Newton-Cotes Integration Formula

Chapter 21

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What is Integration?

- Integrate means “to bring together”, as parts, into a whole; to indicate total amount.
  \[ I = \int_{a}^{b} f(x).dx \]

- The above stands for integral of function \( f(x) \) with respect to the independent variable \( x \) between the limits \( x = a \) to \( x = b \).
What is Integration?

• Graphically integration is simply to find the area under a certain curve between the two integration limits.

\[ I = \int_{a}^{b} f(x).dx = A \]
Newton-Cotes Integration Formulas

Introduction

• The **Newton-Cotes formulas** are the most common numerical integration methods.

• They are based on the strategy of replacing a complicated function with an approximating function that is easy to integrate.

\[
I = \int_{a}^{b} f(x) \, dx \cong \int_{a}^{b} f_n(x) \, dx
\]

\[
f_n(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n
\]
1. Trapezoidal Rule

The trapezoidal rule uses a polynomial of the first degree to replace the function to be integrated.

\[
I = \int_{a}^{b} f(x).dx \approx \int_{a}^{b} f_1(x).dx
\]

\[
f_1(x) = a + \frac{f(b) - f(a)}{b - a} (x - a)
\]

\[
I = \int_{a}^{b} f(x).dx \approx \int_{a}^{b} f_1(x).dx
\]

\[
= \int_{a}^{b} \left\{ a + \frac{f(b) - f(a)}{b - a} (x - a) \right\}.dx
\]

\[
I = (b - a) \frac{f(a) + f(b)}{2}
\]
Error of the Trapezoidal Rule

When we employ the integral under a straight line segment to approximate the integral under a curve, error may be:

\[ E_t = -\frac{1}{12} f''(\xi)(b - a)^3 \]

Where \( \xi \) lies somewhere in the interval from \( a \) to \( b \).
Trapezoidal Rule

Example 21.1
Multiple Trapezoidal Rule

• One way to improve the accuracy of the trapezoidal rule is to divide the integration interval from a to b into a number of segments and apply the method to each segment.

• The areas of individual segments can then be added to yield the integral for the entire interval.
Multiple Trapezoidal Rule
Multiple Trapezoidal Rule

\[ h = \frac{b - a}{n} \quad a = x_0 \quad b = x_n \]

\[ I = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx \]

Substitute into the integrals for \( f(x) \) by \( f_1(x) \) in each segment and integrate:

\[ I = h \frac{f(x_0) + f(x_1)}{2} + h \frac{f(x_1) + f(x_2)}{2} + \cdots + h \frac{f(x_{n-1}) + f(x_n)}{2} \]

\[ I = \frac{h}{2} \left[ f(x_0) + 2 \sum_{i=1}^{n-1} f(x_i) + f(x_n) \right] \]
\[ f(x) \]

\[ f(x_0) \quad f(x_1) \quad f(x_2) \quad f(x_3) \]

\[ x_0 \quad x_1 \quad x_2 \quad x_3 \]

\[ f(x_{n-3}) \quad f(x_{n-2}) \quad f(x_{n-1}) \quad f(x_n) \]

\[ x_{n-3} \quad x_{n-2} \quad x_{n-1} \quad x_n \]

\[ h = \frac{b - a}{n} \]

\[ x_0 = a \quad x_n = b \]
Multiple Trapezoidal Rule

An error for multiple-application trapezoidal rule can be obtained by summing the individual errors for each segment:

\[ \sum f''(\xi_i) \equiv nf'' \]

\[ E_a = -\frac{(b-a)^3}{12n^2} f'' \]
More accurate estimate of an integral is obtained if a high-order polynomial is used to connect the points. The formulas that result from taking the integrals under such polynomials are called **Simpson’s Rules**.
Simpson’s Rules

• Simpson’s 1/3 Rule
  Results when a second-order interpolating polynomial is used.

• Simpson’s 3/8 Rule
  Results when a third-order (cubic) interpolating polynomial is used.
Simpson’s Rules

(a) Simpson’s 1/3 Rule
(b) Simpson’s 3/8 Rule
Simpson’s 1/3 Rule

\[ I = \int_{a}^{b} f(x) \, dx \approx \int_{a}^{b} f_2(x) \, dx \]

\[ a = x_0 \quad b = x_2 \]

\[ I = \int_{x_0}^{x_2} \left[ \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \right] \, dx \]

\[ I \approx \frac{h}{3} \left[ f(x_0) + 4f(x_1) + f(x_2) \right] \quad h = \frac{b-a}{2} \]
Simpson’s 1/3 Rule

- Single segment application of Simpson’s 1/3 rule has a truncation error of:

\[ E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\xi) \quad a < \xi < b \]

- Simpson’s 1/3 rule is more accurate than trapezoidal rule.
The Multiple-Application
Simpson’s 1/3 Rule

• Just as the trapezoidal rule, Simpson’s rule can be improved by dividing the integration interval into a number of segments of equal width.

\[ I \approx 2h \frac{f(x_o) + 4f(x_1) + f(x_2)}{6} + 2h \frac{f(x_2) + 4f(x_3) + f(x_4)}{6} + \cdots + 2h \frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6} \]

with \( h = \frac{b-a}{n} \)

\[ \approx (b - a) \frac{\left\{ f(x_o) + f(x_n) + 4 \sum_{i=1,3,5} f(x_i) + 2 \sum_{j=2,4,6} f(x_j) \right\}}{3n} \]

\[ E_t = -\frac{(b-a)^5}{180n^4} \bar{f}^{(4)}(\xi) \]

\[ I = \frac{(b-a)}{3n} \left[ f(x_0) + f(x_n) + 4 \sum_{i=1,3,5} f(x_i) + 2 \sum_{j=2,4,6} f(x_j) \right] \]
The Multiple-Application Simpson’s 1/3 Rule
Simpson’s 3/8 Rule

If there are 2 extra points between the integration limits \( a \) and \( b \), then a 3rd degree polynomial can be used instead of the parabola to replace the function to be integrated:

\[
I = \int_{a}^{b} f(x) \, dx \approx \int_{a}^{b} f_3(x) \, dx
\]

\[
I \approx \frac{3h}{8} \left[ f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) \right], \quad h = \frac{(b-a)}{3}
\]

\[
E_t = -\frac{(b-a)^5}{6480} f^{(4)}(\xi)
\]

Simpson’s 3/8 Rule
Newton Cotes Integration-Example

Find the integral of:

\[ f(x) = 0.2 + 25 \ x - 200 \ x^2 + 675 \ x^3 - 900 \ x^4 + 400 \ x^5 \]

Between the limits 0 to 0.8, \( f(0) = 0.2 \), \( f(0.8) = 0.232 \),
\[ \text{\( I_{\text{exact}} = 1.640533 \)} \]

1. The trapezoidal rule (ans. 0.1728)

\[ I = (b - a) \frac{f(a) + f(b)}{2} \Rightarrow I = (0.8 - 0) \frac{0.2 + 0.232}{2} = 0.1728 \]
\[ E_t = 1.640533 - 0.1728 = 1.467733 \Rightarrow \varepsilon_t = 89.5\% \]

\[ f''(x) = -400 + 4050x - 10,800x^2 + 8000x^3 \]

\[ \bar{f}''(x) = \int_{0}^{0.8} (-400 + 4050x - 10,800x^2 + 8000x^3) \, dx \]
\[ \bar{f}''(x) = \frac{1}{0.8 - 0} \left( -400(0.8)^2 + 4050(0.8) - 10,800(0.8)^3 + 8000(0.8)^4 \right) = -60 \]
\[ E_a = - \frac{1}{12} (60)(0.8)^3 = 2.56 \]
2. Multiple trapezoidal rule (n=4)  \hspace{1cm} (\text{ans. 1.4848})

\[ f(0)=0.2, \ f(0.2)=1.288, \ f(0.4)=2.456, \ f(0.6)=3.464, \ f(0.8)=0.232 \]

\[ h = \frac{(b-a)}{4} = \frac{(0.8-0)}{4} = 0.2 \]

\[ I = \frac{h}{2} \left[ f(x_0) + 2 \sum_{i=1}^{n-1} f(x_i) + f(x_n) \right] \]

\[ = \frac{0.8}{2} \left[ 0.2 + 2(1.288 + 2.456 + 3.464) + 0.232 \right] = 1.4848 \]
Newton Cotes Integration-Example

3. The Simpson 1/3 rule (ans. 1.367467)

\[ f(0) = 0.2, \quad f(0.4) = 0.2456, \quad f(0.8) = 0.232 \]

\[ h = \frac{b - a}{2} = \frac{0.8 - 0}{2} = 0.4 \]

\[ I = \frac{h}{3} \left[ f(x_0) + 4f(x_1) + f(x_2) \right] \]

\[ = \frac{0.4}{3} \left[ 0.2 + 4 \times 2.456 + 0.232 \right] = 1.367467 \]

\[ E_t = 1.640533 - 1.367467 = 0.2730667 \implies \varepsilon_t = 16.6\% \]

\[ f^{(4)}(x) = -2400 \]

\[ E_a = -\frac{(b - a)^5}{2880} f^{(4)}(\xi) = -\frac{(0.8 - 0)^5}{2880} (-2400) = 0.2730667 \]
Newton Cotes Integration-Example

4. Multiple application of Simpson 1/3 rule (n=4)
   (ans. 1.623467).

\[ f(0)=0.2, \ f(0.2)=1.288, \ f(0.4)=2.456, \ f(0.6)=3.464 \ , f(0.8)=0.232 \]

\[ h = \frac{(b-a)}{n} = \frac{(0.8-0)}{4} = 0.2 \]

\[ I = \frac{h}{3} \left[ f(x_0) + 4 \sum_{i=1,3,5} f(x_i) + 2 \sum_{i=2,4,6} f(x_i) + f(x_n) \right] \]

\[ = \frac{0.2}{3} \left[ 0.2 + 4(1.288+3.464) + 2(2.456)+0.232 \right] = 1.623467 \]

\[ E_t = 1.640533 - 1.623467 = 0.017067 \quad \Rightarrow \quad \varepsilon_t = 1.04\% \]

\[ E_a = -\frac{(b-a)^5}{180n^4} \tilde{f}^{(4)}(\xi) = - \frac{0.8^5}{180(4)^4}(-2400) = 0.017067 \]
5. The Simpson 3/8 rule (ans. 1.519170)

\[ f(0)=0.2, f(0.2667)=1.432724, f(0.5333)=3.487177, f(0.8)=0.232 \]

\[ h = \frac{(b-a)}{3} = \frac{(0.8-0)}{3} = 0.2667 \]

\[ I = I \approx \frac{3h}{8} \left[ f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) \right] \]

\[ = \frac{0.8}{8} \left[ 0.2 + 3 \times 1.432724 + 3 \times 3.487177 + 0.232 \right] = 1.519170 \]

\[ E_t = 1.640533 - 1.51917 = 0.121363 \quad \Rightarrow \quad \varepsilon_t = 7.4\% \]

\[ E_a = -\frac{(b-a)5}{6480} f^{(4)}(\xi) = -\frac{0.8^5}{6480}(-2400) = 0.1213630 \]
Double integral - Example

- Compute the average temperature of a rectangular heated plate which is 8 m long in the x-direction and 6 m wide in the y-direction. The temperature is given as:

$$T(x, y) = 2xy + 2x - x^2 - 2y^2 + 72$$

- (Use 2 segment applications of the trapezoidal rule in each dimension)
Double integral - Example

\[
I = \int_0^8 \int_0^6 (2xy + 2x - x^2 - 2y^2 + 72) \, dx \, dy
\]

**Multiple Trapezoidal rule (n = 2)** → \( I = 2688, T_{avg} = 2688 / (6 \times 8) = 56 \)

**Simpson 1/3 rule** → \( I = 2816, T_{avg} = 2816 / (6 \times 8) = 58.6667 \)

**HW:** Use two points Gauss formula to solve the problem