IF-THEN RULES
AND
FUZZY INFERENCE
Inference

Inference

\textbf{In"fer*ence\textbackslash{}, n. [From Infer\textbackslash{}.]}

1. The act or process of inferring by deduction or induction.

2. That which inferred; a truth or proposition drawn from another which is admitted or supposed to be true; a conclusion; a deduction. --Milton.

Inference is a process of obtaining new knowledge through existing knowledge.
To perform inference, knowledge should be represented in some form.

Representation of knowledge as rules is the most popular form.

\[
\text{if } x \text{ is } A \text{ then } y \text{ is } B
\]

(\text{where } A \text{ and } B \text{ are linguistic values defined by fuzzy sets on universes of discourse } X \text{ and } Y).\]

- A rule is also called a fuzzy implication
- “\text{x is A}” is called the antecedent or premise
- “\text{y is B}” is called the consequence or conclusion
Representation of knowledge

Examples:

- If pressure is high, then volume is small.
- If the road is slippery, then driving is dangerous.
- If an apple is red, then it is ripe.
- If the speed is high, then apply the brake a little.
Knowledge as Rules

- How do you reason?
  - You want to play golf on Saturday or Sunday and you don’t want to get wet when you play.

- Use rules!
  - If it rains, you get wet!
  - If you get wet, you can’t play golf

- If it rains on Saturday and won’t rain on Sunday
  - You play golf on Sunday!

*Fuzzy Thinking: The new Science of Fuzzy Logic, Bart Kosko*
Knowledge as Rules

- Knowledge is rules
- Rules are in black-and-white language
  - Bivalent rules
- AI has so far, after over 30 years of research, not produced smart machines!
  - Because they can’t yet put enough rules in the computer (use 100-1000 rules, need >100k}
  - Throwing more rules at the problem

*Fuzzy Thinking: The new Science of Fuzzy Logic, Bart Kosko*
Forms of reasoning

Generalized Modus Ponens:

Premise: \( x \text{ is } A' \)
Implication: \( \text{if } x \text{ is } A \text{ then } y \text{ is } B \)
Consequence: \( y \text{ is } B' \)

Where \( A, A', B, B' \) are fuzzy sets and \( x \) and \( y \) are symbolic names for objects.
Forms of reasoning

Generalized Modus Tolens:

Premise: \( y \text{ is } B' \)
Implication: \( \text{if } x \text{ is } A \text{ then } y \text{ is } B \)
Consequence: \( x \text{ is } A' \)

Where \( A, A', B, B' \) are fuzzy sets and \( x \) and \( y \) are symbolic names for objects.

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Fuzzy rule as a relation

if $x$ is $A$ then $y$ is $B$

“$x$ is $A$”, “$y$ is $B$” – fuzzy predicates $A(x)$, $B(y)$

if $A(x)$ then $B(y)$

can be represented as a relation

$R(x,y) : A(x) \rightarrow B(y)$

where $R(x,y)$ can be considered a fuzzy set with 2-dimensional membership function

$\mu_R(x,y) = f(\mu_A(x), \mu_B(y))$

where $f$ is fuzzy implication function

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MIN fuzzy implication

- Interprets the fuzzy implication as the minimum operation [Mamdani].

\[ R_C = A \times B \]

\[ = \int_{X \times Y} \mu_A(x) \wedge \mu_B(y) / (x, y) \]

where \( \wedge \) is the min operator

http://if.kaist.ac.kr/lecture/cs670/textbook/
PRODUCT fuzzy implication

- Interprets the fuzzy implication as the product operation [Larsen].

\[
R_p = A \times B \\
= \int_{X \times Y} \mu_A(x) \cdot \mu_B(y) / (x, y)
\]

where \( \cdot \) is the algebraic product operator
EXAMPLE OF FUZZY IMPLICATION

Fuzzy rule:
“If temperature is high, then humidity is fairly high”

Let’s define:
- $T$ – universe of discourse for temperature
- $H$ – universe of discourse for humidity
- $t \in T$, $h \in H$ – variables for temperature and humidity
- Denote “high” as $A$, $A \subseteq T$
- Denote “fairly high” as $B$, $B \subseteq H$

Then the rule becomes:
$R(t,h)$: if $t$ is $A$ then $h$ is $B$ or $R(t,h): R(t) \rightarrow R(h)$

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EXAMPLE OF FUZZY IMPLICATION

if we know $A$ and $B$, we can find $R(t,h)=A \times B$

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<tr>
<th>$t$</th>
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<tbody>
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<td>$\mu_A(t)$</td>
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<td>$\mu_B(h)$</td>
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$R_C(t, h) = A \times B$

$= \int \mu_A(t) \land \mu_B(h) / (t, h)$

Mamdani (min) implication

http://if.kaist.ac.kr/lecture/cs670/textbook/
EXAMPLE OF FUZZY IMPLICATION

we know $R_{C}(t, h)$ for fuzzy rule

“If temperature is high, then humidity is fairly high”

According to this rule, what is the humidity when
“temperature is fairly high” or $t$ is $A'$, $A' \subseteq T$ ?

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EXAMPLE OF FUZZY IMPLICATION

We can use composition of fuzzy relations to find $R(h)$!

$$R(t) \circ R_C(t, h) = R(h)$$

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In order to draw conclusions from a set of rules (rule base) one needs a mechanism that can produce an output from a collection of rules. This is done using the compositional rule of inference.

Consider a single fuzzy rule and its inference

Rule: if $v$ is $A$ then $w$ is $C$
Input: $v$ is $A'$
Result: $C'$

$A \subseteq U$, $C \subseteq W$, $v \in U$, and $w \in C$.

The fuzzy rule is interpreted as an implication

$R: A \rightarrow C$ or $R = A \times C$

When input $A'$ is given to the inference system, the output $C' = A' \circ R$
COMPOSITIONAL RULE OF INFERENCE

\[ C' = A' \circ R \]

"\(\circ\)" is the composition operator. The inference procedure is called “compositional rule of inference”. The inference mechanism is determined by two factors:

1. Implication operators:
   - Mamdani: min
   - Larsen: algebraic product

2. Composition operators:
   - Mamdani: max-min
   - Larsen: max-product
Compositional rule of inference can be represented graphically as a combination of cylindrical extension, intersection and projection of fuzzy sets:

1. Build a cylindrical extension of $A$, $A(x,y)$
2. Determine intersection of $R(x,y)$ and $A(x,y)$
3. Build projection of $R(x,y) \land A(x,y)$
COMPOSITIONAL RULE OF INFEERENCE

(a) Fuzzy Relation R on X and Y

(b) Cylindrical Extension of A

(c) Minimum of (a) and (b)

(d) Projection of (c) onto Y
There are many methods to perform fuzzy inference. Consider a fuzzy rule:

\[ R_1: \text{if } u \text{ is } A_1 \text{ and } v \text{ is } B_1 \text{ then } w \text{ is } C_1 \]

Inputs \( u \) and \( v \) can be:

- **crisp inputs.** Crisp inputs can be treated as fuzzy singletons
- **fuzzy sets** \( A' \) and \( B' \)

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MAMDANI METHOD

This method uses the minimum operation $R_C$ as a fuzzy implication and the max-min operator for the composition.

Suppose a rule base is given in the following form:

$R_i$: if $u$ is $A_i$ and $v$ is $B_i$ then $w$ is $C_i$, $i = 1, 2, \ldots, n$

for $u \in U$, $v \in V$, and $w \in W$.

Then, $R_i = (A_i \text{ and } B_i) \rightarrow C_i$ is defined by

$$\mu_{R_i} = \mu_{(A_i \text{ and } B_i \rightarrow C_i)}(u, v, w)$$
Case 1: Inputs are crisp and treated as fuzzy singletons.

\[ u = u_0, \quad v = v_0 \]

\[ \mu_{C_i}(w) = \left[ \mu_{A_i}(u_0) \text{ and } \mu_{B_i}(v_0) \right] \rightarrow \mu_{C_i}(w) \]

Inference

Result

Example:

if temperature is high and humidity is high then fan speed is high

How to determine the fan speed for temperature 85°F and humidity 93%?
MAMDANI METHOD

Mamdani method uses min operator ($\land$) as fuzzy implication function ($\rightarrow$):

$$\mu_{C_i}(w) = \alpha_i \land \mu_{C_i}(w)$$

where $\alpha_i = \mu_{A_i}(u_0) \land \mu_{B_i}(v_0)$

$\alpha_i$ is called "firing strength", "matching degree", "satisfaction degree"

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MAMDANI METHOD

For multiple rules (for example, two rules $R_1$ and $R_2$):

$$\mu_{C'}(w) = \mu_{C'_1} \vee \mu_{C'_2}$$

$$= [\alpha_1 \land \mu_{C'_1}(w)] \vee [\alpha_2 \land \mu_{C'_2}(w)]$$
MAMDANI METHOD

In general:

\[ \mu_{C'}(w) = \bigvee_{i=1}^{n} \left[ \alpha_i \land \mu_{C_i}(w) \right] = \bigvee_{i=1}^{n} \mu_{C'_i}(w) \]

max \quad min

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Case 2: Inputs are fuzzy sets $A'$, $B'$

$$\mu_{C_i}(w) = \alpha_i \wedge \mu_{C_i}(w)$$

where $\alpha_i = \min \left[ \max \left( \mu_{A'}(u) \wedge \mu_{A_i}(u) \right), \max \left( \mu_{B'}(v) \wedge \mu_{B_i}(v) \right) \right]$
MAMDANI METHOD

For multiple rules,

\[ \mu_{C'}(w) = \bigvee_{i=1}^{n} [\alpha_i \wedge \mu_{C_i}(w)] = \bigvee_{i=1}^{n} \mu_{C_i'}(w) \]

\[ C' = \bigcup_{i=1}^{n} C_i' \]
EXAMPLE OF MAMDANI METHOD

Let the fuzzy rule base consist of one rule:

R: If \( u \) is \( A \) then \( v \) is \( B \)

where \( A=(0, 2, 4) \) and \( B=(3, 4, 5) \) are triangular fuzzy sets

Question 1: What is the output \( B' \) if the input is a crisp value \( u_0=3 \)?

Question 2: What is the output \( B' \) if the input is a fuzzy set \( A'=(0, 1, 2) \)?

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EXAMPLE OF MAMDANI METHOD

Fuzzy inference with input $u_0=3$

Fuzzy inference with input $A'=(0, 1, 2)$.

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This method uses the product operation $R_p$ as a fuzzy implication and the max-product operator for the composition.

Suppose a rule base is given in the following form:

$R_i$: if $u$ is $A_i$ and $v$ is $B_i$ then $w$ is $C_i$, $i = 1, 2, \ldots, n$

for $u \in U$, $v \in V$, and $w \in W$.

Then, $R_i = (A_i \text{ and } B_i) \rightarrow C_i$ is defined by

$$\mu_{R_i} = \mu_{(A_i \text{ and } B_i \rightarrow C_i)}(u, v, w)$$
LARSEN METHOD

Case 1: Inputs are crisp and treated as fuzzy singletons.

\[ u = u_0, \ v = v_0 \]

\[ \mu_{C_i}(w) = [\mu_{A_i}(u_0) \text{ and } \mu_{B_i}(v_0)] \rightarrow \mu_{C_i}(w) \]

Inference if \dots then \dots

result

\[ = [\mu_{A_i}(u_0) \land \mu_{B_i}(v_0)] \cdot \mu_{C_i}(w) \]

\[ = \alpha_i \cdot \mu_{C_i}(w) \quad \text{where } \alpha_i = \mu_{A_i}(u_0) \land \mu_{B_i}(v_0) \]

For multiple rules:

\[ \mu_{C'}(w) = \bigvee_{i=1}^{n} \alpha_i \cdot \mu_{C_i}(w) = \bigvee_{i=1}^{n} \mu_{C_i}(w) \]

\[ C' = \bigcup_{i=1}^{n} C_i' \]

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LARSEN METHOD

Graphical representation of Larsen method with singleton input

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LARSEN METHOD

Case 2: Inputs are fuzzy sets $A'$, $B'$

$$\mu_{C'_i}(w) = \alpha_i \cdot \mu_{C_i}(w)$$

where $\alpha_i = \min[\max(\mu_{A'_i}(u) \land \mu_{A_i}(u)), \max(\mu_{B'_i}(v) \land \mu_{B_i}(v))]$

For multiple rules:

$$\mu_{C''}(w) = \bigvee_{i=1}^{n}[\alpha_i \cdot \mu_{C_i}(w)] = \bigvee_{i=1}^{n} \mu_{C'_i}(w)$$

$$C'' = \bigcup_{i=1}^{n} C'_i$$

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Graphical representation of Larsen method with fuzzy set inputs
EXAMPLE OF LARSEN METHOD

The fuzzy rule base consists of one rule:

\[ R: \text{If } u \text{ is } A \text{ and } v \text{ is } B \text{ then } w \text{ is } C \]

Where \( A = (0, 2, 4) \), \( B = (3, 4, 5) \) and \( C = (3, 4, 5) \) are triangular fuzzy sets.

Question 1: What is the output \( C' \) if the inputs are crisp values \( u_0 = 3, v_0 = 4 \)?

Question 2: What is the output \( C' \) if the inputs are fuzzy sets \( A' = (0, 1, 2) \) and \( B' = (2, 3, 4) \)?
EXAMPLE OF LARSEN METHOD

Larsen method with input $u_0 = 3$, $v_0 = 4$.

Larsen method with input $A' = (0, 1, 2)$, $B' = (2, 3, 4)$. 

[Diagram showing the Larsen method with input values]
The output of Mamdani and Larsen inference methods is a fuzzy set! For practical applications, a crisp value is often needed. The process of converting a fuzzy answer into a crisp value is called defuzzification.
SUMMARY

Inference - the logical process by which new facts are derived from known facts by the application of inference rules.

Fuzzy rules – a convenient way to represent knowledge

A fuzzy rule can be represented as a fuzzy relation connected by a fuzzy implication function

The fuzzy inference procedure is called the compositional rules of inference
SUMMARY

Mamdani and Larsen methods are two very popular methods of fuzzy inference. There are many more inference methods. Defuzzification is needed for the results obtained through fuzzy inference.