What is Optimization?

The Optimization Problem is:

Find values of the variables that minimize or maximize the objective function while satisfying the constraints.
Components of Optimization Problem

Optimization problems are made up of three basic ingredients:

- An **objective function** which we want to minimize or maximize
- A set of **unknowns** or **variables** which affect the value of the objective function
- A set of **constraints** that allow the unknowns to take on certain values but exclude others
Optimization Tree

- Global Optimization
  - Nondifferentiable Optimization
    - Nonlinear Least Squares
    - Nonlinear Equations
    - Unconstrained
      - Continuous
    - Constrained
      - Linear Programming
      - Network Programming
      - Stochastic Programming
    - Bound Constrained
      - Integer Programming
      - Discrete
A linear programming problem is one in which we are to find the maximum or minimum value of a linear expression

$$ax + by + cz + \ldots$$

(called the objective function), subject to a number of linear constraints of the form

$$Ax + By + Cz + \ldots N$$

or

$$Ax + By + Cz + \ldots N.$$  

The largest or smallest value of the objective function is called the optimal value, and a collection of values of $x, y, z, \ldots$ that gives the optimal value constitutes an optimal solution. The variables $x, y, z, \ldots$ are called the decision variables.
LP Properties and Assumptions

**PROPERTIES OF LINEAR PROGRAMS**

1. One objective function
2. One or more constraints
3. Alternative courses of action
4. Objective function and constraints are linear

**ASSUMPTIONS OF LP**

1. Certainty
2. Proportionality
3. Additivity
4. Divisibility
5. Nonnegative variables
Basic Assumptions of LP

• We assume conditions of *certainty* exist and numbers in the objective and constraints are known with certainty and do not change during the period being studied.
• We assume *proportionality* exists in the objective and constraints:
  - constancy between production increases and resource utilization – if 1 unit needs 3 hours then 10 require 30 hours.
• We assume *additivity* in that the total of all activities equals the sum of the individual activities.
• We assume *divisibility* in that solutions need not be whole numbers.
• All answers or variables are *nonnegative* as we are dealing with real physical quantities.
Example

Find the maximum value of
\[ p = 3x + 2y + 4z \]
Subject to:
\[ 4x + 3y + z \geq 3 \]
\[ x + 2y + z \geq 4 \]
\[ x \geq 0, \ y \geq 0, \ z \geq 0 \]
Formulating LP Problems

- Formulating a linear program involves developing a mathematical model to represent the managerial problem.
- The steps in formulating a linear program are:
  1. Completely understand the managerial problem being faced.
  2. Identify the objective and constraints.
  3. Define the decision variables.
  4. Use the decision variables to write mathematical expressions for the objective function and the constraints.
Formulating LP Problems

- One of the most common LP applications is the *product mix problem*
- Two or more products are produced using limited resources such as personnel, machines, and raw materials
- The profit that the firm seeks to maximize is based on the profit contribution per unit of each product
- The company would like to determine how many units of each product it should produce so as to maximize overall profit given its limited resources
The Flair Furniture Company produces inexpensive tables and chairs.

Processes are similar in that both require a certain amount of hours of carpentry work and in the painting and varnishing department.

Each table takes 4 hours of carpentry and 2 hours of painting and varnishing.

Each chair requires 3 hours of carpentry and 1 hour of painting and varnishing.

There are 240 hours of carpentry time available and 100 hours of painting and varnishing.

Each table yields a profit of $70 and each chair a profit of $50.
The company wants to determine the best combination of tables and chairs to produce to reach the maximum profit.

<table>
<thead>
<tr>
<th>DEPARTMENT</th>
<th>HOURS REQUIRED TO PRODUCE 1 UNIT</th>
<th>AVAILABLE HOURS THIS WEEK</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(T) TABLES  (C) CHAIRS</td>
<td></td>
</tr>
<tr>
<td>Carpentry</td>
<td>4  3</td>
<td>240</td>
</tr>
<tr>
<td>Painting and varnishing</td>
<td>2  1</td>
<td>100</td>
</tr>
<tr>
<td>Profit per unit</td>
<td>$70  $50</td>
<td></td>
</tr>
</tbody>
</table>
The objective is to

Maximize profit

The constraints are

1. The hours of carpentry time used cannot exceed 240 hours per week
2. The hours of painting and varnishing time used cannot exceed 100 hours per week

The decision variables representing the actual decisions we will make are

\[ T = \text{number of tables to be produced per week} \]
\[ C = \text{number of chairs to be produced per week} \]
We create the LP objective function in terms of $T$ and $C$

Maximize profit = $70T + 50C$

Develop mathematical relationships for the two constraints

For carpentry, total time used is

(4 hours per table)(Number of tables produced) + (3 hours per chair)(Number of chairs produced)

We know that

Carpentry time used $\leq$ Carpentry time available

$4T + 3C \leq 240$ (hours of carpentry time)
Flair Furniture Company

- Similarly
  - Painting and varnishing time used
    \[ \leq \text{Painting and varnishing time available} \]
  - \[ 2T + 1C \leq 100 \] (hours of painting and varnishing time)
  - This means that each table produced requires two hours of painting and varnishing time

- Both of these constraints restrict production capacity and affect total profit
The values for $T$ and $C$ must be nonnegative

$T \geq 0$ (number of tables produced is greater than or equal to 0)

$C \geq 0$ (number of chairs produced is greater than or equal to 0)

The complete problem stated mathematically

Maximize profit = $70T + 50C$

subject to

$4T + 3C \leq 240$ (carpentry constraint)

$2T + 1C \leq 100$ (painting and varnishing constraint)

$T, C \geq 0$ (nonnegativity constraint)
Max $70T + 50C$

Subject to

$4T + 3C \leq 240$

$2T + 1C \leq 100$

$T \geq 0$

$C \geq 0$

Global optimal solution found.

Objective value: 4100.00

Infeasibilities: 0.00000

Total solver iterations: 2

Model Class: LP

Total variables: 2

Nonlinear variables: 0

Integer variables: 0

Total constraints: 5

Nonlinear constraints: 0

Total nonzeros: 8

Nonlinear nonzeros: 0

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>Reduced Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>30.00000</td>
<td>0.000000</td>
</tr>
<tr>
<td>C</td>
<td>40.00000</td>
<td>0.000000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Row</th>
<th>Slack or Surplus</th>
<th>Dual Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4100.000</td>
<td>1.000000</td>
</tr>
<tr>
<td>2</td>
<td>0.000000</td>
<td>15.000000</td>
</tr>
<tr>
<td>3</td>
<td>0.000000</td>
<td>5.000000</td>
</tr>
<tr>
<td>4</td>
<td>30.00000</td>
<td>0.000000</td>
</tr>
<tr>
<td>5</td>
<td>40.00000</td>
<td>0.000000</td>
</tr>
</tbody>
</table>
Setting Up the LP Problem

Problem Statement. The following problem is developed from the area of chemical or petroleum engineering. However, it is relevant to all areas of engineering that deal with producing products with limited resources.

Suppose that a gas-processing plant receives a fixed amount of raw gas each week. The raw gas is processed into two grades of heating gas, regular and premium quality. These grades of gas are in high demand (that is, they are guaranteed to sell) and yield different profits to the company. However, their production involves both time and on-site storage constraints. For example, only one of the grades can be produced at a time, and the facility is open for only 80 hr/week. Further, there is limited on-site storage for each of the products. All these factors are listed below (note that a metric ton, or tonne, is equal to 1000 kg):

<table>
<thead>
<tr>
<th>Resource</th>
<th>Regular</th>
<th>Premium</th>
<th>Resource Availability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Raw gas</td>
<td>7 m³/tonne</td>
<td>11 m³/tonne</td>
<td>77 m³/week</td>
</tr>
<tr>
<td>Production time</td>
<td>10 hr/tonne</td>
<td>8 hr/tonne</td>
<td>80 hr/week</td>
</tr>
<tr>
<td>Storage</td>
<td>9 tonnes</td>
<td>6 tonnes</td>
<td></td>
</tr>
<tr>
<td>Profit</td>
<td>150/tonne</td>
<td>175/tonne</td>
<td></td>
</tr>
</tbody>
</table>

Develop a linear programming formulation to maximize the profits for this operation.
Maximize $Z = 150x_1 + 175x_2$ (maximize profit)

subject to

$7x_1 + 11x_2 \leq 77$  \hspace{1cm} \text{(material constraint)}

$10x_1 + 8x_2 \leq 80$  \hspace{1cm} \text{(time constraint)}

$x_1 \leq 9$  \hspace{1cm} \text{("regular" storage constraint)}

$x_2 \leq 6$  \hspace{1cm} \text{("premium" storage constraint)}

$x_1, x_2 \geq 0$  \hspace{1cm} \text{(positivity constraints)}
Example

Two Quarrying Sites Company

A Quarrying Company owns two different rock sites that produce aggregate which, after being crushed, is graded into three classes: high, medium and low-grade. The company has contracted to provide a concrete batching plant with 12 tons of high-grade, 8 tons of medium-grade and 24 tons of low-grade aggregate per week. The two sites have different operating characteristics as detailed below.

<table>
<thead>
<tr>
<th>Site</th>
<th>Cost per day ($'000)</th>
<th>Production (tons/day)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>High</td>
</tr>
<tr>
<td>X</td>
<td>180</td>
<td>6</td>
</tr>
<tr>
<td>Y</td>
<td>160</td>
<td>1</td>
</tr>
</tbody>
</table>

How many days per week should each Site be operated to fulfill the batching plant contract?
Example 2: Solution of the Two Quarrying Sites

Translate the verbal description into an equivalent mathematical description.
Determine:
- Variables
- Constraints
- Objective

Formulating the problem (mathematical representation of the problem).

(1) Variables
These represent the "decisions that have to be made" or the "unknowns".
Let
x = number of days per week Site X is operated
y = number of days per week Site Y is operated
Note here that x >= 0 and y >= 0.
Solution of the Two Quarrying Sites (cont.)

(2) Constraints
It is best to first put each constraint into words and then express it in a mathematical form.

*Aggregate production constraints*
balance the amount produced with the quantity required under the batching plant contract

Aggregate

- **High**: \(6x + 1y \geq 12\)
- **Medium**: \(3x + 1y \geq 8\)
- **Low**: \(4x + 6y \geq 24\)

*Days per week constraint*
we cannot work more than a certain maximum number of days a week e.g. for a 5 day week we have:

\[ x \leq 5 \quad y \leq 5 \]

Constraints of this type are often called implicit constraints because they are implicit in the definition of the variables.
(3) Objective
Again in words our objective is (presumably) to minimize cost which is given by $180x + 160y$
Hence we have the complete mathematical representation of the problem as:

\[
\text{minimize} \quad 180x + 160y \\
\text{Subject to} \\
6x + y \geq 12 \\
3x + y \geq 8 \\
4x + 6y \geq 24 \\
x \leq 5 \\
y \leq 5 \\
x, y \geq 0
\]
Case 1: Optimization of Well Treatment

Problem Statement:

Three wells in Gaza are used to pump water for domestic use. The maximum discharge of the wells as well as the properties of water pumped are shown in the table below. The water is required to be treated using chlorine before pumped into the system.

Determine the minimum cost of chlorine treatment per cubic meter per day for the three wells given the cost in $/m3 for the three wells. Total discharge should be equal to 5000 m3/day.
# PROPERTIES OF PUMPED WATER

<table>
<thead>
<tr>
<th>Well No.</th>
<th>Cl (mg/l)</th>
<th>Qmax (m³/hr.)</th>
<th>Cl treatment ($/m³)</th>
<th>max working hours</th>
</tr>
</thead>
<tbody>
<tr>
<td>Well No. 1</td>
<td>200</td>
<td>100</td>
<td>0.05</td>
<td>20</td>
</tr>
<tr>
<td>Well No. 2</td>
<td>500</td>
<td>150</td>
<td>0.12</td>
<td>18</td>
</tr>
<tr>
<td>Well No. 3</td>
<td>300</td>
<td>200</td>
<td>0.08</td>
<td>15</td>
</tr>
</tbody>
</table>
Mathematical Model

• Minimize Cost of Chlorine Treatment per Day

Min \[0.05Q_1 + 0.12Q_2 + 0.08Q_3\]

Subject to:

Well 1: max discharge should be equal to or less than 2000 m\(^3\)/day.
(100 m\(^3\)/hr x 20hr/day)
Q\(_1\) \(\leq\) 2000 ............................... (1)

Well 2: max discharge should be equal to or less than 2700 m\(^3\)/day.
(150 m\(^3\)/hr x 18hr/day)
Q\(_2\) \(\leq\) 2700 ............................... (2)

Well 3: max discharge should be equal to or less than 1800 m\(^3\)/day.
(120 m\(^3\)/hr x 15hr/day)
Q\(_3\) \(\leq\) 1800 ............................... (3)
Wells 1, 2, 3:

- total discharge should be equal to 5000 m$^3$/day.
- \[ Q_1 + Q_2 + Q_3 = 5000 \] .................................... (4)
- discharge of each well should be greater than 0.
- \[ Q_1, Q_2, Q_3 > 0 \] ........................................... (5)
Concrete is to be proportioned for a target compressive strength of 420 kg/cm² at the age of 28 days. The slump should not be less than 5 cm. Max. W/C allowed is 0.5. The aggregate max. particle size is 2.5 cm and F.M. should be less than 5.9. The unit weight of fresh concrete is estimated to be 2400 kg/m³. The aggregate content is estimated to be approx. 1800 kg/m³.

Calculate the concrete composition that fulfills the preceding requirements at minimum cost of ingredients.
## Materials Properties

<table>
<thead>
<tr>
<th>CEMENT</th>
<th>FINE AGGR.</th>
<th>COARSE AGGR.</th>
<th>WATER</th>
</tr>
</thead>
<tbody>
<tr>
<td>Portland Cement Type I</td>
<td>Natural Quartz F.M.=2.8</td>
<td>Gravel, Dmax=2.5cm F.M.=7.2</td>
<td>Tap</td>
</tr>
<tr>
<td>Cost= 120$/ton</td>
<td>Cost= $8/ton</td>
<td>Cost= $13/ton</td>
<td>Cost= $0.3/ton</td>
</tr>
</tbody>
</table>
Mathematical Model

• Minimize Cost of Concrete Composition per m³

Objective Function: \[ C = 0.12C + 0.008S + 0.013G + 0.0003W \]

• Constraints

Subject to:

Strength: \[ f'c = 230(C/W - 0.5) \text{ should be greater than } 420 \text{ kg/cm}^2 \]

\[ 230(C/W - 0.5) \geq 420 \]
\[ c - 2.346w \geq 0 \] ........................... (1)

Water- Cement ratio:

\[ W/C \leq 0.5 \] ................................. (2)

Consistency: \[ W = 0.072C + 20.43(11-F.M.) \text{ for } 5\text{cm slump}, \]

Grading: \[ \text{where } F.M. = (2.8S + 7.2G)/1800, \]
\[ F.M. \leq 5.9 \]
(2.80S + 7.20G)/1800 ≤ 5.9  ........................................ (3)

**Unit Weight:**
Total weights of all ingredients per m3 should be equal to the Unit Weight of fresh concrete.

\[ C + W + S + G = 2400 \]  ........................................ (4)

**Non Negativity:**

\[ C, W, S , G \geq 0 \]  ........................................ (5)
Case Studies (Work in Class)

Case 1

A developer has the alternative of building two, three, and four-bedroom houses. He wishes to establish the number of each, if any, that will maximize his profit, subject to the following constraints:

1- The total budget for the project cannot exceed $9,000,000
2- The total number of units must be at least 350 for the venture to be economically feasible.
3- The maximum percentages of each type, based on an analysis of the market, are:
   - 2-bedroom units, 20% of total
   - 3-bedroom units, 60% of total
   - 4-bedroom units, 40% of total

4- Building costs including land, architectural and engineer. fees, landscaping, and so on are:
   - 2-bedroom unit, $20,000
   - 3-bedroom unit, $25,000
   - 4-bedroom unit, $30,000

5- Net profits after interest, taxes, and so on, are
   - 2-bedroom unit, $2,000
   - 3-bedroom unit, $3,000
   - 4-bedroom unit, $4,000
Case 1: Solution

Let $x_1$, $x_2$, $x_3$ be, respectively, the number of two-, three-, and four-bedroom houses. The profit derived from selling them is, in dollars,

$$Z = 2000x_1 + 3000x_2 + 4000x_3.$$

However, a set of constraints is always present in this type of problem and for the particular illustration given here it is the following:

1. The total budget cannot exceed $9,000 (in dollars). This means, using the building costs given, that

$$20000x_1 + 25000x_2 + 30000x_3 \leq 9000000$$

2. The total number of units must be at least 350,

$$x_1 + x_2 + x_3 \geq 350$$

3. Finally, the market preferences data given can be translated into the following set of constraint inequalities,

$$X_1 \leq 0.2 \ (\text{Total Number of Units}) \ (x_1 + x_2 + x_3)$$

$$X_2 \leq 0.6 \ (\text{T.N.U.})$$

$$X_3 \leq 0.4 \ (\text{T.N.U})$$
Case 2

In order to assure adequate stability under load repetition, a soil mixture for base and sub-base courses in the construction of a certain highway must have a liquid limit, $21 \leq L.L. \leq 28$, and a Plasticity Index, $4 \leq P.I. \leq 6$.

Two materials, A and B, are available as follows:

<table>
<thead>
<tr>
<th>Properties</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>L.L.</td>
<td>35</td>
<td>20</td>
</tr>
<tr>
<td>P.I.</td>
<td>8</td>
<td>3.5</td>
</tr>
<tr>
<td>Cost ($/cu. m )</td>
<td>$.35</td>
<td>$.65</td>
</tr>
</tbody>
</table>

Assume that the L.L. and the P.I. are linear functions of the combinations of the two materials A and B and determine the optimal proportion of base and sub-base.

1. Use graphical analysis.
2. Use the simplex algorithm.
Case 2 - MODEL FORMULATION:

\[
\begin{align*}
\text{MIN} & \quad 0.35 \, X_A + 0.65 \, X_B \\
\text{SUBJECT TO} & \\
\text{L.L.} & \hspace{1cm} 2) \quad 35 \, X_A + 20 \, X_B \geq 21 \\
\text{L.L.} & \hspace{1cm} 3) \quad 35 \, X_A + 20 \, X_B \leq 28 \\
\text{P.I.} & \hspace{1cm} 4) \quad 8 \, X_A + 3.5 \, X_B \geq 4 \\
\text{P.I.} & \hspace{1cm} 5) \quad 8 \, X_A + 3.5 \, X_B \leq 6 \\
\text{Proportionality} & \hspace{1cm} 6) \quad X_A + X_B = 1 \\
\end{align*}
\]

END
Case 2: LINDO OUTPUT

SOLUTION:
LP OPTIMUM FOUND AT STEP 4
OBJECTIVE FUNCTION VALUE 1) .4900000

VARIABLE VALUE REDUCED COST
XA .533333 .000000
XB .466667 .000000

ROW SLACK OR SURPLUS DUAL PRICES
2) 7.000000 .000000
3) .000000 .020000
4) 1.900000 .000000
5) .100000 .000000
6) .000000 -1.050000

NO. ITERATIONS= 4

RANGE(SENSITIVITY) ANALYSIS:

Y ?
RANGES IN WHICH THE BASIS IS UNCHANGED

OBJ COEFFICIENT RANGES

VARIABLE CURRENT ALLOWABLE ALLOWABLE
COEF INCREASE DECREASE
XA .350000 .300000 INFINITY
XB .650000 INFINITY .300000

RIGHTHAND SIDE RANGES

ROW CURRENT ALLOWABLE ALLOWABLE
RHS INCREASE DECREASE
2 21.000000 7.000000 INFINITY
3 28.000000 .333333 6.333333
4 4.000000 1.900000 INFINITY
5 6.000000 INFINITY .100000
6 1.000000 .400000 .040000

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Case 3

A Mat Foundation (500 m³) need to be cast. Three concrete batching plants could be used to deliver concrete to the site. The relevant data associated with these plants are given below. Formulate the LP problem to minimize the cost of casting the Mat Foundation given that the casting time allowed is 6 hours.

<table>
<thead>
<tr>
<th>Concrete Batch Plant</th>
<th>Cost ($/m³)</th>
<th>Transport Time (min.)</th>
<th>Daily Production (m³/day)</th>
<th>No. of Available Transit Mixers (Size)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>120</td>
<td>30</td>
<td>300</td>
<td>2 10 m³</td>
</tr>
<tr>
<td>B</td>
<td>110</td>
<td>45</td>
<td>350</td>
<td>5 10 m³</td>
</tr>
<tr>
<td>C</td>
<td>130</td>
<td>50</td>
<td>400</td>
<td>3 10 m³</td>
</tr>
</tbody>
</table>