1, 1), (2, 2), (1, 2), (2, 1)\}$. Find $X/\mathcal{E}$.
Example 2. Let $X = \{1, 2, 3\}$ and let $E = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}$. Find $X/E$.

Solution.

Example 3. Let $\Box = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x^2 = y^2\}$. Show that $\Box$ is an equivalence relation on $\mathbb{Z}$ and Find $\mathbb{Z}/\Box$.

Solution.
Theorem 3.3. (Properties of equivalence classes)

Let $\mathcal{E}$ be an equivalence relation on a nonempty set $X$. Then

(a) each equivalence class $x/\mathcal{E}$ is a nonempty subset of $X$,

(b) $x/\mathcal{E} \cap y/\mathcal{E} \neq \emptyset$ if and only if $(x, y) \in \mathcal{E}$,

(c) $(x, y) \in \mathcal{E}$ if and only if $x/\mathcal{E} = y/\mathcal{E}$.

Proof.
**Theorem 3.4.** \((X/\mathcal{E} \text{ is a partition of } X)\)

Let \(\mathcal{E}\) be an equivalence relation on a nonempty set \(X\). Then \(X/\mathcal{E}\) is a partition of \(X\).

**Proof.**

**Example 1.** Let \(X = \{1, 2, 3, 4\}\) and let \(\mathcal{E} = \{(1, 1), (2, 2), (3, 3), (4, 4), (2, 3), (3, 2)\}\). Find \(X/\mathcal{E}\).

**Solution.**

**Example 2.** Let \(X = \mathbb{Z}\) and let \(\mathcal{E} = \{(x, y) : x - y = 4k \text{ for some } k \in \mathbb{Z}\}\). Find \(X/\mathcal{E}\).

**Solution.**

**Remark.** Theorem 3.4 shows that an equivalence relation on \(X \neq \emptyset\) gives rise to a partition on \(X\). We shall show that the converse is also true.
Relations Induced by Partitions

**Definition.** (Relations induced by partitions)
Let $\mathcal{P}$ be a partition of a nonempty set $X$. We define a relation $X/\mathcal{P}$ on $X$ by $(x,y) \in X/\mathcal{P}$ if and only if there exists a set $A \in \mathcal{P}$ such that $x$ and $y$ are in $A$. That is

$$X/\mathcal{P} = \{(x,y) : x \in A \land y \in A \text{ for some } A \in \mathcal{P}\}.$$

**Example 1.** Let $X = \{0,1,2\}$ and let $\mathcal{P} = \{\{0,2\}, \{1\}\}$. Find the relation $X/\mathcal{P}$. Is $X/\mathcal{P}$ an equivalence relation?

**Solution.**

**Example 2.** Let $X = \mathbb{Z}$ and let $\mathcal{P} = \{E,D\}$. Find the relation $X/\mathcal{P}$. Is $X/\mathcal{P}$ an equivalence relation?

**Solution.**

**Theorem 3.5.** ($X/\mathcal{P}$ is an equivalence relation)
Let $\mathcal{P}$ be a partition of a nonempty set $X$. Then the relation $X/\mathcal{P}$ is an equivalence relation on $X$.

**Proof.**
Lemma 3.1. \((x/Q = A)\)

Let \(\mathcal{P}\) be a partition of a nonempty set \(X\) and let \(Q = X/\mathcal{P}\). If \(x \in A\) for some \(A \in \mathcal{P}\), then \(x/Q = A\).

\[\text{Proof.}\]

\[\square\]

Theorem 3.6. \((X/(X/\mathcal{P}) = \mathcal{P})\)

Let \(\mathcal{P}\) be a partition of a nonempty set \(X\). Then the equivalence classes induced by the relation \(X/\mathcal{P}\) are precisely the sets in \(\mathcal{P}\). Symbolically \(X/(X/\mathcal{P}) = \mathcal{P}\).
Proof.

Example 3. Let $X = \{1, 2, 3, 4\}$ and consider the partition $P = \{\{1\}, \{2, 3, 4\}\}$. Find $X/P$ and $X/(X/P)$.

Solution.

Exercise 3.3 (1-11)

Additional Exercises

1. Let $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x + y \text{ is even}\}$. Prove that $R$ is an equivalence relation and determine the distinct equivalence classes.

2. Prove that $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : 3x - 7y = 2k \text{ for some } k \in \mathbb{Z}\}$ is an equivalence relation and find $\mathbb{Z}/R$.

3. Let $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : 3x + 5y = 2k \text{ for some } k \in \mathbb{Z}\}$. Prove that $R$ is an equivalence relation and find $\mathbb{Z}/R$.

4. Let $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : 3x + 7y = 2k \text{ for some } k \in \mathbb{Z}\}$. Determine whether $R$ is an equivalence relation or not.

5. Let $R = \{(x, y) \in \mathbb{Q} \times \mathbb{Q} : x + y = k \text{ for some } k \in \mathbb{Z}\}$. Prove or disprove that $R$ is an equivalence relation.
3.4 Functions

The concept of function is one of the most basic ideas in every branch of mathematics.

Definition. (Functions)
Let $X$ and $Y$ be sets. A function from $X$ to $Y$ is a triple $(f, X, Y)$, where $f$ is a relation from $X$ to $Y$ satisfying

1. $\text{Dom}(f) = X$,
2. if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

Example 1. Let $X = \{1, -1\}$, $Y = \{0\}$, and $f = \{(1, 0), (-1, 0)\}$. Is $f$ a function?

Solution.

Example 2. Let $X = \{1, 2, 3\}$, $Y = \{5, 10, 15, 20\}$, and $f = \{(1, 5), (2, 20)\}$. Is $f$ a function?

Solution.

Example 3. Let $X = \{1, 2, 3\}$, $Y = \{5, 10, 15, 20\}$, and $f = \{(1, 5), (2, 20), (3, 15), (3, 10)\}$. Is $f$ a function?

Solution.

Notation.

1. If $(f, X, Y)$ is a function from $X$ to $Y$, then we write $f : X \rightarrow Y$.
2. If $(f, X, Y)$ is a function from $X$ to $Y$ and $(x, y) \in f$, then we write $y = f(x)$.

Definition. (Image and pre-image)
Let $f : X \rightarrow Y$ be a function. If $y = f(x)$, then we say that $y$ is the image of $x$ under $f$ and $x$ is a pre-image of $y$ under $f$.

Definition. (Range)
If $f : X \rightarrow Y$ is a function, then $Y$ is called the range of $f$. 
Remark. Note that \( \text{Im}(f) = \{ y \in Y : (x, y) \in f \text{ for some } x \in X \} \subseteq \text{range of } f \). In general, range of \( f \neq \text{Im}(f) \).

Example. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be defined by \( f(x) = |x| = \) the greatest integer \( \leq x \). Find \( \text{Range}(f) \) and \( \text{Im}(f) \).

Solution.

**Theorem 3.6.** (Each set containing the image can be a range) Let \( f : X \rightarrow Y \) be a function and let \( W \) be a set containing the image of \( f \); that is \( \text{Im}(f) \subseteq W \). Then \( f : X \rightarrow W \) is a function.

**Proof.** Exercise

Example. Let \( f(x) = |x| = \) the greatest integer less than or equal to \( x \). Then \( f : \mathbb{R} \rightarrow \mathbb{R}, f : \mathbb{R} \rightarrow \mathbb{Q}, \) and \( f : \mathbb{R} \rightarrow \mathbb{Z} \)

**Theorem 3.7.** (Equality of functions)
Let \( f : X \rightarrow Y \) and \( g : X \rightarrow Y \) be functions. Then \( f = g \) if and only if \( f(x) = g(x) \) for all \( x \in X \).

**Proof.**

Example. Let \( X = \{1, 2, 3\} \) and let \( Y = \{-1, -2, -3\} \). Define \( f : X \rightarrow Y \) by \( f = \{(1, -1), (2, -2), (3, -3)\} \) and \( g : X \rightarrow Y \) by \( g(x) = -x \). Is \( f = g \)?

Solution.
Some special functions

Example 1. Let $X$ be a nonempty set. Then the diagonal relation $\Delta_x$ on $X$ is a function from $X$ to $X$. It is called the identity function on $X$.

Example 2. Let $X \neq \emptyset$ and $Y \neq \emptyset$ be two sets and let $b$ be any fixed element of $Y$. Then the relation $C_b = \{(x, b) : x \in X\}$ is a function from $X$ to $Y$. $C_b$ is called a constant function.

Example 3. Let $A$ be a subset of a nonempty set $X$ and define
$$
\chi_A = \{(x, y) \in X \times \{0, 1\} : y = 1 \text{ if } x \in A \land y = 0 \text{ if } x \notin A\};
$$
that is
$$
\chi_A(x) = \begin{cases} 
1, & \text{if } x \in A \\
0, & \text{if } x \in X - A.
\end{cases}
$$
Then $\chi_A$ is a function from $X$ to $\{0, 1\}$ and it is called the characteristic function of $A$ in $X$.

Example 4. Let $X$ and $Y$ be two sets. Then the $X$-projection function $p_x : X \times Y \to X$ is given by $p_x(x, y) = x$. The $Y$-projection function $p_y : X \times Y \to Y$ is given by $p_y(x, y) = y$.

Unions of functions

Example 1. Let $f : \{1, 2\} \to \{0\}$ be given by $f = \{(1, 0), (2, 0)\}$ and let $g : \{2, 3\} \to \{5\}$ be given by $g = \{(2, 5), (3, 5)\}$. Find $f \cup g$. Is $f \cup g$ a function?

Solution.

Example 2. Let $f : \{1, 2\} \to \{1, 4\}$ be given by $f = \{(1, 1), (2, 4)\}$ and let $g : \{3, 4\} \to \{5\}$ be given by $g = \{(3, 5), (4, 5)\}$. Find $f \cup g$. Is $f \cup g$ a function?

Solution.

Example 3. Let $f : (-\infty, 1) \to \mathbb{R}$ be given by $f(x) = 2x + 1$ and let $g : [1, \infty) \to \mathbb{R}$ be given by $g(x) = x - 2$. Find $f \cup g$. Is $f \cup g$ a function?
Solution.

**Theorem 3.8. (The union of two functions)**

Let \( f : A \rightarrow C \) and \( g : B \rightarrow D \) be two functions such that \( f(x) = g(x) \), \( \forall x \in A \cap B \). Then the union \( h \) of \( f \) and \( g \) define the function \( h = f \cup g : A \cup B \rightarrow C \cup D \), where

\[
\begin{align*}
  h(x) &= \begin{cases} 
    f(x), & \text{if } x \in A \\
    g(x), & \text{if } x \in B.
  \end{cases}
\end{align*}
\]

*Proof.*

\( \square \)
Example. Let \( f : (-\infty, 0] \to \mathbb{R} \) be given by \( f(x) = -x^2 \) and let \( g : [0, \infty) \to \mathbb{R} \) be given by \( g(x) = x^2 \). Find \( f \cup g \). Is \( f \cup g \) a function?

Solution.

Exercise 3.4 (1-17)
3.5 Images and Inverse Images of Sets

The concept of image and preimage of an element can be extended to subsets.

**Definition.** (Images and inverse images of sets)
Let \( f : X \rightarrow Y \) be a function, and let \( A \) and \( B \) be subsets of \( X \) and \( Y \) respectively.

1. The image of \( A \) under \( f \), denoted by \( f(A) \), is the set
   \[
   f(A) = \{ y \in Y : y = f(x) \text{ for some } x \in A \} = \{ f(x) : x \in A \}.
   \]

2. The inverse image of \( B \) under \( f \), denoted by \( f^{-1}(B) \), is the set
   \[
   f^{-1}(B) = \{ x \in X : y = f(x) \text{ for some } y \in B \} = \{ x \in X : f(x) \in B \}.
   \]

**Remarks.**

1. \( f^{-1} \) in the above definition does NOT denote the inverse function of the function \( f \).

2. If \( f : X \rightarrow Y \), then \( f(X) = \operatorname{Im}(f) \) and \( f^{-1}(Y) = \operatorname{Dom}(f) \).

**Example 1.** Let \( f = \{(0, 5), (1, 6), (2, 7), (3, 5), (4, 5)\} \) be a function from \( X = \{0, 1, 2, 3, 4\} \) to \( Y = \{5, 6, 7\} \). Find \( f(A) \) and \( f^{-1}(B) \) if \( A = \{0, 1, 2\} \) and \( B = \{5, 6\} \).

**Solution.**

**Example 2.** Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be defined by \( f(x) = x^2 \). Find \( f([-1, 1]) \) and \( f^{-1}([-2, 0]) \).

**Solution.**

**Remarks.**

1. \( x \in A \Rightarrow f(x) \in f(A) \) but \( f(x) \in f(A) \nRightarrow x \in A \).

2. \( x \in f^{-1}(B) \Leftrightarrow f(x) \in B \).
Example. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$, and let $A = \{2\}$. Then $f(A) = \{4\}$. Thus $f(-2) = 4 \in f(A)$ but $-2 \notin A$.

Theorem 3.9. (Basic properties of images and inverse images)
Let $f : X \to Y$ be a function. Then

(a) $f(\emptyset) = \emptyset$,

(b) $f(\{a\}) = \{f(a)\}$, $\forall a \in X$,

(c) if $A \subseteq B \subseteq X$, then $f(A) \subseteq f(B)$,

(d) if $C \subseteq D \subseteq Y$, then $f^{-1}(C) \subseteq f^{-1}(D)$.

Proof.
Example. Let $f : X \to Y$ be a function such that $f(X) = Y$, and let $B$ and $C$ be subsets of $Y$. Prove that if $f^{-1}(B) = f^{-1}(C)$, then $B = C$.

Solution.

**Theorem 3.10.** (Images of unions and intersections)
Let $f : X \to Y$ be a function and let $\{A_\gamma : \gamma \in \Gamma\}$ be a family of subsets of $X$. Then

(a) $f \left( \bigcup_{\gamma \in \Gamma} A_\gamma \right) = \bigcup_{\gamma \in \Gamma} f(A_\gamma)$,

(b) $f \left( \bigcap_{\gamma \in \Gamma} A_\gamma \right) \subseteq \bigcap_{\gamma \in \Gamma} f(A_\gamma)$.

Proof.
Example. Let \( X = \{1, 2\} \), \( Y = \{3\} \), \( A = \{1\} \), \( B = \{2\} \), and let \( f : X \to Y \) be given by \( f(1) = f(2) = 3 \). Then \( f(A \cap B) = f(\phi) = \phi \) but \( f(A) \cap f(B) = \{3\} \). Thus \( f(A) \cap f(B) \not\subset f(A \cap B) \).

Theorem 3.11. (Inverse images of unions and intersections)
Let \( f : X \to Y \) be a function and let \( \{B_\gamma : \gamma \in \Gamma\} \) be a family of subsets of \( Y \). Then

\[
\begin{align*}
(a) \quad f^{-1}\left( \bigcup_{\gamma \in \Gamma} B_\gamma \right) &= \bigcup_{\gamma \in \Gamma} f^{-1}(B_\gamma), \\
(b) \quad f^{-1}\left( \bigcap_{\gamma \in \Gamma} B_\gamma \right) &= \bigcap_{\gamma \in \Gamma} f^{-1}(B_\gamma).
\end{align*}
\]

Proof. \(\square\)
**Theorem 3.12.** (Inverse images of differences)

Let \( f : X \to Y \) be a function and let \( B \) and \( C \) be any subsets of \( Y \). Then \( f^{-1}(B - C) = f^{-1}(B) - f^{-1}(C) \).

**Proof.**

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**Exercise 3.5 (1-15)**
3.6 Injective, Surjective, and Bijective Functions

Injective, Surjective, و Bijective Temperatur

Definition. (Injection)

1. A function \( f : X \to Y \) is said to be injective or one-to-one provided that if \( x_1, x_2 \in X \) with \( f(x_1) = f(x_2) \), then \( x_1 = x_2 \).

2. An injective function is called an injection.

Remark. \( f : X \to Y \) is injective \( \iff (\forall x_1)(\forall x_2)[f(x_1) = f(x_2) \implies x_1 = x_2] \)

Example 1. Let \( X \subseteq Y \) and \( f : X \to Y \) be a function given by \( f(x) = x \). Show that \( f \) is an injective function.

Solution.

Example 2. Let \( f : Z \to \mathbb{N} \cup \{0\} \) be a function given by \( f(n) = n^2 \). Show that \( f \) is not an injection.

Solution.

Example 3. Let \( f : X \to Y \) be a function, and let \( A \subseteq X \). Prove that if \( f \) is injective, then \( f^{-1}(f(A)) = A \).

Solution.
**Definition.** (Surjection)

(1) A function \( f : X \to Y \) is said to be surjective or onto provided that \( \forall y \in Y \) there exists at least one \( x \in X \) such that \( f(x) = y \). In other words, \( f : X \to Y \) is surjective if and only if \( f(X) = Y = \text{Im}(f) \).

(2) A surjective function is called a surjection.

**Example 1.** Let \( f : \mathbb{R} \to [0, \infty) \) be a function given by \( f(x) = x^2 \). Show that \( f \) is a surjective function.

**Solution.**

**Example 2.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a function given by \( f(x) = x^2 \). Then \( f \) is not a surjective function since \( f(x) \neq -1 \) for all \( x \in \mathbb{R} \).

**Example 3.** Prove that the \( X \)-projection function \( p_x : X \times Y \to X \) is surjective.

**Solution.**

**Example 4.** Let \( f : X \to Y \) be a function and let \( B \subseteq Y \). Prove that if \( f \) is surjective, then \( f(f^{-1}(B)) = B \).

**Solution.**
Definition. (Bijection)

1. A function $f : X \rightarrow Y$ is said to be bijective if it is both injective and surjective.

2. A bijection is also called a one-to-one correspondence.

Example 1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function given by $f(x) = 2x + 3$. Show that $f$ is a bijective function.

Solution.

Example 2. Let $f : X \rightarrow Y$ be an injection and let $A_1, A_2$ be subsets of $X$. Show that

$$f(A_1) \cap f(A_2) \subseteq f(A_1 \cap A_2).$$

Solution.
Theorem 3.13. *(Image of intersections under injective functions)*

Let \( f : X \to Y \) be an injection and let \( \{ A_\gamma : \gamma \in \Gamma \} \) be a family of subsets of \( X \). Then

\[
f \left( \bigcap_{\gamma \in \Gamma} A_\gamma \right) = \bigcap_{\gamma \in \Gamma} f(A_\gamma).
\]

**Proof.**

**Example.** Let \( X = \{a, b, c, d\} \), \( Y = \{1, 2, 3, 4\} \), and consider the function \( f = \{(a, 1), (b, 2), (c, 1), (d, 4)\} \).

Find \( f^{-1} \). Is \( f^{-1} \) a function?

**Solution.**
Theorem 3.14. (The inverse of a bijection is a bijective function)
Let \( f : X \rightarrow Y \) be a bijection. Then \( f^{-1} : Y \rightarrow X \) is a bijection.

Proof.
3.7 Composition of Functions

Definition. (Composition of functions)
Let \( f : X \to Y \) and \( g : Y \to Z \) be two functions. The composition of these two functions is the function \( g \circ f : X \to Z \) defined by \((g \circ f)(x) = g(f(x))\). In other words,
\[
g \circ f = \{(x, z) \in X \times Z : \exists y \in Y \text{ such that } (x, y) \in f \land (y, z) \in g\}
\]

Example. Let \( f : \mathbb{R} \to \mathbb{R} \) and let \( g : \mathbb{R} \to \mathbb{R} \) be two functions given by \( f(x) = x^2 + 2 \) and \( g(x) = \sqrt{x^2 + 4} \). Find \((g \circ f)(x)\) and \((f \circ g)(x)\).

Solution.

Remark. Function composition is NOT commutative; that is, in general, \( f \circ g \neq g \circ f \).

Theorem 3.15. (Composition is associative)
Function composition is associative; that is, if \( f : X \to Y \), \( g : Y \to Z \) and \( h : Z \to W \), then \((h \circ g) \circ f = h \circ (g \circ f)\).

Proof.
Theorem 3.16. (Criterion for injection and surjection)
Let $f : X \rightarrow Y$ be a function.

(a) If there exists a function $g : Y \rightarrow X$ such that $g \circ f = I_X$, where $I_X : X \rightarrow X$ is the identity function on $X$, then $f$ is injective.

(b) If there exists a function $h : Y \rightarrow X$ such that $f \circ h = I_Y$, where $I_Y : Y \rightarrow Y$ is the identity function on $Y$, then $f$ is surjective.

Proof.

Example. Let $f : X \rightarrow Y$ be a function. Prove that $f \circ I_X = f = I_Y \circ f$.

Solution.

Exercise 3.7 (1-12)