Chapter 1

Mathematical Modeling

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Chapter 1: Mathematical Model

• A mathematical model is a description of a system using mathematical concepts. Mathematical models are used in the natural sciences (such as physics, biology, earth science, meteorology) and engineering disciplines. A model may help to explain a system to study the effects of different components, and to make predictions about behavior.

• Mathematical models can take many forms, including but not limited to dynamical systems, statistical models, differential equations.

http://en.wikipedia.org/wiki/Mathematical_modeling
Mathematical Model

• A formulation or equation that expresses the essential features of a physical system or process in mathematical terms.

• Generally, it can be represented as a functional relationship of the form

\[
\text{Dependent variable} = f\left(\text{independent variable, parameters, forcing functions}\right)
\]
## Chapter 1: Mathematical Model

\[
\text{Dependent variable} = f \left( \text{independent variable, parameters, forcing functions} \right)
\]

<table>
<thead>
<tr>
<th>متغير تابع</th>
<th>متغير مستقل</th>
<th>عامل متغير</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Dependent variable =</strong></td>
<td>A characteristic that usually reflects the behavior or state of the system</td>
<td></td>
</tr>
<tr>
<td><strong>Independent variables =</strong></td>
<td>Are usually dimensions, such as time and space</td>
<td></td>
</tr>
<tr>
<td><strong>Parameters =</strong></td>
<td>Are reflective of system’s properties or compositions</td>
<td></td>
</tr>
<tr>
<td><strong>Forcing functions =</strong></td>
<td>Are external influences acting on the system</td>
<td></td>
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</table>
Simple Mathematical Model

Example: Newton’s Second Law

(The time rate of change of momentum of a body is equal to the resultant force acting on it)

\[ F = ma \quad \text{or} \quad a = \frac{F}{m} \]

- \(a = \) acceleration (m/s\(^2\)) \text{ ....the dependent variable}
- \(m = \) mass of the object (kg) \text{ ....the parameter representing a property of the system.}
- \(f = \) force acting on the body (N)
Complex Mathematical Model

Example: Newton’s Second Law

\[ m \frac{dv}{dt} = F \]

\[ F = F_D + F_U \]

\[ F_D = mg \]

\[ F_U = -cv \]

\[ F_D = \text{downward force due to gravity} \]

\[ F_U = \text{upward force due air resistance} \]

Where:

\[ c = \text{drag coefficient (kg/s)}, \]

\[ v = \text{falling velocity (m/s)} \]
Complex Mathematical Model

At rest: \( v = 0 \) at \( t = 0 \),

Calculus can be used to solve the equation

\[
m \frac{dv}{dt} = F_D + F_U
\]
\[
= mg - cv
\]
\[
\frac{dv}{dt} = g - \frac{c}{m} v
\]

At rest: \( v = 0 \) at \( t = 0 \),

Calculus can be used to solve the equation

\[
v(t) = \frac{gm}{c} \left[ 1 - e^{-\left(\frac{c}{m}\right)t} \right]
\]
Analytical solution to Newton's Second Law

A parachutist with a mass of 68.1 kg jumps out of a stationary hot-air balloon. Compute the velocities $\nu$ in an increment of 2 seconds prior to the opening of the chute. Use a drag coefficient value of 12.5 kg/s, and tabulate your values.

$$\nu(t) = \frac{gm}{c} \left[ 1 - e^{-\left(\frac{c}{m}\right)t} \right]$$
Analytical solution to Newton's Second Law

\[ v(t) = \frac{gm}{c} \left[ 1 - e^{-\left(\frac{c}{m}\right)t} \right] \]

\[ v(t) = \frac{9.8(68.1)}{12.5} \left[ 1 - e^{-\left(\frac{12.5}{68.1}\right)t} \right] \]

\[ v(t) = 53.3904 \left[ 1 - e^{-0.18355t} \right] \]

<table>
<thead>
<tr>
<th>t (s)</th>
<th>v (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>16.405</td>
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<tr>
<td>4</td>
<td>27.7693</td>
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<td>6</td>
<td>35.6418</td>
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<td>47.4902</td>
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<tr>
<td>∞</td>
<td>53.3904</td>
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</table>
Analytical solution to Newton's Second Law
Numerical Solution to Newton's Second Law

- Numerical solution: approximates the exact solution by arithmetic operations

\[
\frac{dv}{dt} \approx \frac{\Delta v}{\Delta t} = \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i}
\]

- Suppose

\[
\frac{dv}{dt} = g - \frac{c}{m}v
\]

\[
\frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i} = g - \frac{c}{m}v
\]

\[
v(t_{i+1}) = v(t_i) + \left(g - \frac{c}{m}v(t_i)\right)(t_{i+1} - t_i)
\]  (4)

New value = old value + slope \times step size
Numerical Solution to Newton's Second Law

\[ v(t_{i+1}) = v(t_i) + \left[ g - \frac{c}{m} v(t_i) \right] (t_{i+1} - t_i) \]

At \( t_i = 0 \), the velocity of the parachutist is zero. Using this information and the parameter values from Example 1, Eq. 4 can be used to compute the velocity at \( t_{i+1} = 2 \) seconds, that is

\[ v = 0 + \left[ 9.8 - \frac{12.5}{68.1} (0) \right] (2 - 0) = 19.60 \text{ m/s} \]
Numerical Solution to Newton's Second Law

For the next interval (from $t = 2$ to $4$ s), the computation is repeated, with the following result

$$v = 19.60 + \left[ 9.8 - \frac{12.5}{68.1} (19.60) \right] (4 - 2) = 32.00 \text{ m/s}$$

The calculation is continued in a similar fashion to obtain additional values as shown in the next viewgraph:

$$v(t_{i+1}) = v(t_i) + \left[ 9.8 - \frac{12.5}{68.1} v(t_i) \right] (t_{i+1} - t_i)$$

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Analytical Solution vs. Numerical Solution

![Graph showing approximate and exact solutions with a terminal velocity marker and a table with time and velocity values.]

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