Chapter 2
Discrete Random Variables (R.V)
Part 1
2.1 Random Variables

- A random variable over a sample space is a function that maps every sample point (i.e. outcome) to a real number.

- The main purpose of using a random variable is so that we can define certain probability functions that make it easy to compute the probabilities of various events.
Ex: Roll a fair coin once:
Outcomes={H,T}

\[
X = \begin{cases} 
1 & \text{if head} \\
0 & \text{if Tail} 
\end{cases}
\]
Ex: Toss a coin twice. Let $Y$ denote the number of heads.

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<th>Compound Event</th>
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<td>$(Y=0)$</td>
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<td>$(Y=2)$</td>
<td>$(\text{Head, Head})$</td>
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The probability mass function of a fair die

\[
P_Y(y) = \begin{cases} 
1/4 & y = 0 \\
1/2 & y = 1 \\
1/4 & y = 2 
\end{cases}
\]
The R.V \( N \) has PMF

\[
P_N(n) = \begin{cases} 
c(1/2)^n & \text{n = 0, 1, 2} \\
0 & \text{else}
\end{cases}
\]

(a) Find the value of the constant \( c \)?

\[
P(N = 0) + P(N = 1) + P(N = 2) = 1 \\
c \times 1 + c \times (1/2) + c \times (1/4) = 1 \\
c = 4/7
\]

(a) \( P[N \leq 1] \)

\[
P[N \leq 1] = P[N = 0] + P[N = 1] = \frac{4}{7} + \left(\frac{4}{7} \times \frac{1}{2}\right) = \frac{6}{7}
\]
**PROBLEM 2.2.2**

\[ P_x(x) = \begin{cases} 
1/8 & x = 0 \\
3/8 & x = 1 \\
3/8 & x = 2 \\
1/8 & x = 3 \\
0 & \text{else} 
\end{cases} \]

*Find :* \( P[X=0] \), \( P[X<3] \)

\[ P[X = 0] = 1/8 \]

\[ P[X < 3] = P[X = 0] + P[X = 1] + P[X = 2] = 7/8 \]
The R.V N has PMF

\[ P_v(v) = \begin{cases} \frac{c v^2}{30} & v = 1, 2, 3, 4 \\ 0 & \text{else} \end{cases} \]

(a) Find the value of the constant c?

\[ P(V = 1) + P(V = 2) + P(V = 3) + P(V = 4) = 1 \]
\[ c + 4c + 9c + 16c = 1 \]
\[ c = \frac{1}{30} \]

(a) \( P[V \in \{ U^2 | u=1,2,3,4,\ldots \}] \)

\[ P[V = 1] + P[V = 4] = \left( \frac{1}{30} \right) * 1 + \left( \frac{1}{30} \right) * 16 = \frac{17}{30} \]
(c) Find the probability that $V$ is even?

$$P(V = 2) + P(V = 4) = \left(\frac{1}{30} \times 4\right) + \left(\frac{1}{30} \times 16\right) = \frac{20}{30}$$
In basketball: when a player is fouled the player is awarded “One and One” one free throw and if it goes in the player is awarded another one.

Find PMF of $Y$, the number of point scored in “1 and 1”
The throw goes in with probability $p$

$Y = \{0,1,2\}$,

$b1 \Rightarrow y = 0$

$g1b2 \Rightarrow y = 1$, , $g1g2 \Rightarrow y = 2$

$P_Y(y) = \begin{cases} 
1 - p & y = 0 \\
p(1-p) & y = 1 \\
p^2 & y = 2 \\
0 & \text{else}
\end{cases}$
You are manager of a ticket agency that sell concert tickets. You assume that people will call three times in an attempt to buy tickets and then give up. You want to make sure that you are able to serve at least 95% of the people who want tickets.

Let $P$ be the probability that the caller gets through to your ticket agency. What is the min value of $P$ necessary to meet your goal?
prob. that the caller fails = 1 - 0.95 = 0.05

\[ P[F] = (1-p)(1-p)(1-p) = (1-p)^3 \]

\[ (1-p)^3 \leq 0.05 \]

\[ p = 0.63 \]
2.3 Families of discrete R.V.

[1] Bernoulli

\[
P_X(x) = \begin{cases} 
1 - p & x = 0 \\
p & x = 1 \\
0 & \text{else}
\end{cases}
\]

Ex: test one circuit (rejected=0.2 or acceptable=0.8)
R.V. \( X=\# \) of rejected

\[
P_X(x) = \begin{cases} 
0.8 & x = 0 \\
0.2 & x = 1 \\
0 & \text{else}
\end{cases}
\]
2.3 Families of discrete R.V


\[ P_X(x) = \begin{cases} 
  p(1 - p)^{x-1} & \text{if } x = 1, 2, 3, \ldots, \\
  0 & \text{else} 
\end{cases} \]

Ex: test one circuit (rejected=(1-p) or acceptable=p)
R.V \( X = \# \) of test until get the first acceptable

\[ P_X(x) = \binom{n}{x} p^x (1 - p)^{n-x} \quad , \quad x = 0, \ldots, n \]

Ex: test 10 circuits (rejected=0.2 or acceptable=0.8)
R.V \( X \)=# of rejected in the 10 tests

\[ P_X(x) = \binom{10}{x} (0.2)^x (0.8)^{10-x} \]
Ex: test circuits until you find 5 rejected, after you done 105 tests the 5 rejected is found. (rejected=0.2 or acceptable=0.8)

\[
P[A] \downarrow \text{from binomial} = \binom{105-1}{5-1} (0.2)^{5-1} (0.8)^{(105-1)-(5-1)}
\]

\[
P[B] = p = 0.2
\]
Prob. of all except the final one =

\[ P[A]P[B] = \binom{105 - 1}{5 - 1} (0.2)^{5-1} (0.8)^{(105-1)-(5-1)} \times (0.2) \]

\[ = \binom{105 - 1}{5 - 1} (0.2)^5 (0.8)^{(105-5)} \]

\[ Pascal = P[(k-1) \text{ rejects in (L-1) attempts, rejected on attempt L}] \]

\[ P_X(x) = \begin{cases} \frac{1}{l-k+1} & \text{x} = k, k+1, \ldots, l \\ 0 & \text{else} \end{cases} \]

Ex: roll die \{1(k), 2, 3, 4, 5, 6(L)\}

\[ P_X(x) = \begin{cases} \frac{1}{6-1+1} = \frac{1}{6} & \text{x} = 1, 2, \ldots, 6 \\ 0 & \text{else} \end{cases} \]
Poisson R.V

\[ P_x(x) = \begin{cases} 
\alpha^x \frac{e^{-\alpha}}{x!} & \text{for } x = 0, 1, 2, \ldots \\
0 & \text{else}
\end{cases} \]

\[ \alpha = \lambda T \]

\( \lambda \): average rate
The probability that the message will be received is \( p \). The message is received at least once, a system transmits the message \( n \) times.

(a) What is the PMF of \( K \): the number of times the receiver receives the same message?

\[
P_K(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, \ldots, n \quad /// \text{binomial}
\]

(b) Assume \( p=0.8 \), What is the min value of \( n \) that produces a probability of 0.95 of receiving the message at least once?

\[
P[\text{at least One}] = 1 - P[\text{zero}]
\]

\[
P_K(0) = \binom{n}{0} p^0 (1-p)^{n-0} = 0.2^n
\]

\[
P[\text{at least One}] = 1 - 0.2^n = 0.95
\]

\[
0.2^n = 0.05
\]

\[
n \ln 0.2 = \ln 0.05
\]

\[
n = 1.86 \approx 2
\]
A child throws a Frisbee, the child’s dog catches it with probability $p$, when the dog catches the Frisbee it run away. The child continues to throw Frisbee until the dog catches it. Let $X$ denote the number of times the Frisbee is thrown.

(a) What is PMF of $X$?

$X$ is a geometric R.V.

$$P_X(x) = p(1 - p)^{x-1}, \ x = 1, 2, 3, ....$$

(b) Assume $p=0.2$, What is the probability that the child will throw the Frisbee more than four times?

$$P[\text{more 4 times}] = P[x = 5] + P[x = 6] + P[x = 7] + \ldots$$

$$= 1 - \{ P[x = 4] + P[x = 3] + P[x = 2] + P[x = 1] \}$$

$$= 1 - \{0.2 \times 0.8^3 + 0.2 \times 0.8^2 + 0.2 \times 0.8^1 + 0.2 \times 0.8^0 \}$$

$$= 0.41$$
When a two-way paging system transmits a message, the probability that the message is received by the pager it is sent to is $p$.

When the pager receives the message, it transmits an acknowledge signal (ACK) to the paging system. If the paging system does not receive the ACK, it sends the message again.

(a) What is PMF of $N$: the number of times the system sends the same message?

$N$ is a geometric R.V.

$$P_N(n) = p(1 - p)^{n-1}, \quad n = 1, 2, 3, \ldots$$

if $n = 6$ so 5 fails and 1 success

(b) The paging company wants to limit the number of times it has to send the same message. It has a goal of $P[N \leq 3] \geq 0.95$, what is the min value of $p$ necessary to achieve the goal?

\[ p^3 - 3p^2 + 3p = 0.95 \]
\[ p^3 - 3p^2 + 3p - 0.95 = 0 \]

solve for \( p \Rightarrow p_{1,2} = 1.184 \pm j0.319 \text{(not accept)} \]

\[ P_3 = 0.6315 \]

\[ p = 0.6315 \]
The number of buses that arrive at a bus stop in $T$ minutes is a Poisson R.V $B$ with expected value $T/5$

(a) What is the PMF of $B$, the number of buses that arrive in $T$ minutes?

For Poisson R.V $\alpha = \text{expected value} = T/5 = \lambda T$

$\lambda = 1/5$

$P_B(b) = \begin{cases} (T/5)^b \frac{e^{-(T/5)}}{b!} & \text{x = 0,1,2,.....} \\ 0 & \text{else} \end{cases}$

$\alpha = \lambda T$

$\lambda : \text{average rate}$

(b) What is the probability that in a 2 minute interval, three buses will arrive?

$T = 2, b = 3$

$P_B(b = 3) = (2/5)^3 \frac{e^{-(2/5)}}{3!} = 7.15e - 3$
(d) How much time should you allow so that with probability 0.99 at least one bus arrives?

\[ P[\text{at least one}] = P[B \geq 1] = 1 - P[B = 0] = 0.99 \]
\[ P[B = 0] = 0.01 \]
\[ P_B(b = 1) = (T / 5)^0 \frac{e^{-(T/5)}}{1!} = e^{-(T/5)} = 0.01 \]
\[ \ln 0.01 = -(T / 5) \]
\[ T = 23 \text{ min} \]
A Zipf (n,\(\alpha=1\)) R.V X has PMF

\[
P_x(x) = \begin{cases} \frac{c_n}{x^\alpha} & x = 1, 2, \ldots, n \\ 0 & \text{else} \end{cases}
\]

The constant \(C(n)\) is set so that

\[
\sum_{x=1}^{n} P_x(x) = 1
\]

Calculate \(c(n)\) for \(n=1, 2, \ldots, 6\)

\[
\begin{align*}
\text{for } n = 1, & \quad \sum_{x=1}^{1} C(1)/1 = 1 \Rightarrow C(1) = 1 \\
\text{for } n = 2, & \quad \sum_{x=1}^{2} C(2)/x = 1 \Rightarrow \frac{C(2)}{1} + \frac{C(2)}{2} = 1 \Rightarrow C(2) = \frac{2}{3} \\
\text{for } n = 3, & \quad \sum_{x=1}^{3} C(3)/x = 1 \Rightarrow \frac{C(3)}{1} + \frac{C(3)}{2} + \frac{C(3)}{3} = 1 \Rightarrow C(2) = \frac{6}{11}
\end{align*}
\]
A radio station gives a pair of concert tickets to the sixth caller who knows the birthday of the performer. For each person who calls, the probability is 0.75 of knowing the birthday. All calls are independent.

(a) What is the PMF of \( L \), the number of calls necessary to find the winner?

\[
P_{L}(l) = \binom{L-1}{6-1} (0.75)^6 (1-0.75)^{L-6}
\]

(b) What is the probability of finding the winner on the tenth call?

\[
P_{L}(L = 10) = \binom{10-1}{6-1} (0.75)^6 (1-0.75)^{10-6} = 0.0876
\]
(b) What is the probability that the station will need nine or more calls to find the winner?

\[ P_L(\text{nine or more}) = P_L(L \geq 9) = 1 - P_L(L < 9) = 1 - \{P_L(L = 6) + P_L(L = 7) + P_L(L = 8)\} = \]
Thank You!