Chapter 10:
MOMENTS OF INERTIA
APPLICATIONS

Many structural members like beams and columns have cross sectional shapes like I, H, C, etc..

Why do they usually not have solid rectangular, square, or circular cross sectional areas?

What primary property of these members influences design decisions?

How can we calculate this property?
Many structural members are made of tubes rather than solid squares or rounds.

Why?

What parameters of the cross sectional area influence the designer’s selection?

How can we determine the value of these parameters for a given area?
Consider a plate submerged in a liquid. The pressure of a liquid at a distance $z$ below the surface is given by $p = \gamma z$, where $\gamma$ is the specific weight of the liquid.

The force on the area $dA$ at that point is $dF = p \, dA$. The moment about the x-axis due to this force is $z \, (dF)$. The total moment is

$$\int_A z \, dF = \int_A \gamma z^2 \, dA = \gamma \int_A (z^2 \, dA).$$

This sort of integral term also appears in solid mechanics when determining stresses and deflection. This integral term is referred to as the **moment of inertia** of the area of the plate about an axis.
Consider three different possible cross sectional shapes and areas for the beam RS. All have the same total area and, assuming they are made of same material, they will have the same mass per unit length.

For the given vertical loading P on the beam, which shape will develop less internal stress and deflection? Why?

The answer depends on the MoI of the beam about the x-axis. It turns out that Section A has the highest MoI because most of the area is farthest from the x axis. Hence, it has the least stress and deflection.
For the differential area \( dA \), shown in the figure:

\[
\begin{align*}
    d I_x &= y^2 \, dA, \\
    d I_y &= x^2 \, dA, \quad \text{and,} \\
    d J_O &= r^2 \, dA,
\end{align*}
\]

where \( J_O \) is the polar moment of inertia about the pole \( O \) or \( z \) axis.

The moments of inertia for the entire area are obtained by integration.

\[
\begin{align*}
    I_x &= \int_A y^2 \, dA; \\
    I_y &= \int_A x^2 \, dA \\
    J_O &= \int_A r^2 \, dA = \int_A (x^2 + y^2) \, dA = I_x + I_y
\end{align*}
\]

The MoI is also referred to as the second moment of an area and has units of length to the fourth power (\( \text{m}^4 \) or \( \text{in}^4 \)).
10.3 RADIUS OF GYRATION OF AN AREA

For a given area $A$ and its MoI, $I_x$, imagine that the entire area is located at distance $k_x$ from the $x$ axis.

Then, $I_x = k_x^2 A$ or $k_x = \sqrt{\frac{I_x}{A}}$. This $k_x$ is called the radius of gyration of the area about the $x$ axis. Similarly;

$$k_y = \sqrt{\frac{I_y}{A}} \text{ and } k_O = \sqrt{\frac{J_O}{A}}$$

The radius of gyration has units of length and gives an indication of the spread of the area from the axes. This characteristic is important when designing columns.
10.2 Parallel-Axis Theorem for an Area

Since the moment of inertia of \( dA \) about the \( x \) axis is \( dI_x = (y' + d_y)^2 \, dA \), then for the entire area,

\[
I_x = \int_A (y' + d_y)^2 \, dA
= \int_A y'^2 \, dA + 2d_y \int_A y' \, dA + d_y^2 \int_A dA
\]

The first integral represents the moment of inertia of the area about the centroidal axis, \( \bar{I}_x \). The second integral is zero since the \( x' \) axis passes through the area’s centroid \( C \); i.e., \( \int y' \, dA = \bar{y} \int dA = 0 \) since \( \bar{y} = 0 \). Realizing that the third integral represents the total area \( A \), the final result is therefore

\[
I_x = \bar{I}_x' + Ad_y^2 \quad I_y = \bar{I}_y' + Ad_x^2 \quad J_0 = \bar{J}_C + Ad_y^2
\]
Determine the moment of inertia for the rectangular area shown in Fig. 10–5 with respect to (a) the centroidal \( x' \) axis, (b) the axis \( x_b \) passing through the base of the rectangle, and (c) the pole or \( z' \) axis perpendicular to the \( x'-y' \) plane and passing through the centroid \( C \).

**Solution (Case 1)**

**Part (a).** The differential element shown in Fig. 10–5 is chosen for integration. Because of its location and orientation, the *entire element* is at a distance \( y' \) from the \( x' \) axis. Here it is necessary to integrate from \( y' = -h/2 \) to \( y' = h/2 \). Since \( dA = b \ dy' \), then

\[
\bar{I}_{x'} = \int_A y'^2 \, dA = \int_{-h/2}^{h/2} y'^2 (b \, dy') = b \int_{-h/2}^{h/2} y'^2 \, dy
\]

\[
= \frac{1}{12} bh^3
\]

**Ans.**

**Part (b).** The moment of inertia about an axis passing through the base of the rectangle can be obtained by using the result of part (a) and applying the parallel-axis theorem, Eq. 10–3.

\[
I_{x_b} = \bar{I}_{x'} + Ad_y^2
\]

\[
= \frac{1}{12} bh^3 + bh \left( \frac{h}{2} \right)^2 = \frac{1}{3} bh^3
\]

**Ans.**
**Part (c).** To obtain the polar moment of inertia about point $C$, we must first obtain $I_{y'}$, which may be found by interchanging the dimensions $b$ and $h$ in the result of part (a), i.e.,

$$I_{y'} = \frac{1}{12} hb^3$$

Using Eq. 10–2, the polar moment of inertia about $C$ is therefore

$$I_C = I_{x'} + I_{y'} = \frac{1}{12} bh(h^2 + b^2)$$
10.4 MoI FOR AN AREA BY INTEGRATION

For simplicity, the area element used has a differential size in only one direction (dx or dy). This results in a single integration and is usually simpler than doing a double integration with two differentials, dx·dy.

The step-by-step procedure is:

1. Choose the element dA: There are two choices: a vertical strip or a horizontal strip. Some considerations about this choice are:

   a) The element parallel to the axis about which the MoI is to be determined usually results in an easier solution. For example, we typically choose a horizontal strip for determining \( I_x \) and a vertical strip for determining \( I_y \).
b) If \( y \) is easily expressed in terms of \( x \) (e.g., \( y = x^2 + 1 \)), then choosing a vertical strip with a differential element \( dx \) wide may be advantageous.

2. Integrate to find the MoI. For example, given the element shown in the figure above:

\[
I_y = \int x^2 \, dA = \int x^2 \, y \, dx \quad \text{and} \quad I_x = \int dI_x = \int \left( \frac{1}{3} \right) y^3 \, dx \quad (\text{using the information for a rectangle about its base from the inside back cover of the textbook}).
\]

Since in this case the differential element is \( dx \), \( y \) needs to be expressed in terms of \( x \) and the integral limit must also be in terms of \( x \). As you can see, choosing the element and integrating can be challenging. It may require a trial and error approach plus experience.
EXAMPLE

**Given:** The shaded area shown in the figure.

**Find:** The MoI of the area about the x- and y-axes.

**Plan:** Follow the steps given earlier.

**Solution**

\[
I_x = \int y^2 \, dA
\]

\[
dA = (4 - x) \, dy = (4 - y^2/4) \, dy
\]

\[
I_x = \int_0^4 y^2 \left(4 - \frac{y^2}{4}\right) \, dy
\]

\[
= \left[ \frac{4}{3} y^3 - \frac{1}{20} y^5 \right]_0^4 = 34.1 \text{ in}^4
\]
In the above example, it will be difficult to determine $I_y$ using a horizontal strip. However, $I_x$ in this example can be determined using a vertical strip. So,

$$I_x = \int (1/3) y^3 \, dx = \int (1/3) (2\sqrt{x})^3 \, dx.$$
CONCEPT QUIZ

1. A pipe is subjected to a bending moment as shown. Which property of the pipe will result in lower stress (assuming a constant cross-sectional area)?

A) Smaller $I_x$  B) Smaller $I_y$
C) Larger $I_x$  D) Larger $I_y$
GROUP PROBLEM SOLVING

**Given:** The shaded area shown.

**Find:** $I_x$ and $I_y$ of the area.

**Plan:** Follow the steps described earlier.

**Solution**

\[
I_x = \int (1/3) y^3 \ dx
\]

\[
= \int_{0}^{8} (1/3) x \ dx = \left[ \frac{x^2}{6} \right]_{0}^{8}
\]

\[= 10.7 \text{ in}^4\]
\[ I_y = \int x^2 \, dA = \int x^2 y \, dx \]
\[ = \int x^2 \left( x^{1/3} \right) \, dx \]
\[ = 0 \int_0^8 x^{7/3} \, dx \]
\[ = \left[ \left( \frac{3}{10} \right) x^{10/3} \right]_0^8 \]
\[ = 307 \text{ in}^4 \]