Finite Element Method

Chapter 6

Part I

Review of the Basic Elasticity Theory
Review of the Basic Elasticity Theory

Stresses & Strains

\[ \sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx} \] for stresses,

\[ \varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx} \] for strains.
Review of the Basic Elasticity Theory

Two-dimensional State of Stress

\[ \sigma_y + \frac{\partial \sigma_y}{\partial y} dy \quad \tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} dy \]

\[ \sigma_x + \frac{\partial \sigma_x}{\partial x} dx \]

\[ \tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx \]
Review of the Basic Elasticity Theory

Equations of Equilibrium:

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + X_b = 0
\]

\[
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + Y_b = 0
\]

\[
\sum M_C = 0
\]

\[
\tau_{xy} \frac{dy}{2} \left( \tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} \right) dx - \tau_{yx} \frac{dx}{2} - \tau_{yx} \frac{dy}{2} \left( \tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \right) dy = 0
\]

\[
\tau_{xy} = \tau_{yx}
\]
Review of the Basic Elasticity Theory

Differential Equations of Equilibrium for 2D Problems:

\[
\begin{align*}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X_b &= 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y_b &= 0 \\
\tau_{xy} &= \tau_{yx}
\end{align*}
\]
Similarly Equilibrium for 3D Problems:

\[
\begin{align*}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X_b &= 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y_b &= 0 \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z_b &= 0
\end{align*}
\]

\[\tau_{xy} = \tau_{yx}\]
\[\tau_{xz} = \tau_{zx}\]
\[\tau_{yz} = \tau_{zy}\]

Review of the Basic Elasticity Theory
Review of the Basic Elasticity Theory

Strains for 2D Problem

\[ \varepsilon_x = \frac{\partial u}{\partial x} \]
\[ \varepsilon_y = \frac{\partial v}{\partial y} \]
\[ \gamma_{xy} = \gamma_1 + \gamma_2 \approx \tan \gamma_1 + \tan \gamma_2 \]
\[ \approx \frac{(\partial v / \partial x) \, dx}{dx \left(1 + \partial u / \partial x\right)} + \frac{(\partial u / \partial y) \, dy}{dy \left(1 + \partial v / \partial y\right)} \]
\[ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \]
Review of the Basic Elasticity Theory

Obtain the compatibility equation by differentiating $\gamma_{xy}$ with respect to both $x$ and $y$

\[
\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y} \frac{\partial u}{\partial y} + \frac{\partial^2}{\partial x \partial y} \frac{\partial v}{\partial x} = \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2}
\]

or

\[
\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0
\]

Similarly

\[
\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} - \frac{\partial^2 \gamma_{xz}}{\partial z \partial x} = 0
\]

\[
\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} - \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} = 0
\]
Three-dimensional state of stress and strain

Strains for 3D Problem

\[
\begin{align*}
\varepsilon_x &= \frac{\partial u}{\partial x} \\
\varepsilon_y &= \frac{\partial v}{\partial y} \\
\varepsilon_z &= \frac{\partial w}{\partial z} \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\
\gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\
\gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}
\end{align*}
\]
Poisson’s Ratio

- For a slender bar subjected to axial loading:
  \[ \varepsilon_x = \frac{\sigma_x}{E} \quad \sigma_y = \sigma_z = 0 \]

- The elongation in the x-direction is accompanied by a contraction in the other directions. Assuming that the material is isotropic (no directional dependence),
  \[ \varepsilon_y = \varepsilon_z \neq 0 \]

- Poisson’s ratio is defined as
  \[ \nu = \left| \frac{\text{lateral strain}}{\text{axial strain}} \right| = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_z}{\varepsilon_x} \]
Three-dimensional state of stress and strain

Stress/Strain Relationships

\[
\varepsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E}
\]
\[
\varepsilon_y = -\nu \frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E}
\]
\[
\varepsilon_z = -\nu \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} + \frac{\sigma_z}{E}
\]
\[
\gamma_{xy} = \frac{\tau_{xy}}{G}, \quad \gamma_{yz} = \frac{\tau_{yz}}{G}, \quad \gamma_{zx} = \frac{\tau_{zx}}{G}
\]
\[
G = \frac{E}{2(1+\nu)}
\]
Three-dimensional state of stress and strain

Stress/Strain Relationships

\[
\begin{align*}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{xy} \\
\gamma_{yz} \\
\gamma_{zx}
\end{bmatrix} &= \frac{1}{E} \begin{bmatrix}
1 & -v & -v & 0 & 0 & 0 \\
-v & 1 & -v & 0 & 0 & 0 \\
-v & -v & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2(1 + \nu) & 0 & 0 \\
0 & 0 & 0 & 0 & 2(1 + \nu) & 0 \\
0 & 0 & 0 & 0 & 0 & 2(1 + \nu)
\end{bmatrix} \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{xy} \\
\tau_{yz} \\
\tau_{zx}
\end{bmatrix}
\end{align*}
\]
Three-dimensional state of stress and strain

Stress/Strain Relationships

\[
\begin{align*}
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{xy} \\
\tau_{yz} \\
\tau_{zx}
\end{bmatrix}
&= \frac{E}{(1 + \nu)(1 - 2\nu)}
\begin{bmatrix}
1 - \nu & \nu & \nu & 0 & 0 & 0 \\
1 - \nu & 1 - \nu & 0 & 0 & 0 & 0 \\
1 - \nu & 0 & 1 - \nu & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1 - 2\nu}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1 - 2\nu}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1 - 2\nu}{2}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{xy} \\
\gamma_{yz} \\
\gamma_{zx}
\end{bmatrix}
\end{align*}
\]
Two-dimensional state of stress and strain

Stresses

\[ \{ \sigma \} = \begin{cases} 
\sigma_x \\
\sigma_y \\
\tau_{xy} 
\end{cases} \]

equilibrium equations

\[
\begin{align*}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X_b &= 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y_b &= 0 \\
\tau_{xy} &= \tau_{yx}
\end{align*}
\]
Two-dimensional state of stress and strain

Principal Stresses

\[ \sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sigma_{\text{max}} \]

\[ \sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sigma_{\text{min}} \]

Principal Angle \( \theta_p \)

\[ \tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \]
2D Strains Problem

\[\begin{align*}
\varepsilon_x &= \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y} \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\end{align*}\]

\[\{\varepsilon\} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}\]
Plane Stress and Plane Strain

**Plane stress** is defined to be a state of stress in which the normal stress and shear stresses directed perpendicular to the plane are assumed to be zero.

Generally, members that are thin (\textit{z dimension is small} compared to the in-plane x and y dimensions) and whose loads act only in the x-y plane can be considered to be under plane stress.
Plane Stress and Plane Strain

Plane stress
Recall Stress Strain Relationship

\[
\begin{align*}
\begin{pmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{xy} \\
\gamma_{yz} \\
\gamma_{zx}
\end{pmatrix}
= \frac{1}{E}
\begin{bmatrix}
1 & -\nu & -\nu & 0 & 0 & 0 \\
-\nu & 1 & -\nu & 0 & 0 & 0 \\
-\nu & -\nu & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2(1 + \nu) & 0 & 0 \\
0 & 0 & 0 & 0 & 2(1 + \nu) & 0 \\
0 & 0 & 0 & 0 & 0 & 2(1 + \nu)
\end{bmatrix}
\begin{pmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{xy} \\
\tau_{yz} \\
\tau_{zx}
\end{pmatrix}
\end{align*}
\]
State of Plane stress

\[ \sigma_z = \tau_{xz} = \tau_{yz} = 0 \]

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} = \frac{1}{E} \begin{bmatrix}
1 & -\nu & 0 \\
-\nu & 1 & 0 \\
0 & 0 & 2(1+\nu)
\end{bmatrix} \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix}
\]

\[ \gamma_{xz} = \gamma_{yz} = 0, \quad \text{but} \quad \varepsilon_z \neq 0 \]
**State of Plane stress**

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} = \frac{E}{1 - \nu^2}
\begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1-\nu}{2}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
\]

\[
\{\sigma\} = [D] \{\varepsilon\}
\]

\[
[D] = \frac{E}{1 - \nu^2}
\begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1-\nu}{2}
\end{bmatrix}
\]

\(D:\) *is called stress/strain matrix*  
(or *constitutive matrix*)
State of Plane stress

Partial differential equations governing the plane stress

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1 + \nu}{2} \left( \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial x \partial y} \right)
\]

\[
\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{1 + \nu}{2} \left( \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} \right)
\]
Plane strain is defined to be a state of strain in which the strain normal to the x-y plane $\varepsilon_z$ and the shear strains $\gamma_{xz}$ and $\gamma_{yz}$ are assumed to be zero.

Constant cross-sectional area subjected to loads that act only in the x and/or y directions and do not vary in the z direction. Dams, retaining walls, and culvert boxes are good examples
Plane Stress and Plane Strain

Plane strain
State of Plane Strain

\[ \varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0 \]

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{xy} \\
\tau_{yz} \\
\tau_{zx}
\end{bmatrix} = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix}
1 - \nu & \nu & \nu & 0 & 0 & 0 \\
\nu & 1 - \nu & \nu & 0 & 0 & 0 \\
\nu & \nu & 1 - \nu & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1 - 2\nu}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1 - 2\nu}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1 - 2\nu}{2}
\end{bmatrix} \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{xy} \\
\gamma_{yz} \\
\gamma_{zx}
\end{bmatrix}
\]
State of Plane Strain

\[
\begin{align*}
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix}
&= \frac{E}{(1 + \nu)(1 - 2\nu)}
\begin{bmatrix}
1 - \nu & \nu & 0 \\
\nu & 1 - \nu & 0 \\
0 & 0 & \frac{1 - 2\nu}{2}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
\end{align*}
\]

\(\tau_{xz} = \tau_{yz} = 0, \quad \text{but} \quad \sigma_z \neq 0\)

\[
\{\sigma\} = [D] \{\varepsilon\}
\]

\[
[D] = \frac{E}{(1 + \nu)(1 - 2\nu)}
\begin{bmatrix}
1 - \nu & \nu & 0 \\
\nu & 1 - \nu & 0 \\
0 & 0 & \frac{1 - 2\nu}{2}
\end{bmatrix}
\]
The von Mises Stress

The von Mises stress is the effective or equivalent stress for 2-D and 3-D stress analysis.

For a ductile material, the stress level is considered to be safe, if

\[ \sigma_e \leq \sigma_Y \]

Where: \( \sigma_e \) is the von Mises stress and \( \sigma_Y \) the yield stress of the material.

This is a generalization of the 1-D (axial) result to 2-D and 3-D situations.
The von Mises Stress

The von Mises stress for 3D problems is defined by

$$\sigma_e = \frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

For 2-D problems,

$$\sigma_1^P = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2^P = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_e = \sqrt{(\sigma_x + \sigma_y)^2 - 3(\sigma_x \sigma_y - \tau_{xy}^2)}$$
Averaged Stresses:

Stresses are usually averaged at nodes in FEA software packages to provide more accurate stress values. This option should be turned off at nodes between two materials or other geometry discontinuity locations where stress discontinuity does exist.