Deflection Energy Method
Energy Method

The conservation of energy principle

This principle states that the work done by all the external forces acting on a structure $U_e$ is transformed into internal work or strain energy $U_i$, which is developed when the structure deforms.

This principle can be stated mathematically as:

$$U_e = U_i$$
Energy Method

External Work - Force

When a force $F$ undergoes a displacement $dx$ in the same direction as the force, the work done is $dU_e = F \, dx$

If the total displacement is $x$ the work become

$$U_e = \int_{0}^{x} F \, dx$$

$$U_e = \frac{1}{2} P \Delta \quad \text{The force applied gradually}$$
Energy Method

External Work - Force

If $P$ is already applied to the bar and that another force $F'$ is now applied, the work done by $P$ when the bar undergoes the further deflection $\Delta'$ is then

$$U_{e'} = P \Delta'$$
The work of a moment is defined by the product of the magnitude of the moment $M$ and the angle $d\theta$ then:  
$$dU_e = M \, d\theta$$

If the total angle of rotation is $\theta$ the work become:
$$U_e = \int_{0}^{\theta} M \, d\theta$$

If the moment is **applied gradually** to a structure from zero to $M$
$$U_e = \frac{1}{2} M \, \theta$$

If the moment is **already applied** to the structure and other loadings further distort the structure by an amount $\theta'$
$$U_e = M \, \theta'$$
Energy Method

Strain Energy – Axial Force

\[ \sigma = E \varepsilon \]
\[ \sigma = \frac{N}{A} \]
\[ \Delta = \frac{NL}{AE} \]
\[ \varepsilon = \frac{\Delta}{L} \]

\( N \) = internal normal force in a truss member caused by the real load
\( L \) = length of member
\( A \) = cross-sectional area of a member
\( E \) = modulus of elasticity of a member

\[
\begin{align*}
P &= N \\
U &= \frac{1}{2} PA \\
U_i &= \frac{N^2 L}{2AE}
\end{align*}
\]
Energy Method

Strain Energy – Bending

\[ \frac{d\theta}{dx} = \frac{M}{EI} \]

\[ U = \frac{1}{2} M\theta \]

\[ \frac{dU_i}{dx} = \frac{M^2 dx}{2EI} \]

\[ U_i = \int_0^L \frac{M^2 dx}{2EI} \]
**Principle of Work and Energy**

**Example**

Consider finding the displacement $\Delta$ at the point where the force $P$ is applied to the cantilever beam in

$$
U_e = \frac{1}{2} P \Delta,
$$

$$
U_i = \int_0^L \frac{M^2}{2EI} \, dx = \int_0^L \left(\frac{-Px}{2EI}\right)^2 \, dx = \frac{P^2 L^3}{6EI}
$$

$$
\frac{1}{2} P \Delta = \frac{P^2 L^3}{6EI}
$$

$$
\Delta = \frac{P L^3}{3EI}
$$
Principle of Virtual Work

\[ \sum P \Delta = \sum u \delta \]

- Work of External Loads
- Work of Internal Loads

\[ 1. \Delta = \sum u. dL \]

Virtual Load

Real displacement

Apply virtual load \( P' \) first.

Then apply real loads \( P_1, P_2, P_3 \).
Method of Virtual Work: Trusses

External Loading

1. \[ \Delta = \sum u \cdot dL \]

1. \[ \Delta = \sum n \cdot \frac{NL}{AE} \]

1 = external virtual unit load acting on the truss joint in the stated direction of \( \Delta \)

\( u \) = internal virtual normal force in a truss member caused by the external virtual unit load

\( \Delta \) = external joint displacement caused by the real load on the truss

\( N \) = internal normal force in a truss member caused by the real load

\( L \) = length of member

\( A \) = cross-sectional area of a member

\( E \) = modulus of elasticity of a member
Method of Virtual Work: Trusses

Temperature

1. \[ \Delta = \sum n \cdot \alpha \cdot \Delta T \cdot L \]

1 = external virtual unit load acting on the truss joint in the stated direction of
n = internal virtual normal force in a truss member caused by the external virtual unit load
\( \Delta \) = external joint displacement caused by the temperature change.
\( \alpha \) = coefficient of thermal expansion of member
\( \Delta T \) = change in temperature of member
L = length of member
Method of Virtual Work: Trusses
Fabrication Errors and Camber

1. \[ \Delta = \sum n \cdot \Delta L \]

1 = external virtual unit load acting on the truss joint in the stated direction of
n = internal virtual normal force in a truss member caused by the external virtual unit load
\( \Delta = \) external joint displacement caused by fabrication errors
\( \Delta L = \) difference in length of the member from its intended size as caused by a fabrication error.
Example 1

The cross sectional area of each member of the truss show, is $A = 400\text{mm}^2$ & $E = 200\text{GPa}$.

a) Determine the vertical displacement of joint C if a 4-kN force is applied to the truss at C
A virtual force of 1 kN is applied at C in the vertical direction.

Solution

$$1. \Delta = \sum n \cdot \frac{NL}{AE}$$
<table>
<thead>
<tr>
<th>Member</th>
<th>$n$ (KN)</th>
<th>$N$ (KN)</th>
<th>$L$ (m)</th>
<th>$nNL$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>0.667</td>
<td>2</td>
<td>8</td>
<td>10.67</td>
</tr>
<tr>
<td>AC</td>
<td>-0.833</td>
<td>2.5</td>
<td>5</td>
<td>-10.41</td>
</tr>
<tr>
<td>CB</td>
<td>-0.833</td>
<td>-2.5</td>
<td>5</td>
<td>10.41</td>
</tr>
</tbody>
</table>

**Sum**: 10.67

1. $\Delta = \sum \frac{nNL}{AE} = \frac{10.67}{AE} = \frac{10.67 \text{ (kN)}}{400 \times 10^{-6} \text{ (m}^2\text{)} \times 200 \times 10^6 \text{ (kN/m}^2\text{)}}$

$\Delta = 0.000133m = 0.133 \text{ mm}$
Text book Example 8-14
Determine vertical displacement at C
A = 0.5 in²
E = 29 (10)³ ksi
Virtual-Work Equation. Arranging the data in tabular form, we have

<table>
<thead>
<tr>
<th>Member</th>
<th>( n ) (k)</th>
<th>( N ) (k)</th>
<th>( L ) (ft)</th>
<th>( n NL ) (k²·ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( AB )</td>
<td>0.333</td>
<td>4</td>
<td>10</td>
<td>13.33</td>
</tr>
<tr>
<td>( BC )</td>
<td>0.667</td>
<td>4</td>
<td>10</td>
<td>26.67</td>
</tr>
<tr>
<td>( CD )</td>
<td>0.667</td>
<td>4</td>
<td>10</td>
<td>26.67</td>
</tr>
<tr>
<td>( DE )</td>
<td>-0.943</td>
<td>-5.66</td>
<td>14.14</td>
<td>75.42</td>
</tr>
<tr>
<td>( FE )</td>
<td>-0.333</td>
<td>-4</td>
<td>10</td>
<td>13.33</td>
</tr>
<tr>
<td>( EB )</td>
<td>-0.471</td>
<td>0</td>
<td>14.14</td>
<td>0</td>
</tr>
<tr>
<td>( BF )</td>
<td>0.333</td>
<td>4</td>
<td>10</td>
<td>13.33</td>
</tr>
<tr>
<td>( AF )</td>
<td>-0.471</td>
<td>-5.66</td>
<td>14.14</td>
<td>37.71</td>
</tr>
<tr>
<td>( CE )</td>
<td>1</td>
<td>4</td>
<td>10</td>
<td>40</td>
</tr>
</tbody>
</table>

\[ \Sigma 246.47 \]

Thus

\[ 1 \text{ k} \cdot \Delta_{C_v} = \sum \frac{nNL}{AE} = \frac{246.47 \text{ k}^2 \cdot \text{ft}}{AE} \]

Converting the units of member length to inches and substituting the numerical values for \( A \) and \( E \), we have

\[ 1 \text{ k} \cdot \Delta_{C_v} = \frac{(246.47 \text{ k}^2 \cdot \text{ft})(12 \text{ in./ft})}{(0.5 \text{ in}^2)(29(10^3) \text{ k/in}^2)} \]

\[ \Delta_{C_v} = 0.204 \text{ in.} \]

\textit{Ans.}
Example 3

Determine the vertical displacement of joint C of the steel truss shown in Fig. 9–10a. Due to radiant heating from the wall, member AD is subjected to an increase in temperature of $\Delta T = +120^\circ$F. Take $\alpha = 0.6(10^{-5})$/°F and $E = 29(10^3)$ ksi. The cross-sectional area of each member is indicated in the figure.
\[ 1 \cdot \Delta_{C_v} = \sum \frac{nNL}{AE} + \sum n\alpha \Delta T \ L \]
\[ = \frac{(0.75)(120)(6)(12)}{2[29(10^3)]} + \frac{(1)(80)(8)(12)}{2[29(10^3)]} \]
\[ + \frac{(-1.25)(-100)(10)(12)}{1.5[29(10^3)]} + (1)[0.6(10^{-5})](120)(8)(12) \]

\[ \Delta_{C_v} = 0.658 \text{ in.} \quad \text{Ans.} \]
Method of Virtual Work: Beam

1. \( \Delta = \sum u \cdot dL \)

\[
d\theta = \frac{M}{EI} \, dx
\]

1. \( \Delta = \int_0^L m \frac{M}{EI} \, dx \)

1 = external virtual unit load acting on the beam in the stated direction of \( \Delta \)

m = internal virtual moment in a beam caused by the external virtual unit load

\( \Delta \) = external joint displacement caused by the real load on the truss

M = internal moment in a beam caused by the real load

L = length of beam

I = moment of inertia of cross-sectional

E = modulus of elasticity of the beam
Method of Virtual Work: Beam

Similarly the rotation angle at any point on the beam can be determine, a unit couple moment is applied at the point and the corresponding internal moment \( m_\theta \) have to be determine.

\[
\theta = \int_0^L \frac{m_\theta M}{EI} \, dx
\]
Example 4

Determine the displacement at point B of a steel beam
\( E = 200 \, \text{Gpa} \), \( I = 500(10^6) \, \text{mm}^4 \)
Solution

\[ M = -6x^2 \]

real load

\[ M = -1x \]

virtual unit force

\[
1.\Delta = \int_0^L m \frac{M}{EI} \, dx = \int_0^1 (-1x) \times (-6x^2) \frac{dx}{EI} = \int_0^1 6x^3 \frac{dx}{EI} = \left[ \frac{6x^4}{4EI} \right]_0^1
\]

\[
\Delta = \frac{15(10^3)}{EI} = \frac{15(10^3)}{200(10^6) \times 500(10^6)(10^{-12})} = 0.15m
\]
Another Solution

Real Load

Virtual Load

\[ \Delta_B = \frac{-2000}{200(10^6) \times 500(10^6)(10^{-12})} \times -7.5 \]

\[ \Delta_B = 0.15m \]
Example 5

Determine the tangential rotation at point $A$ of the steel beam shown in Fig. 9–14a. Take $E = 200$ GPa, $I = 60(10^6)$ mm$^4$. 

(a)

(b)

$\theta_m = -1$

virtual unit couple

$1 \text{kN-m}$
Real Moment $M$. Using the same $x$ coordinate, the internal moment $M$ is formulated as shown in Fig. 9–14c.

Virtual-Work Equation. The tangential rotation at $A$ is thus

\[
(1 \text{ kN} \cdot \text{m}) \cdot \theta_A = \int_0^L \frac{m_o M}{EI} \, dx
\]

\[
= \int_0^3 \frac{(-1) \left( \frac{-x^3}{3} \right)}{EI} \, dx
\]

\[
= \frac{1}{3EI} \int_0^3 x^3 \, dx
\]

\[
= \frac{6.75}{200(10^6) \text{ kN/m}^2(60(10^6) \text{ mm}^4)(10^{-12} \text{ m}^4/\text{mm}^4)}
\]

or

\[
\theta_A = 0.000563 \text{ rad}
\]

Ans.
Example 6

Determine the Slope $\theta$ at point B of a steel beam $E = 200 \text{ Gpa}, I = 60(10^6) \text{ mm}^4$
Virtual Load

Solution

virtual unit couple
1. $\theta = \int_0^L m_\theta \frac{M}{EI} dx = \int_0^5 (0) \times (-3x) \, dx + \int_5^{10} (-1) \times (-3x) \, dx = \int_5^{10} \frac{3x \, dx}{EI} = \left[ \frac{3x^2}{2EI} \right]_5^{10}$

$\theta_B = \frac{3(10^2) - 3(5^2)}{2 \times 200(10^6) \times 60(10^{-6})} = 0.0094 \text{ rad}$
Another Solution

Real Load

Virtual Load

$$\theta_B = \frac{-112}{EI} \times -1 = \frac{112}{EI}$$

$$= \frac{112}{200 \times 10^6 \times 60 \times 10^{-6}} = 0.0094 \text{rad}$$
Example 7

Determine the displacement at $D$ of the steel beam in Fig. 9–16a. Take $E = 29(10^3)$ ksi, $I = 800 \text{ in}^4$. 

(a)
Virtual-Work Equation. Applying the equation of virtual work to the beam using the data in Figs. 9–16b and 9–16c, we have

\[ 1 \cdot \Delta_D = \int_0^L \frac{mM}{EI} \, dx \]

\[ = \int_0^{15} \frac{(-1x_1)(0)}{EI} \, dx_1 + \int_0^{10} \frac{(0.75x_2 - 15)(7x_2)}{EI} \, dx_2 \]

\[ + \int_0^{10} \frac{(-0.75x_3)(80 - 1x_3)}{EI} \, dx_3 \]

\[ \Delta_D = \frac{0}{EI} - \frac{3500}{EI} - \frac{2750}{EI} = -\frac{6250 \text{ k} \cdot \text{ft}^3}{EI} \]

or

\[ \Delta_D = \frac{-6250 \text{ k} \cdot \text{ft}^3(12)^3 \text{ in}^3/\text{ft}^3}{29(10^3) \text{ k}/\text{in}^2(800 \text{ in}^4)} \]

\[ = -0.466 \text{ in.} \quad \text{Ans.} \]
Example 8

Determine the horizontal displacement of point C on the frame shown in Fig. 9–17a. Take $E = 29(10^3)$ ksi and $I = 600 \text{ in}^4$ for both members.
$m_2 = 1.25x_2$

$v_2$

$n_2$

$x_2$

$1k$

$1.25k$

$1k$

$8$ ft

$m_1 = 1x_1$

$v_1$

$n_1$

$x_1$

$1k$

$1k$

$1.25k$

$1.25k$

virtual loadings

$M_2 = 25x_2$

$N_2$

$x_2$

$25k$

$25k$

$25k$

$8$ ft

$40x_1 - 2x_1^2$

$N_1$

$v_1$

$x_1$

$1.25k$

$40k$

$40k$

$25k$

real loadings

$25k$

$5$ ft
**Real Moments** $M$. In a similar manner the support reactions and real moments are computed as shown in Fig. 9–17c.

**Virtual-Work Equation.** Using the data in Fig. 9–17b and 9–17c, we have

$$1 \cdot \Delta_{C_h} = \int_0^L \frac{mM}{EI} \, dx = \int_0^{10} \frac{(1x_1)(40x_1 - 2x_1^2)}{EI} \, dx_1 + \int_0^8 \frac{(1.25x_2)(25x_2)}{EI} \, dx_2$$

$$\Delta_{C_h} = \frac{8333.3}{EI} + \frac{5333.3}{EI} = \frac{13666.7 \text{ k} \cdot \text{ft}^3}{EI}$$  \hspace{1cm} \text{(1)}$$

If desired, the integrals $\int mM/dx$ can also be evaluated graphically using the table on the inside front cover. The moment diagrams for the frame in Fig. 9–17b and 9–17c are shown in Fig. 9–17d and 9–17e, respectively. Thus, using the formulas for similar shapes in the table yields

$$\int mM \, dx = \frac{5}{12}(10)(200)(10) + \frac{1}{3}(10)(200)(8)$$

$$= 8333.3 + 5333.3 = 13666.7 \text{ k}^2 \cdot \text{ft}^3$$

This is the same as that calculated in Eq. (1). Thus

$$\Delta_{C_h} = \frac{13666.7 \text{ k} \cdot \text{ft}^3}{[29(10^3) \text{ k/ln}^2((12)^2 \text{ in}^2/\text{ft}^2)][600 \text{ in}^4/(12)^4 \text{ in}^4]}$$

$$= 0.113 \text{ ft} = 1.36 \text{ in.} \hspace{1cm} \text{Ans.}$$
Example 9

Determine the tangential rotation at point C of the frame shown in Fig. 9–18a. Take $E = 200$ GPa, $I = 15(10^6)$ mm$^4$. 

![Diagram showing a frame with a 5 kN load at point C, a 3 m segment, a 60° angle at point B, and a 2 m segment.](image)
virtual loads

real loads

\[ M_1 = -2.5x_1 \]

\[ M_2 = 7.5 \]
Virtual-Work Equation. Using the data in Fig. 9–18b and 9–18c, we have

\[ 1 \cdot \theta_c = \int_0^L \frac{m_\theta M}{EI} \, dx = \int_0^3 \frac{(-1)(-2.5x_1)}{EI} \, dx_1 + \int_0^2 \frac{(1)(7.5)}{EI} \, dx_2 \]

\[ \theta_c = \frac{11.25}{EI} + \frac{15}{EI} = \frac{26.25 \text{ kN} \cdot \text{m}^2}{EI} \]

or

\[ \theta_c = \frac{26.25 \text{ kN} \cdot \text{m}^2}{200(10^6) \text{ kN/m}^2[15(10^6) \text{ mm}^4](10^{-12} \text{ m}^4/\text{mm}^4)} \]

\[ = 0.00875 \text{ rad} \]

\text{Ans.}
Example 4

Determine the horizontal deflection at A
Solution

Real Load

Virtual Load
\[
\Delta_A = \frac{-100}{EI} \times 0 + \frac{-500}{EI} \times 2.5 = \frac{-1250}{EI}
\]

\[
= \frac{-1250}{200(10^9) \times 200 \times 10^{-6}} = -0.031 \text{m}
\]
Virtual Strain Energy
Caused by: Axial Load

\[ U_n = \frac{nNL}{AE} \]

\( n \) = internal virtual axial load caused by the external virtual unit load
\( N \) = internal normal force in the member caused by the real load
\( L \) = length of member
\( A \) = cross-sectional area of the member
\( E \) = modulus of elasticity of the material
Virtual Strain Energy

Caused by: Shear

\[ dy = \gamma \, dx \]
\[ \gamma = \frac{\tau}{G} \]
\[ dy = \left( \frac{\tau}{G} \right) \, dx \]
\[ \tau = K \left( \frac{V}{A} \right) \]
\[ dy = K \left( \frac{V}{GA} \right) \, dx \]
\[ dU_s = v \, dy = v K \left( \frac{V}{A} \right) \, dx \]

\[ U_s = \int_0^L K \left( \frac{vV}{GA} \right) \, dx \]
Virtual Strain Energy
Caused by: Shear

\[ U_S = \int_0^L K \left( \frac{\nu V}{G A} \right) dx \]

\(\nu\) = internal virtual shear in the member caused by the external virtual unit load
\(V\) = internal shear in the member caused by the real load
\(G\) = shear modulus of elasticity for the material
\(A\) = cross-sectional area of the member
\(K\) = form factor for the cross-sectional area

- \(K=1.2\) for rectangular cross sections
- \(K=10/9\) for circular cross sections
- \(K=1.0\) for wide-flange and I-beams where \(A\) is the area of the web.
Virtual Strain Energy
Caused by: Torsion

\[ \gamma = c d\theta / dx \]
\[ \gamma = \tau / G \]
\[ \tau = \frac{Tc}{J} \]

\[ d\theta = \frac{\gamma}{c} dx = \frac{\tau}{Gc} dx = \frac{T}{GJ} dx \]
\[ dU_t = t d\theta = \frac{tT}{GJ} dx \]

\[ U_t = \frac{tTL}{GJ} \]
Virtual Strain Energy
Caused by: Torsion

\[ U_t = \frac{tTL}{GJ} \]

\( t \) = internal virtual torque caused by the external virtual unit load
\( T \) = internal shear in the member caused by the real load
\( G \) = shear modulus of elasticity for the material
\( J \) = polar moment of inertia of the cross-sectional
\( L \) = member length

Dr. Mohammed Arafa
Structural Analysis I
Virtual Strain Energy

Caused by: Temperature

\[ d\theta = \frac{\alpha \Delta T_m}{c} \, dx \]

\[ U_{\text{temp}} = \int_{0}^{L} \frac{m \alpha \Delta T_m}{c} \, dx \]
Virtual Strain Energy
Caused by: Temperature

\[ U_{\text{temp}} = \int_{0}^{L} \frac{m \alpha \Delta T}{c} m \, dx \]

- \( m \) = internal virtual moment in the beam caused by the external virtual unit load or unit moment
- \( \alpha \) = coefficient of thermal expansion
- \( \Delta T_m \) = change in temperature between the mean temperature and the temperature at the top or the bottom of the beam
- \( c \) = mid-depth of the beam
Example 10

Determine the horizontal displacement of point C on the frame shown in Fig. 9-22a. Take $E = 29(10^3)$ ksi, $G = 12(10^3)$ ksi, $I = 600$ in$^4$, and $A = 80$ in$^2$ for both members. The cross-sectional area is rectangular. Include the internal strain energy due to axial load and shear.
**Solution**

Here we must apply a horizontal unit load at $C$. The necessary free-body diagrams for the real and virtual loadings are shown in Fig. 9-22b and 9-22c.

![Diagram](image)
**Bending.** The virtual strain energy due to bending has been determined in Example 9–8. There it was shown that

\[
U_b = \int_0^L \frac{mM}{EI} \, dx = \frac{13\,666.7 \, k^2 \cdot ft^3}{EI} = \frac{13\,666.7 \, k^2 \cdot ft^3 \left(12^3 \, in^3/1 \, ft^3\right)}{[29(10^3) \, k/in^2](600 \, in^4)} = 1.357 \, in. \cdot k
\]

**Axial load.** From the data in Fig. 9–22b and 9–22c, we have

\[
U_a = \sum \frac{nNL}{AE}
\]

\[
= \frac{1.25 \, k(25 \, k)(120 \, in.)}{80 \, in^2[29(10^3) \, k/in^2]} + \frac{1 \, k(0)(96 \, in.)}{80 \, in^2[29(10^3) \, k/in^2]}
\]

\[
= 0.001616 \, in. \cdot k
\]
Shear. Applying Eq. 9–21 with \( K = 1.2 \) for rectangular cross sections, and using the shear functions shown in Fig. 9–22b and 9–22c, we have

\[
U_s = \int_0^L K \left( \frac{vV}{GA} \right) dx
\]

\[
= \int_0^{10} \frac{1.2(1)(40 - 4x_1)}{GA} \, dx_1 + \int_0^8 \frac{1.2(-1.25)(-25)}{GA} \, dx_2
\]

\[
= \frac{540 \text{ k}^2 \cdot \text{ft}(12 \text{ in./ft})}{[12(10^3) \text{ k/in}^2](80 \text{ in}^2)} = 0.00675 \text{ in.} \cdot \text{k}
\]

Applying the equation of virtual work, we have

\[
1 \text{ k} \cdot \Delta_{C_h} = 1.357 \text{ in.} \cdot \text{k} + 0.001616 \text{ in.} \cdot \text{k} + 0.00675 \text{ in.} \cdot \text{k}
\]

\[
\Delta_{C_h} = 1.37 \text{ in. \hspace{1cm} Ans.}
\]

Including the effects of shear and axial load contributed only a 0.6% increase in the answer to that determined only from bending.
Example 11

The beam shown in Fig. 9–23a is used in a building subjected to two different thermal environments. If the temperature at the top surface of the beam is $80^\circ$F and at the bottom surface is $160^\circ$F, determine the vertical deflection of the beam at its midpoint due to the temperature gradient. Take $\alpha = 6.5(10^{-6})/{^\circ}$F.
\[ 1 \text{ lb} \cdot \Delta C_v = \int_0^L \frac{m \alpha \Delta T_m}{c} \, dx \]

\[ = 2 \int_0^{60 \text{ in.}} \frac{\left(\frac{1}{2} x\right)6.5(10^{-6})/{}^\circ \text{F}(40^\circ \text{F})}{5 \text{ in.}} \, dx \]

\[ \Delta C_v = 0.0936 \text{ in.} \]

The result indicates a very negligible deflection.