Chapter 8 Fuzzy Logic

Formal language is a language in which the syntax is precisely given and thus is different from informal language like English and French. The study of the formal languages is the content of mathematics known as mathematical logic. The mathematical logic is called classical logic in this chapter. The classical logic considers the binary logic which consists of truth and false. The fuzzy logic is a generalization of the classical logic and deals with the ambiguity in the logic. In this chapter, we summarize the classical logic and then study the fuzzy logic.

8.1 Classical logic

8.1.1 Proposition logic

Definition (Proposition) As in our ordinary informal language, “sentence” is used in the logic. Especially, a sentence having only “true (1)” or “false (0)” as its truth value is called “proposition”.

Example 8.1 The following sentences are propositions.

Smith hits 30 home runs in one season. (true)
2 + 4 = 7 (false)
For every x, if f(x) = sin x, then f′(x) = cos x. (true)
It rains now. (true)

Example 8.2 The followings are not propositions.

Why are you interested in the fuzzy theory?
He hits 5 home runs in one season.
x + 5 = 0
x + y = z

In the second example, we do not know who is “He” and thus cannot determine whether the sentence is true (1) or false (0). If “He” is replaced by “Tom”, we have

Tom hits 5 home runs in one season.

Now we can evaluate the truth value of the above sentence. In the same way,
x + 5 = 0
is not a proposition. If the variable x is replaced by –5, then the sentence

−5 + 5 = 0

has its value “true”. A variable is used as a symbol representing an element in a universal set.

Definition (Logic variable) As we know now, a proposition has its value (true or false). If we represent a proposition as a variable, the variable can have the value true or false. This type of variable is called as a “proposition variable” or “logic variable”.

We can combine prepositional variables by using “connectives”. The basic connectives are negation, conjunction, disjunction, and implication.

1) Negation

Let’s assume that prepositional variable P represents the following sentence.
P: 2 is rational.
In this case, the true value of \( P \) is true.

\[ P = \text{true} \]

But its negation is false and represents as follows.

\[ \overline{P} = \text{false} \]

The truth table representing the values of negation is given in Table 8.1.

<table>
<thead>
<tr>
<th>( a )</th>
<th>( \bar{a} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

2) Conjunction

If \( a \) and \( b \) are propositional variables, their conjunction is represented as follows and is interpreted as “\( a \) AND \( b \)”.

\[ a \land b \]

The truth value of the above conjunction is determined according to the values of \( a \) and \( b \) (Table 8.2).

<table>
<thead>
<tr>
<th>( a )</th>
<th>( b )</th>
<th>( a \land b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Example 8.3 Suppose there are two propositions \( a \) and \( b \). We can see their conjunction is 0.

\[ a: 2 + 2 = 4 \]

\[ b: 3 + 2 = 7 \]

\[ c: a \land b \]

then, \( a = 1, b = 0, \text{ and } c = 0. \]

3) Disjunction

The disjunction of two propositions \( a \) and \( b \) is represented as follows

\[ a \lor b \]

The disjunction is interpreted as “\( a \) OR \( b \)” But it has two different meanings: “exclusive OR” and “inclusive OR”. The exclusive OR is used in which two events could not happen simultaneously.
Are you awake or asleep? 
The inclusive OR is used when two events can occur simultaneously.

Are you wearing a shirt or sweater? 
In general, if we say the disjunction, we mean the “inclusive OR” (Table 8.3).

Table 8.3  Truth table of disjunction

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>a ∨ b</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

4) Implication
The proposition “if a, then b.” is represented as follows.

\[ a \rightarrow b \]

**Example 8.4** Consider the following propositions. We study the truth value of each proposition in varying the value of prepositional variables a and b.

1) \( a \rightarrow b \) where \( a: 2 + 2 = 4, b: 3 + 3 = 6 \)

2) \( a \rightarrow b \) where \( a: 2 + 2 = 4, b: 3 + 3 = 7 \)

3) \( a \rightarrow b \) where \( a: 2 + 2 = 5, b: 3 + 3 = 6 \)

4) \( a \rightarrow b \) where \( a: 2 + 2 = 5, b: 3 + 3 = 7 \)

We can see that the above propositions are true except for the second. □

The truth values of implication are summarized in Table 8.4. In the table we see that the value of implication can be represented by \( \overline{a} ∨ b \).

Table 8.4  Truth table of implication

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>a → b</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

8.1.2 Logic function
The “logic function” is a combination of propositional variables by using connectives. Values of the
logic function can be evaluated according to the values of propositional variables and the truth values of connectives. As we know, a logic function having only one propositional variable has two kinds of values: true and false. A logic function containing two variables has \(4 (=2^2)\) different combinations of values: (true, true), (true, false), (false, true), (false, false). Similarly a function having \(n\) variables can have \(2^n\) different combinations.

If a logic function has one or two prepositional (logic) variables, it is called a “logic primitive”. By using the logic primitive, we can represent an (algebraic) expression which is called a “logic formula”.

**Definition (Logic formula)** The logic formula is defined as following:

1. Truth values 0 and 1 are logic formulas
2. If \(\nu\) is a logic variable, \(\nu\) and \(\overline{\nu}\) are a logic formulas
3. If \(a\) and \(b\) represent a logic formulas, \(a \land b\) and \(a \lor b\) are also logic formulas.
4. The expressions defined by the above (1), (2), and (3) are logic formulas.

Any logic formula defines a logic function, and it has its truth value. Properties of logic formulas are summarized in Table 8.5.

<table>
<thead>
<tr>
<th></th>
<th>Properties of classical logic</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>Involution (\overline{\nu} = \nu)</td>
</tr>
<tr>
<td>(2)</td>
<td>Commutativity (a \land b = b \land a) (a \lor b = b \lor a)</td>
</tr>
<tr>
<td>(3)</td>
<td>Associativity ((a \land b) \land c = a \land (b \land c)) ((a \lor b) \lor c = a \lor (b \lor c))</td>
</tr>
<tr>
<td>(4)</td>
<td>Distributivity (a \lor (b \land c) = (a \lor b) \land (a \lor c)) (a \land (b \lor c) = (a \land b) \lor (a \land c))</td>
</tr>
<tr>
<td>(5)</td>
<td>Idempotency (a \land a = a) (a \lor a = a)</td>
</tr>
<tr>
<td>(6)</td>
<td>Absorption (a \lor (a \land b) = a) (a \land (a \lor b) = a)</td>
</tr>
<tr>
<td>(7)</td>
<td>Absorption by 0 and 1 (a \land 0 = 0) (a \lor 1 = 1)</td>
</tr>
<tr>
<td>(8)</td>
<td>Identity (a \land 1 = a) (a \lor 0 = a)</td>
</tr>
<tr>
<td>(9)</td>
<td>De Morgan’s law (\overline{a \land b} = \overline{a} \lor \overline{b}) (\overline{a \lor b} = \overline{a} \land \overline{b})</td>
</tr>
</tbody>
</table>
Some of important logic formulas and their values are given in the following:

<table>
<thead>
<tr>
<th>(1) Negation</th>
<th>$\overline{a} = 1 - a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2) Conjunction</td>
<td>$a \land b = \text{Min}(a, b)$</td>
</tr>
<tr>
<td>(3) Disjunction</td>
<td>$a \lor b = \text{Max}(a, b)$</td>
</tr>
<tr>
<td>(4) Implication</td>
<td>$a \rightarrow b = \overline{a} \lor b$</td>
</tr>
</tbody>
</table>

### 8.1.3 Tautology and inference rule

**Definition (Tautology)** A “tautology” is a logic formula whose value is always true regardless of its logic variables. A “contradiction” is one which is always false. □

**Example 8.5** Consider the following logic formula.

$$\overline{a} \rightarrow b$$

This proposition means that if the value of $(a \rightarrow b)$ is false then $b$ is false. Let’s evaluate its value with different values of logic variables $a$ and $b$ in Table 8.6.

**Table 8.6 Truth value of tautology $(a \rightarrow b) \rightarrow \overline{b}$**

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$a \rightarrow b$</th>
<th>$(a \rightarrow b)$</th>
<th>$\overline{b}$</th>
<th>$(a \rightarrow b) \rightarrow \overline{b}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
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<td>1</td>
<td>1</td>
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</tbody>
</table>

In Table 8.6, we see that the logic formula is always true and thus it is a tautology. That is, if $(a \rightarrow b)$ is true, then $\overline{b}$ is always true. □

**Example 8.6** Let’s consider another example.

$$(a \land (a \rightarrow b)) \rightarrow b$$

The truth values of this proposition are evaluated in Table 8.7.

**Table 8.7 Truth value of tautology $(a \land (a \rightarrow b)) \rightarrow b$**

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$a \land (a \rightarrow b)$</th>
<th>$((a \land (a \rightarrow b))) \rightarrow b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
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<tr>
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<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
We can see that this proposition has also true value regardless of the values of a and b. This tautology means that

“If a is true and \( (a \rightarrow b) \) is true, then \( b \) is true.”

or

“If a exists and the relation \( (a \rightarrow b) \) is true, then \( b \) exists.” □

The interpretations in the above example show a logic procedure of tautology, and thus we can obtain a correct conclusion when we follow the logic procedure. Therefore, the tautology is used as a rule of “deductive inference”. There are some important inference rules using tautologies.

\[
\begin{align*}
(a \land (a \rightarrow b)) & \rightarrow b \quad : \text{modus ponens} \\
(b \land (a \rightarrow b)) & \rightarrow \overline{a} \quad : \text{modus tollens} \\
((a \rightarrow b) \land (b \rightarrow c)) & \rightarrow (a \rightarrow c) \quad : \text{hypothetical syllogism}
\end{align*}
\]

8.1.4 Predicate logic

A “predicate” is a group of words like

“is a man”

“is green”

“is less than”

“belongs to”

They can be applied to one or more names of individuals (objects) to yield meaningful sentences; for example,

“Socrates is a man.”

“Two is less than four.”

“That hat belongs to me.”

“He is John.”

The names of the individuals are called individual constants.

Definition (Predicate logic) “Predicate logic” is a logic which represents a proposition with the predicate and individual (object). □

Example 8.7 The following propositions are “predicate propositions” and consist of predicates and objects.
“Socrates is a man”

predicate: “is a man”

object: “Socrates”

“Two is less than four”

predicate: “is less than”

object: “two”, “four”

Sometimes, the objects can be represented by “variable”, and then, in that case, the “predicate proposition” can be evaluated if an element in the universal set is instantiated to the variable.

Example 8.8 Let’s consider the following examples.

(1) x is a man.

(2) y is green.

(3) z is less than w.

(4) p belongs to q.

If an individual is mapped to a variable, the sentence can have its meaning and then we can evaluate the value of proposition. There are examples of individual constants for the above propositions with variables.

(1) Tom is a man.

(2) His face is green.

(3) Two is less than four.

(4) This hat belongs to me.

In the first sentence, Tom is an element in the universal set “man”. The element Tom is instantiated to the variable x, and then we know the value of the proposition is true. In the second sentence, his face is an object (element) corresponded to the variable y, and we can evaluate the truth value of the proposition.

In the formal language of this chapter, we denote predicates by letters. For example, the sentence “x satisfies P” can be written by P(x).

Example 8.9 For example, the above predicate propositions can be represented in the following way.

\[
\begin{align*}
&\text{is\_a\_man}(x), \quad \text{is\_a\_man}(\text{Tom}) \\
&\text{is\_green}(y), \quad \text{is\_green}(\text{his face}) \\
&\text{is\_less\_than}(z, w), \quad \text{is\_less\_than}(\text{two, four}) \\
&\text{belongs\_to}(p, q), \quad \text{belongs\_to}(\text{this hat, me})
\end{align*}
\]

The number of individual constants to which a given predicate is called number of places of the predicate. For instance, “is a man” is a one-place predicate, and “is less than” is a two-place predicate.

A one-place predicate determines a set of things: namely those things for which it is true. Similarly, a two-place predicate determines a set of pairs of things; that is, a two-place “relation”. In general, an n-place predicate determines an n-place relation. We may think of the predicate as denoting the relation.
Example 8.9  For example, the predicate “is man” determines the set of men, and the predicated “is south of” determines the set of pairs $(x, y)$ of cities such that $x$ is south of $y$. For instance, the relation holds when $x = $ Sydney and $y = $ Tokyo, but not when $x = $ New York and $y = $ Seoul. Different predicates may determine the same relation. For example, “$x$ is south of $y$” and “$y$ is north of $x$.” □

8.1.5 Quantifier
The phrase “for all” is called the “universal quantifier” and is denoted symbolically by $\forall$. The phrase “there exists”, “there is a”, or “for some” is called the “existential quantifier” and is denoted symbolically by $\exists$.

The universal quantifier is kind of an iterated conjunction. Suppose there are only finitely-many individuals. That is, the variable $x$ takes only the values $a_1, a_2, \ldots a_n$. Then the sentence $\forall x P(x)$ has the same meaning as the conjunction $P(a_1) \land P(a_2) \land \ldots \land P(a_n)$.

The existential quantifier is kind of an iterated disjunction. If there are only finitely-many individuals $a_1, a_2, \ldots a_n$, then the sentence $\exists x P(x)$ has the same meaning as the disjunction $P(a_1) \lor P(a_2) \lor \ldots \lor P(a_n)$.

Of course, if the number of individuals is infinite, such an interpretation of the quantifier is not possible, since infinitely long sentences are not allowed.

According to De Morgan’s laws, $P(a_1) \lor P(a_2) \lor \ldots \lor P(a_n)$ is equivalent to $\sim [\sim P(a_1) \land \sim P(a_2) \land \ldots \land \sim P(a_n)]$ where the symbol $\sim$ represents the negative operator. This suggests the possibility of defining the existential quantifier from the universal quantifier. We shall do this; $\exists x P(x)$ will be an abbreviation for $\sim \forall x \sim P(x)$. Of course we could also define the universal quantifier from the existential quantifier; $\forall x P(x)$ has the same meaning as $\sim \exists x \sim P(x)$.

8.2 Fuzzy logic
8.2.1 Fuzzy expression
In the fuzzy expression(formula), a fuzzy proposition can have its truth value in the interval $[0,1]$. The fuzzy expression function is a mapping function from $[0,1]$ to $[0,1]$.

$$f : [0,1] \rightarrow [0,1]$$

If we generalize the domain in n-dimension, the function becomes as follows:

$$f : [0,1]^n \rightarrow [0,1]$$

Therefore we can interpret the fuzzy expression as an n-ary relation from n fuzzy sets to $[0,1]$. In the fuzzy logic, the operations such as negation ($\sim$ or $\neg$), conjunction ($\land$) and disjunction ($\lor$) are used as in the classical logic.

Definition(Fuzzy logic)  Then the fuzzy logic is a logic represented by the fuzzy expression(formula) which satisfies the followings.

1) Truth values, 0 and 1, and variable $x_i (\in [0,1], i = 1, 2, \ldots, n)$ are fuzzy expressions.
2) If $f$ is a fuzzy expression, $\sim f$ is also a fuzzy expression.
3) If $f$ and $g$ are fuzzy expressions, $f \land g$ and $f \lor g$ are also fuzzy expressions. □

8.2.2 Operators in fuzzy expression
There are some operators in the fuzzy expression such as $\neg$ (negation), $\land$ (conjunction), $\lor$ (disjunction), and $\rightarrow$ (implication). However the meaning of operators may be different according to the literature. If we follow Lukasiewicz’s definition, the operators are defined as follows for $a, b \in [0,1]$.
Negation \( \overline{a} = 1 - a \)

Conjunction \( a \land b = \text{Min} \ (a, b) \)

Disjunction \( a \lor b = \text{Max} \ (a, b) \)

Implication \( a \rightarrow b = \text{Min} \ (1, 1 + b - a) \)

The properties of fuzzy operators are summarized in Table 8.9.

**Table 8.9  The properties of fuzzy operators**

<table>
<thead>
<tr>
<th>Property</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Involution</td>
<td>( \overline{\overline{a}} = a )</td>
</tr>
</tbody>
</table>
| Commutativity     | \( a \land b = b \land a \) \[
|                  | \( a \lor b = b \lor a \)                                                  |
| Associativity     | \( (a \land b) \land c = a \land (b \land c) \)                          |
|                  | \( (a \lor b) \lor c = a \lor (b \lor c) \)                               |
| Distributivity    | \( a \lor (b \land c) = (a \lor b) \land (a \lor c) \)                    |
|                  | \( a \land (b \lor c) = (a \land b) \lor (a \land c) \)                   |
| Idempotency       | \( a \land a = a \)                                                        |
|                  | \( a \lor a = a \)                                                         |
| Absorption        | \( a \lor (a \land b) = a \)                                              |
|                  | \( a \land (a \lor b) = a \)                                              |
| Absorption by 0   | \( a \land 0 = 0 \)                                                        |
|                  | \( a \lor 1 = 1 \)                                                         |
| Identity          | \( a \land 1 = a \)                                                        |
|                  | \( a \lor 0 = a \)                                                         |
| De Morgan’s law   | \( a \land b = \overline{\overline{a}} \lor \overline{b} \)              |
|                  | \( a \lor b = \overline{a} \land \overline{\overline{b}} \)              |

But we have to notice that the “law of contradiction” and “law of excluded middle” are not verified in the fuzzy logic.

**Example 8.11**  We can see that the two properties are not satisfied in the following examples.

(1) **law of contradiction**

Assume \( a \) is in [0,1].

\[ a \land \overline{a} = \text{Min}[a, \overline{a}] = \text{Min}[a, 1-a] \]
\[
\begin{align*}
\bar{a} &= \begin{cases} a & \text{if } 0 \leq a \leq 0.5 \\ 1-a & \text{if } 0.5 \leq a < 1 \end{cases} \\
\therefore 0 < a \land \bar{a} \leq 0.5 \\
\text{Then } a \land \bar{a} \neq 0 \\
\end{align*}
\]

(2) law of excluded middle

Suppose \(a\) is in \([0,1]\).

\[
\begin{align*}
a \lor \bar{a} &= \text{Max}[a, \bar{a}] = \text{Max}[a, 1-a] \\
&= \begin{cases} a & \text{if } 0.5 \leq a \lt 1 \\ 1-a & \text{if } 0 \lt a \leq 0.5 \end{cases} \\
\therefore 0.5 \leq a \lor \bar{a} < 1 \\
\text{Then } a \lor \bar{a} = 1 & \text{ if } a = 0 \text{ or } 1 \\
& a \lor \bar{a} < 1 & \text{otherwise} 
\end{align*}
\]

8.2.3 Some examples of fuzzy logic operations

Example 8.12 When \(a = 1, b = 0\)

(1) \(\bar{a} = 0\)

(2) \(a \land b = \text{Min}(1, 0) = 0\)

(3) \(a \lor b = \text{Max}(1, 0) = 1\)

(4) \(a \rightarrow b = \text{Min}(1, 1-1+0) = 0\) \(\square\)

Example 8.13 When \(a = 1, b = 1\)

(1) \(\bar{a} = 0\)

(2) \(a \land b = \text{Min}(1,1) = 1\)

(3) \(a \lor b = \text{Max}(1,1) = 1\)

(4) \(a \rightarrow b = \text{Min}(1, 1-1+1) = 1\) \(\square\)

Example 8.14 When \(a = 0.6, b = 0.7\)

(1) \(\bar{a} = 0.4\)

(2) \(a \land b = \text{Min}(0.6,0.7) = 0.6\)

(3) \(a \lor b = \text{Max}(0.6, 0.7) = 0.7\)

(4) \(a \rightarrow b = \text{Min}(1, 1-0.6+0.7) = \text{Min}(1, 1.1) = 1\) \(\square\)

8.3 Linguistic variable

8.3.1 Definition of linguistic variable

When we consider a variable, in general, it takes numbers as its value. If the variable takes linguistic
terms, it is called “linguistic variable”.

**Definition (Linguistic variable)** The linguistic variable is defined by the following quintuple.

\[ \text{Linguistic variable} = (x, T(x), U, G, M) \]

- **x**: name of variable
- **T(x)**: set of linguistic terms which can be a value of the variable
- **U**: set of universe of discourse which defines the characteristics of the variable
- **G**: syntactic grammar which produces terms in T(x)
- **M**: semantic rules which map terms in T(x) to fuzzy sets in U

**Example 8.15** Let’s consider a linguistic variable “X” whose name is “Age”.

\[ X = (\text{Age}, T(\text{Age}), U, G, M) \]

- **Age**: name of the variable X
- **T(Age)**: \{young, very young, very very young, …\} Set of terms used in the discussion of age
- **U**: [0,100] universe of discourse
- **G(Age)**: \( T^{i+1} = \{\text{young}\} \cup \{\text{very}\} \)
- **M(young)**: \( \{ (u, \mu_{\text{young}}(u)) | u \in [0,100] \} \)

\[ \mu_{\text{young}}(u) = \begin{cases} 1 & \text{if } u \in [0,25] \\ \frac{25 - u}{25}^2 & \text{if } u \in [25,100] \end{cases} \]

In the above example, the term “young” is used as a basis in the T(Age), and thus this kind of term is called a “primary term”. When we add modifiers to the primary terms, we can define new terms (fuzzy terms). In many cases, when the modifier “very” is added, the membership is obtained by square operation. For example, the membership function of the term “very young” is obtained from that of “young”.

\[ \mu_{\text{very young}}(u) = (\mu_{\text{young}}(u))^2 \]

The fuzzy linguistic terms often consist of two parts:

1) Fuzzy predicate (primary term): expensive, old, rare, dangerous, good, etc
2) Fuzzy modifier: very, likely, almost impossible, extremely unlikely, etc

The modifier is used to change the meaning of predicate and it can be grouped into the following two classes:

1) Fuzzy truth qualifier or fuzzy truth value: quite true, very true, more or less true, mostly false, etc
2) Fuzzy quantifier: many, few, almost, all, usually, etc

In the following sections, we will introduce the fuzzy predicate, fuzzy modifier, and fuzzy quantifier.
8.3.2 Fuzzy predicate

As we know now, a predicate proposition in the classical logic has the following form.

“x is a man.”
“y is P.”

x and y are variables, and “man” and “P” are crisp sets. The sets of individuals satisfying the predicates are written by “man(x)” and “P(y)”.

Definition (Fuzzy predicate) If the set defining the predicates of individual is a fuzzy set, the predicate is called a fuzzy predicate.

Example 8.16 For example,

“z is expensive.”
“w is young.”

The terms “expensive” and “young” are fuzzy terms. Therefore the sets “expensive(z)” and “young(w)” are fuzzy sets.

When a fuzzy predicate “x is P” is given, we can interpret it in two ways.

(1) P(x) is a fuzzy set. The membership degree of x in the set P is defined by the membership function \( \mu_{P(x)} \).

(2) \( \mu_{P(x)} \) is the satisfactory degree of x for the property P. Therefore, the truth value of the fuzzy predicate is defined by the membership function.

\[ \text{Truth value} = \mu_{P(x)} \]

8.3.3 Fuzzy modifier

As we know, a new term can be obtained when we add the modifier “very” to a primary term. In this section we will see how semantic of the new term and membership function can be defined.

Example 8.17 Let’s consider a linguistic variable “Age” in Fig 8.1. Linguistic terms “young” and “very young” are defined in the universal set U.

\[ U = \{ u | u \in [0,100] \} \]

The variable Age takes a value in the set T(Age).

\[ T(Age) = \{ \text{young, very young, very very young, … } \} \]
Fig 8.1  Linguistic variable “Age”

In the figure, the term “young” is represented by a membership function $\mu_{\text{young}}(u)$. When we represent the term “very young”, we can use the square of $\mu_{\text{young}}(u)$ as follows.

$$
\mu_{\text{very young}}(u) = (\mu_{\text{young}}(u))^2
$$

The graph of membership function of “very young” is given in the figure.  □

8.4 Fuzzy qualifier
8.4.1 Fuzzy truth values

Baldwin defined fuzzy truth qualifier in the universal set $V = \{v \mid v \in [0,1]\}$ as follows.

$$T = \{\text{true, very true, fairly true, absolutely true, \ldots, absolutely false, fairly false, false}\}$$

The qualifiers in $T$ define “fuzzy truth values” and they can be defined by the membership functions. If we take baldwin’s membership function $\mu_{\text{true}}(v)$, the truth qualifiers are represented in the followig membership functions(Fig 8.2).
Fig 8.2  Baldwin’s truth graph

\begin{align*}
\mu_{\text{true}}(v) &= v & v \in [0,1] \\
\mu_{\text{very true}}(v) &= (\mu_{\text{true}}(v))^2 & v \in [0,1] \\
\mu_{\text{fairly true}}(v) &= (\mu_{\text{true}}(v))^{1/2} & v \in [0,1] \\
\mu_{\text{false}}(v) &= 1 - \mu_{\text{true}}(v) & v \in [0,1] \\
\mu_{\text{very false}}(v) &= (\mu_{\text{false}}(v))^2 & v \in [0,1] \\
\mu_{\text{fairly false}}(v) &= (\mu_{\text{false}}(v))^{1/2} & v \in [0,1] \\
\mu_{\text{absolutely true}}(v) &= \begin{cases} 1 & \text{for } v = 1 \\ 0 & \text{otherwise} \end{cases} \\
\mu_{\text{absolutely false}}(v) &= \begin{cases} 1 & \text{for } v = 0 \\ 0 & \text{otherwise} \end{cases}
\end{align*}

**Example 8.18** Let’s consider a predicate using the primary term “young” and fuzzy truth qualifier “very false”.

\[ P = \text{“Tom is young is very false.”} \]

Suppose the term “young” is defined by the function \( \mu_{\text{young}} \).

\[ \mu_{\text{young}}(u) = \begin{cases} 1 & u \in [0,25] \\ \frac{1}{(1 + \frac{u-25}{5})^{-2}} & u \in [25,100] \end{cases} \]

The term “very false” can be defined by the following.

\[ \mu_{\text{very false}} = (1 - \mu_{\text{true}}(u))^2 = (1 - \mu_{\text{young}}(u))^2 = \begin{cases} 0 & u \in [0,25] \\ (1 - (1 + \frac{u-25}{5})^{-2})^2 & u \in [25,100] \end{cases} \]

Therefore, if Tom has age less than 25, the truth value of the predicate \( P \) is 0. If he is in \([25,100]\), the truth value is calculated from \( \mu_{\text{very false}} \).

**8.4.2 Examples of fuzzy truth qualifier**

**Example 8.19** Let’s consider a predicate \( p \) in the following.
P = “20 is young.”

Assume the terms “young” and “very young” are defined as shown in Fig 8.3.

We see the membership degree of 20 in “young” is 0.9. Therefore, the truth value of the predicate P is 0.9. Now we can modify the predicate P by using fuzzy truth qualifiers as follows.

\[ P_1 = \text{“20 is young is true.”} \]
\[ P_2 = \text{“20 is young is fairly true.”} \]
\[ P_3 = \text{“20 is young is very true.”} \]
\[ P_4 = \text{“20 is young is false.”} \]

The truth values are changed according to the qualifiers as shown in Fig 8.4.

---

**Fig 8.3** Fuzzy sets “young” and “very young”
Fig 8.4 The truth values of fuzzy proposition

We know already $\mu_{\text{young}}(20)$ is 0.9. That is, the truth value of P is 0.9. For the predicate P₁, we use the membership function “true” in the figure and obtain the truth value 0.9. For P₂, the membership function “fairly true” is used and 0.95 is obtained. In the same way, we can calculate for P₃ and P₄ and summarize the truth values in the following.

For P₁: 0.9
For P₂: 0.95
For P₃: 0.81
For P₄: 0.1

8.5 Representation of fuzzy rule
8.5.1 Inference and knowledge representation

In general, the “inference” is a process to obtain new information by using existing knowledge. The representation of knowledge is an important issue in the inference. When we consider the representation methods, the following rule type “if-then” is the most popular form.

“If x is a, then y is b.”

The rule is interpreted as an “implication” and consists of the “antecedent (if part)” and “consequent (then part)”. If a rule is given in the above form and we have a fact in the following form,

“x is a”

Then we can infer and obtain new result:

“y is b”

Based on the above discussion, we can summarize two types of “reasoning”.

1) Modus ponens
   
   Fact: x is a
   Rule: If x is a, then y is b
   Result: y is b

2) Modus tollens
   
   Fact: y is \( \overline{b} \)
Rule: If $x$ is $a$ then $y$ is $b$

Result: $x$ is $a$.

The modus ponens is used in the forward inference and the modus tollens is in the backward one.

### 8.5.2 Representation of fuzzy predicate by fuzzy relation

We saw that a fuzzy predicate is considered as a fuzzy set. In this section, we will see how the fuzzy predicate is used in fuzzy inference. When there is a fuzzy predicate proposition such that “$x$ is $P$”, it is represented by fuzzy set $P(x)$ and whose membership function is by $\mu_{P(x)}(x)$. We know also a fuzzy relation is one type of fuzzy sets, and thus we can represent a predicate by using a relation.

“$R(x) = P$”

$P$ is a fuzzy set and $R(x)$ is a relation that consists of elements in $P$. The membership function of the predicate is represented by $\mu_{P(x)}(x)$ which shows the membership degree of $x$ in $P$. The predicate represented by a relation will be used in the representation of fuzzy rule and premise.

### 8.5.3 Representation of fuzzy rule

When we consider fuzzy rules, the general form is given in the following.

If $x$ is $A$, then $y$ is $B$.

The fuzzy rule may include fuzzy predicates in the antecedent and consequent, and it can be rewritten as in the form.

If $A(x)$, then $B(y)$

This rule can be represented by a relation $R(x, y)$.

$R(x, y)$: If $A(x)$, then $B(y)$

Or

$R(x, y)$: $A(x) \rightarrow B(y)$

If there are a rule and facts involving fuzzy sets, we can execute two types of reasoning.

1) Generalized modus ponens (GMP)

Fact: $x$ is $A'$

Rule: If $x$ is $A$ then $y$ is $B$

Result: $y$ is $B'$

2) Generalized modus tollens (GMT)

Fact: $y$ is $B'$

Rule: If $x$ is $A$ then $y$ is $B$

Result: $x$ is $A'$

In the above reasoning, we see that the facts ($A'$ and $B'$) are not exactly same with the antecedents ($A$ and $B$) in the rules; the results may be also different from the consequents. Therefore, we call this kind of inference as “fuzzy (approximate) reasoning or inference”.

In general, when we execute the fuzzy (approximate) reasoning, we apply the “compositional rule of inference” which shows the procedure of the reasoning. The operation used in the reasoning is denoted by the notation “$\circ$”, and thus the result is represented by the output of the composition when we use the
Example 8.20 We have knowledge such as:

If $x$ is $A$ then $y$ is $B$

$x$ is $A'$

From the above knowledge, how can we apply the inference procedure to get new information about $z$?

(1) We apply the implication operator to get implication relation

$$R(x, y) = A \times B.$$ Here, the cartesian product $A \times B$ is used.

$$R(x, y): A(x) \rightarrow B(y)$$

(2) We manipulate the fact into the form and the apply the generalized modus ponens

$$R(x) = R(y) \circ R(x, y)$$

In this step, composition operator $\circ$ is used.

Therefore, there are two issues in the fuzzy reasoning: determination of the “implication relation” $R(x,y)$ and selection of the “composition operator”. These issues will be developed in the next chapter.
[SUMMARY]

- Classical logic
  - proposition: a sentence having truth value true (1) or false (0)
  - truth value: proposition $\rightarrow \{0, 1\}$
  - logic variable: variable representary a proposition

- Logic operation
  - negation (NOT): $\bar{P}$
  - conjunction (AND): $a \land b$
  - disjunction (OR): $a \lor b$
  - implication ($\rightarrow$): $a \rightarrow b$

- Logic function
  - logic function
  - logic primitive
  - logic formula

- Tautology
  - tautology: logic formula whose value is always true
  - inference: developing new facts by using the tautology

- Deductive inference
  - modus ponens
  - modus tollens
  - hypothetical syllogism
 Predicate logic
- predicate
- predicate logic
- predicate proposition: proposition consist of predicate and object
- evaluation of proposition

 Quantifier
- universal quantifier: ∀ (for all)
- existential quantifier: ∃ (there exists)

 Fuzzy logic
- fuzzy logic formula
- fuzzy proposition
- truth value: fuzzy proposition \( \rightarrow [0, 1] \)

 Fuzzy logic operation
- negation (NOT): \( \bar{P} \)
- conjunction (AND): \( a \land b \)
- disjunction (OR): \( a \lor b \)
- implication (\( \rightarrow \)): Min(1, \( 1+b-a \))

 Linguistic variable
- linguistic variable
- linguistic terms (fuzzy sets)

 Fuzzy predicate
- fuzzy predicate: predicate represented by fuzzy sets
- fuzzy truth value \([0, 1]\)
- fuzzy modifier

 Fuzzy truth qualifier
- fuzzy truth value: true, very true, fairly true, etc.
- $\mu_{\text{very true}}(v) = (\text{true}(v))^2$
- value of “P is very true” is 0.81 when value of P is 0.9

- Fuzzy rule
  - rule representation: if $A(x)$ then $B(y)$
  - rule is a relation $R(x, y)$
  - inference: $R(y) = R(x) \circ R(x, y)$

- Fuzzy reasoning (inference)
  - approximate reasoning: facts may not equal to antecedents
  - generalized modus ponens
  - generalized modus tollens
8.1 Evaluate the following propositions.

(1) \[2 + 6 = 9\]
(2) \[3 + 4 = 7\]
(3) \[x + 5 = 7\]
(4) Einstein is a man.
(5) The man has no head.

8.2 Develop the truth value of “exclusive OR”.

8.3 Evaluate the following implication propositions.

(1) \[a \rightarrow b\] where \[a : 2 + 2 = 5, \quad b : 3 + 3 = 6\]
(2) \[a \rightarrow b\] where \[a : 3 + 4 = 6, \quad b : 4 + 2 = 5\]

8.4 Show truth values of the logic formula.

\[
(\overline{b} \land (a \rightarrow b)) \rightarrow \overline{a}
\]

8.5 Evaluate the following predicate propositions.

(1) Sophie is a woman.
(2) \[x\] is greater than 5 where \[x = 6\].
(3) \[y\] is green where \[y\] is a tree.
(4) \[p\] belongs to me where \[p\] is a bag.

8.6 Evaluate the following fuzzy logic formulas where \[a = 0.5\] and \[b = 0.7\].

(1) \[\overline{a} = 0.4\]
(2) \[a \land b\]
8.7 Define components for the linguistic variable X whose name is Temperature.

\[ X = (\text{Temperature}, T(\text{Temperature}), U, G, M) \]

8.8 Determine the truth value of the following propositions P1 and P2.

\[ P_1 = \text{"P is very true"} \]
\[ P_2 = \text{"P is false"} \]

where

\[ P = \text{"30 is high"}, \]

the truth value of P is 0.3,

\[ \mu_{\text{very\ true}} = (\mu_{\text{true}})^2 \]

8.9 Show the following rule and fact in the form of relation and generalized modus ponens.

Rule: If A(x) then B(y) : R(x, y)

Fact: x is A' : R(x)

Result: y is B(y) : R(y) = R(x) \circ R(x, y)