Appendix D

The Decibel

Telephone engineers who were concerned with the power loss across the cascaded circuits used to transmit telephone signals introduced the decibel. Figure D.1 defines the problem.

There, \( p_i \) is the power input to the system, \( p_1 \) is the power output of circuit A, \( p_2 \) is the power output of circuit B, and \( p_o \) is the power output of the system. The power gain of each circuit is the ratio of the power out to the power in. Thus

\[
\sigma_A = \frac{p_1}{p_i}, \quad \sigma_B = \frac{p_2}{p_1}, \quad \text{and} \quad \sigma_C = \frac{p_o}{p_2}.
\]

The overall power gain of the system is simply the product of the individual gains, or

\[
\frac{p_o}{p_i} = \frac{p_1}{p_i} \frac{p_2}{p_1} \frac{p_o}{p_2} = \sigma_A \sigma_B \sigma_C.
\]

The multiplication of power ratios is converted to addition by means of the logarithm; that is,

\[
\log_{10} \frac{p_o}{p_i} = \log_{10} \sigma_A + \log_{10} \sigma_B + \log_{10} \sigma_C.
\]

This log ratio of the powers was named the bel, in honor of Alexander Graham Bell. Thus we calculate the overall power gain, in bels, simply by summing the power gains, also in bels, of each segment of the transmission system. In practice, the bel is an inconveniently large quantity. One-tenth of a bel is a more useful measure of power gain; hence the decibel. The number of decibels equals 10 times the number of bels, so

\[
\text{Number of decibels} = 10 \log_{10} \frac{p_o}{p_i}.
\]

When we use the decibel as a measure of power ratios, in some situations the resistance seen looking into the circuit equals the resistance loading the circuit, as illustrated in Fig. D.2.

When the input resistance equals the load resistance, we can convert the power ratio to either a voltage ratio or a current ratio:

\[
\frac{p_o}{p_i} = \frac{v_{out}^2 / R_L}{v_{in}^2 / R_{in}} = \left( \frac{v_{out}}{v_{in}} \right)^2
\]

or

\[
\frac{p_o}{p_i} = \frac{i_{out}^2 R_L}{i_{in}^2 R_{in}} = \left( \frac{i_{out}}{i_{in}} \right)^2.
\]
These equations show that the number of decibels becomes

\[
\text{Number of decibels} = 20 \log_{10} \frac{v_{\text{out}}}{v_{\text{in}}}
\]

\[
= 20 \log_{10} \frac{i_{\text{out}}}{i_{\text{in}}}. \tag{D.1}
\]

The definition of the decibel used in Bode diagrams (see Appendix E) is borrowed from the results expressed by Eq. D.1, since these results apply to any transfer function involving a voltage ratio, a current ratio, a voltage-to-current ratio, or a current-to-voltage ratio. You should keep the original definition of the decibel firmly in mind because it is of fundamental importance in many engineering applications.

When you are working with transfer function amplitudes expressed in decibels, having a table that translates the decibel value to the actual value of the output/input ratio is helpful. Table D.1 gives some useful pairs. The ratio corresponding to a negative decibel value is the reciprocal of the positive ratio. For example, -3 dB corresponds to an output/input ratio of 1/1.41, or 0.707. Interestingly, -3 dB corresponds to the half-power frequencies of the filter circuits discussed in Chapters 14 and 15.

The decibel is also used as a unit of power when it expresses the ratio of a known power to a reference power. Usually the reference power is 1 mW and the power unit is written dBm, which stands for “decibels relative to one milliwatt.” For example, a power of 20 mW corresponds to ±13 dBm.

AC voltmeters commonly provide dBm readings that assume not only a 1 mW reference power but also a 600 Ω reference resistance (a value commonly used in telephone systems). Since a power of 1 mW in 600 Ω corresponds to 0.7746 V (rms), that voltage is read as 0 dBm on the meter. For analog meters, there usually is exactly a 10 dB difference between adjacent ranges. Although the scales may be marked 0.1, 0.3, 1, 3, 10, and so on, in fact 3.16 V on the 3 V scale lines up with 1 V on the 1 V scale.

Some voltmeters provide a switch to choose a reference resistance (50, 135, 600, or 900 Ω) or to select dBm or dBV (decibels relative to one volt).

### Table D.1 Some dB-Ratio Pairs

<table>
<thead>
<tr>
<th>dB</th>
<th>Ratio</th>
<th>dB</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.00</td>
<td>30</td>
<td>31.62</td>
</tr>
<tr>
<td>3</td>
<td>1.41</td>
<td>40</td>
<td>100.00</td>
</tr>
<tr>
<td>6</td>
<td>2.00</td>
<td>60</td>
<td>10^3</td>
</tr>
<tr>
<td>10</td>
<td>3.16</td>
<td>80</td>
<td>10^4</td>
</tr>
<tr>
<td>15</td>
<td>5.62</td>
<td>100</td>
<td>10^5</td>
</tr>
<tr>
<td>20</td>
<td>10.00</td>
<td>120</td>
<td>10^6</td>
</tr>
</tbody>
</table>
Appendix E

Bode Diagrams

As we have seen, the frequency response plot is a very important tool for analyzing a circuit's behavior. Up to this point, however, we have shown qualitative sketches of the frequency response without discussing how to create such diagrams. The most efficient method for generating and plotting the amplitude and phase data is to use a digital computer; we can rely on it to give us accurate numerical plots of \( |H(j\omega)| \) and \( \theta(j\omega) \) versus \( \omega \). However, in some situations, preliminary sketches using Bode diagrams can help ensure the intelligent use of the computer.

A Bode diagram, or plot, is a graphical technique that gives a feel for the frequency response of a circuit. These diagrams are named in recognition of the pioneering work done by H. W. Bode.\(^1\) They are most useful for circuits in which the poles and zeros of \( H(s) \) are reasonably well separated.

Like the qualitative frequency response plots seen thus far, a Bode diagram consists of two separate plots: One shows how the amplitude of \( H(j\omega) \) varies with frequency, and the other shows how the phase angle of \( H(j\omega) \) varies with frequency. In Bode diagrams, the plots are made on semilog graph paper for greater accuracy in representing the wide range of frequency values. In both the amplitude and phase plots, the frequency is plotted on the horizontal log scale, and the amplitude and phase angle are plotted on the linear vertical scale.

### E.1 Real, First-Order Poles and Zeros

To simplify the development of Bode diagrams, we begin by considering only cases where all the poles and zeros of \( H(s) \) are real and first order. Later we will present cases with complex and repeated poles and zeros. For our purposes, having a specific expression for \( H(s) \) is helpful. Hence we base the discussion on

\[
H(s) = \frac{K(s + z_i)}{s(s + p_i)},
\]

from which

\[
H(j\omega) = \frac{K(j\omega + z_i)}{j\omega(j\omega + p_i)}.
\]

The first step in making Bode diagrams is to put the expression for \( H(j\omega) \) in a standard form, which we derive simply by dividing out the poles and zeros:

\[
H(j\omega) = \frac{Kz_i(1 + j\omega/p_i)}{p_i(j\omega)(1 + j\omega/p_i)}.
\]

---

Next we let $K_o$ represent the constant quantity $K z_i / p_1$, and at the same time we express $H(j\omega)$ in polar form:

$$H(j\omega) = \frac{K_o |1 + j\omega z_i|}{|\omega| |\omega|^{90^\circ} |1 + j\omega p_1| / \beta_1}$$

$$= \frac{K_o |1 + j\omega z_i|}{|\omega| |1 + j\omega p_1|} (\psi_1 - 90^\circ - \beta_1). \quad (E.4)$$

From Eq. E.4,

$$|H(j\omega)| = \frac{K_o |1 + j\omega z_i|}{|\omega| |1 + j\omega p_1|}, \quad (E.5)$$

$$\theta(\omega) = \psi_1 - 90^\circ - \beta_1. \quad (E.6)$$

By definition, the phase angles $\psi_1$ and $\beta_1$ are

$$\psi_1 = \tan^{-1} \omega z_i; \quad (E.7)$$

$$\beta_1 = \tan^{-1} \omega p_1. \quad (E.8)$$

The Bode diagrams consist of plotting Eq. E.5 (amplitude) and Eq. E.6 (phase) as functions of $\omega$.

### E.2 Straight-Line Amplitude Plots

The amplitude plot involves the multiplication and division of factors associated with the poles and zeros of $H(s)$. We reduce this multiplication and division to addition and subtraction by expressing the amplitude of $H(j\omega)$ in terms of a logarithmic value: the decibel (dB).\(^2\) The amplitude of $H(j\omega)$ in decibels is

$$A_{\text{dB}} = 20 \log_{10} |H(j\omega)|. \quad (E.9)$$

To give you a feel for the unit of decibels, Table E.1 provides a translation between the actual value of several amplitudes and their values in decibels. Expressing Eq. E.5 in terms of decibels gives

$$A_{\text{dB}} = 20 \log_{10} \frac{K_o |1 + j\omega z_i|}{|\omega| |1 + j\omega p_1|}$$

$$= 20 \log_{10} K_o + 20 \log_{10} |1 + j\omega z_i|$$

$$- 20 \log_{10} \omega - 20 \log_{10} |1 + j\omega p_1|. \quad (E.10)$$

\(^2\) See Appendix D for more information regarding the decibel.
The key to plotting Eq. E.10 is to plot each term in the equation separately and then combine the separate plots graphically. The individual factors are easy to plot because they can be approximated in all cases by straight lines.

The plot of $20 \log_{10} K_o$ is a horizontal straight line because $K_o$ is not a function of frequency. The value of this term is positive for $K_o > 1$, zero for $K_o = 1$, and negative for $K_o < 1$.

Two straight lines approximate the plot of $20 \log_{10} |1 + j\omega/z_1|$. For small values of $\omega$, the magnitude $|1 + j\omega/z_1|$ is approximately 1, and therefore

$$20 \log_{10} |1 + j\omega/z_1| \to 0 \quad \text{as} \quad \omega \to 0. \quad (E.11)$$

For large values of $\omega$, the magnitude $|1 + j\omega/z_1|$ is approximately $\omega/z_1$, and therefore

$$20 \log_{10} |1 + j\omega/z_1| \to 20 \log_{10} (\omega/z_1) \quad \text{as} \quad \omega \to \infty. \quad (E.12)$$

On a log frequency scale, $20 \log_{10} (\omega/z_1)$ is a straight line with a slope of 20 dB/decade (a decade is a 10-to-1 change in frequency). This straight line intersects the 0 dB axis at $\omega = z_1$. This value of $\omega$ is called the corner frequency. Thus, on the basis of Eqs. E.11 and E.12, two straight lines can approximate the amplitude plot of a first-order zero, as shown in Fig. E.1.

The plot of $-20 \log_{10} \omega$ is a straight line having a slope of $-20$ dB/decade that intersects the 0 dB axis at $\omega = 1$. Two straight lines approximate the plot of $-20 \log_{10} |1 + j\omega/p|$. Here the two straight lines

![Figure E.1](image-url)
intersect on the 0 dB axis at \( \omega = p_1 \). For large values of \( \omega \), the straight line
\[ 20 \log_{10}(\omega/p_1) \] has a slope of \(-20 \text{ dB/decade}\). Figure E.2 shows the straight-line approximation of the amplitude plot of a first-order pole.

![Figure E.2](image)

**Figure E.2** A straight-line approximation of the amplitude plot of a first-order pole.

Figure E.3 shows a plot of Eq. E.10 for \( K_o = \sqrt{10} \), \( z_1 = 0.1 \) rad/s, and \( p_1 = 5 \) rad/s. Each term in Eq. E.10 is labeled on Fig. E.3, so you can verify that the individual terms sum to create the resultant plot, labeled \( 20 \log_{10}|H(j\omega)| \).

Example E.1 illustrates the construction of a straight-line amplitude plot for a transfer function characterized by first-order poles and zeros.

![Figure E.3](image)

**Figure E.3** A straight-line approximation of the amplitude plot for Eq. E.10.
Example E.1

For the circuit in Fig. E.4:

a) Compute the transfer function, \( H(s) \).

b) Construct a straight-line approximation of the Bode amplitude plot.

c) Calculate \( 20 \log_{10}|H(j\omega)| \) at \( \omega = 50 \text{ rad/s} \) and \( \omega = 1000 \text{ rad/s} \).

d) Plot the values computed in (c) on the straight-line graph; and

e) Suppose that \( v_{t}(t) = 5 \cos(500t + 15^\circ) \text{ V} \), and then use the Bode plot you constructed to predict the amplitude of \( v_{v}(t) \) in the steady state.

Solution

a) Transforming the circuit in Fig. E.4 into the \( s \)-domain and then using \( s \)-domain voltage division gives

\[
H(s) = \frac{(R/L)s}{s^2 + (R/L)s + \frac{1}{LC}}.
\]

Substituting the numerical values from the circuit, we get

\[
H(s) = \frac{110s}{s^2 + 110s + 1000} = \frac{110s}{(s + 10)(s + 100)}.
\]

b) We begin by writing \( H(j\omega) \) in standard form:

\[
H(j\omega) = \frac{0.11 j\omega}{[1 + j(\omega/10)][1 + j(\omega/100)]}.
\]

The expression for the amplitude of \( H(j\omega) \) in decibels is

\[
A_{dB} = 20 \log_{10}|H(j\omega)|
\]

\[
= 20 \log_{10}0.11 + 20 \log_{10}|j\omega|
\]

\[ - 20 \log_{10} \left| 1 + \frac{j\omega}{10} \right| - 20 \log_{10} \left| 1 + \frac{j\omega}{100} \right|.
\]

Figure E.5 shows the straight-line plot. Each term contributing to the overall amplitude is identified.

\[
\begin{align*}
100 \text{ mH} &
\end{align*}
\]

\[
10 \text{ mF}
\]

\[
11 \Omega
\]

\[
\text{Figure E.5 ▲ The straight-line amplitude plot for the transfer function of the circuit in Fig. E.4.}
\]
d) See Fig. E.5.

e) As we can see from the Bode plot in Fig. E.5, the value of $A_{dB}$ at $\omega = 500$ rad/s is approximately $-12.5$ dB. Therefore,

$$|A| = 10^{-12.5/20} = 0.24$$

and

$$V_{mo} = |A|V_{ni} = (0.24)(5) = 1.19 \text{ V}.$$
E.4 Straight-Line Phase Angle Plots

We can also make phase angle plots by using straight-line approximations. The phase angle associated with the constant $K_0$ is zero, and the phase angle associated with a first-order zero or pole at the origin is a constant $\pm 90^\circ$. For a first-order zero or pole not at the origin, the straight-line approximations are as follows:

- For frequencies less than one tenth the corner frequency, the phase angle is assumed to be zero.
- For frequencies greater than 10 times the corner frequency, the phase angle is assumed to be $\pm 90^\circ$.
- Between one tenth the corner frequency and 10 times the corner frequency, the phase angle plot is a straight line that goes through $0^\circ$ at one-tenth the corner frequency, $\pm 45^\circ$ at the corner frequency, and $\pm 90^\circ$ at 10 times the corner frequency.

In all these cases, the plus sign applies to the first-order zero and the minus sign to the first-order pole. Figure E.7 depicts the straight-line approximation for a first-order zero and pole. The dashed curves show the exact variation of the phase angle as the frequency varies. Note how closely the
straight-line plot approximates the actual variation in phase angle. The maximum deviation between the straight-line plot and the actual plot is approximately 6°.

Figure E.8 depicts the straight-line approximation of the phase angle of the transfer function given by Eq. B.1. Equation B.6 gives the equation for the phase angle; the plot corresponds to \( z_1 = 0.1 \text{ rad/s} \), and \( p_1 = 5 \text{ rad/s} \).

An illustration of a phase angle plot using a straight-line approximation is given in Example E.2.
Example E.2

a) Make a straight-line phase angle plot for the transfer function in Example E.1.

b) Compute the phase angle $\theta(\omega)$ at $\omega = 50$, 500, and 1000 rad/s.

c) Plot the values of (b) on the diagram of (a).

d) Using the results from Example E.1(e) and (b) of this example, compute the steady-state output voltage if the source voltage is given by $v_s(t) = 10 \cos(500t - 25^\circ)$ V.

Solution

a) From Example E.1,

$$H(j\omega) = \frac{0.11(j\omega)}{[1 + j(\omega/10)][1 + j(\omega/100)]}$$

$$= \frac{0.11|j\omega|}{[1 + j(\omega/10)][1 + j(\omega/100)] \cos(\psi_1 - \beta_1 - \beta_2)}.$$

Therefore,

$$\theta(\omega) = \psi_1 - \beta_1 - \beta_2,$$

where $\psi_1 = 90^\circ$, $\beta_1 = \tan^{-1}(\omega/10)$, and $\beta_2 = \tan^{-1}(\omega/100)$. Figure E.9 depicts the straight-line approximation of $\theta(\omega)$.

b) We have

$$H(j50) = 0.96/-15.25^\circ,$$

$$H(j500) = 0.22/-77.54^\circ,$$

$$H(j1000) = 0.11/-83.72^\circ.$$

Thus,

$$\theta(j50) = -15.25^\circ,$$

$$\theta(j500) = -77.54^\circ,$$

and

$$\theta(j1000) = -83.72^\circ.$$

c) See Fig. E.9.

d) We have

$$V_m = |H(j500)|V_m$$

$$= (0.22)(10)$$

$$= 2.2 \text{ V},$$

and

$$\theta_\theta = \theta(\omega) + \theta_i$$

$$= -77.54^\circ - 25^\circ$$

$$= -102.54^\circ.$$

Thus,

$$v_o(t) = 2.2 \cos(500t - 102.54^\circ) \text{ V}.$$