Chapter 6

Fatigue Failure Resulting from Variable Loading

Mohammad Suliman Abuhaiba, Ph.D., PE
Chapter Outline

- Introduction to Fatigue in Metals
- Approach to Fatigue Failure in Analysis and Design
- Fatigue-Life Methods
  - The Stress-Life Method
  - The Strain-Life Method
  - The Linear-Elastic Fracture Mechanics Method
- The Endurance Limit
- Fatigue Strength
- Endurance Limit Modifying Factors
- Stress Concentration and Notch Sensitivity
- Characterizing Fluctuating Stresses
- Fatigue Failure Criteria for Fluctuating Stress
- Torsional Fatigue Strength under Fluctuating Stresses
- Combinations of Loading Modes
- Varying, Fluctuating Stresses; Cumulative Fatigue Damage
- Surface Fatigue Strength
- Stochastic Analysis

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Introduction to Fatigue in Metals

- Loading produces stresses that are variable, repeated, alternating, or fluctuating
- Maximum stresses well below yield strength
- Failure occurs after many stress cycles
- Failure is by sudden ultimate fracture
- No visible warning in advance of failure
Stages of Fatigue Failure

- **Stage I**: Initiation of micro-crack due to cyclic plastic deformation
- **Stage II**: Progresses to macro-crack that repeatedly opens & closes, creating bands called *beach marks*
Stages of Fatigue Failure

- **Stage III**: Crack has propagated far enough that remaining material is insufficient to carry the load, and fails by simple ultimate failure.
Fatigue Fracture Example

- AISI 4320 drive shaft
- B: crack initiation at stress concentration in keyway
- C: Final brittle failure
Fatigue-Life Methods

- Three major fatigue life models
- Methods predict life in number of cycles to failure, \( N \), for a specific level of loading
Fatigue-Life Methods

1. Stress-life method
2. Strain-life method
3. Linear-elastic fracture mechanics method
1. Stress-Life Method

- Test specimens subjected to repeated stress while counting cycles to failure
- Pure bending with no transverse shear
- completely reversed stress cycling

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**$S-N$ Diagram**

- **Low cycle**
- **High cycle**
- **Finite life**
- **Infinite life**

**Diagram Details:**
- **Fatigue strength $S_f$, kpsi**
- **Number of stress cycles, $N$**

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S-N Diagram for Steel

- **Stress levels below** $S_e$: infinite life
- **$10^3$ to $10^6$ cycles**: finite life
- **Below $10^3$ cycles**: low cycle
  - Yielding usually occurs before fatigue

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S-N Diagram for Nonferrous Metals

- no endurance limit
- Fatigue strength $S_f$
- $S$-$N$ diagram for aluminums

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2. Strain-Life Method

- Detailed analysis of plastic deformation at localized regions
- Useful for explaining nature of fatigue
- Fatigue failure begins at a local discontinuity
2. Strain-Life Method

- When stress at discontinuity exceeds elastic limit, plastic strain occurs.
- Cyclic plastic strain can change elastic limit, leading to fatigue.

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Relation of Fatigue Life to Strain

- **Figure 6–13**: relationship of fatigue life to true-strain amplitude

- **Fatigue ductility coefficient** $\varepsilon'_F = \text{true strain at which fracture occurs in one reversal (point A in Fig. 6–12)}$

- **Fatigue strength coefficient** $\sigma'_F = \text{true stress corresponding to fracture in one reversal (point A in Fig. 6–12)}$

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Relation of Fatigue Life to Strain

Fig. 6–13

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Relation of Fatigue Life to Strain

- Equation of plastic-strain line in Fig. 6–13
  \[ \frac{\Delta \varepsilon_p}{2} = \varepsilon'_F (2N)^c \]  
  \hspace{1cm} (6–1)

- Equation of elastic strain line in Fig. 6–13
  \[ \frac{\Delta \varepsilon_e}{2} = \frac{\sigma'_F}{E} (2N)^b \]  
  \hspace{1cm} (6–2)

\[ \frac{\Delta \varepsilon}{2} = \frac{\Delta \varepsilon_e}{2} + \frac{\Delta \varepsilon_p}{2} \]

\[ \frac{\Delta \varepsilon}{2} = \frac{\sigma'_F}{E} (2N)^b + \varepsilon'_F (2N)^c \]  
  \hspace{1cm} (6–3)
Relation of Fatigue Life to Strain

- Fatigue ductility exponent $c$ = slope of plastic-strain line
- $2N$ stress reversals = $N$ cycles
- Fatigue strength exponent $b$ = slope of elastic-strain line
Relation of Fatigue Life to Strain

\[
\frac{\Delta \varepsilon}{2} = \frac{\sigma'_F}{E} (2N)^b + \varepsilon'_F (2N)^c
\]

- **Manson-Coffin**: relationship between fatigue life and total strain
- **Table A–23**: values of coefficients & exponents
- Equation has **limited use for design**
  - Values for total strain at discontinuities are not readily available

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The Endurance Limit

Fig. 6–17

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The Endurance Limit

Simplified estimate of endurance limit for steels for the rotating-beam specimen, $S'_e$

$$S'_e = \begin{cases} 
0.5S_{ut} & S_{ut} \leq 200 \text{ kpsi (1400 MPa)} \\
100 \text{ kpsi} & S_{ut} > 200 \text{ kpsi} \\
700 \text{ MPa} & S_{ut} > 1400 \text{ MPa} 
\end{cases} \quad (6-8)$$

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Fatigue Strength

- For design, an approximation of idealized S-N diagram is desirable.
- To estimate fatigue strength at $10^3$ cycles, start with Eq. (6-2)

$$\frac{\Delta \varepsilon_e}{2} = \frac{\sigma'_F}{E} (2N)^b$$  \hspace{1cm} (6-2)

- Define specimen fatigue strength at a specific number of cycles as

$$(S'_f)_N = E \frac{\Delta \varepsilon_e}{2}$$

$$(S'_f)_N = \sigma'_F (2N)^b$$  \hspace{1cm} (6-9)
Fatigue Strength

- At 10³ cycles, \((S'_f)_{10^3} = \sigma'_F (2 \cdot 10^3)^b = f S_{ut}\)

- \(f = \text{fraction of } S_{ut} \text{ represented by } (S'_f)_{10^3}\)

\[ f = \frac{\sigma'_F}{S_{ut}} (2 \cdot 10^3)^b \quad (6-10) \]

- SAE approximation for steels with \(H_B \leq 500\),

\[ \sigma'_F = S_{ut} + 50 \text{ kpsi} \]

\[ \sigma'_F = S_{ut} + 345 \text{ MPa} \]

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Fatigue Strength

- To find $b$, substitute endurance strength and corresponding cycles into Eq. (6–9) and solve for $b$

$$b = -\frac{\log \left( \frac{\sigma'_F}{S'_e} \right)}{\log \left( 2N_e \right)}$$

(6–12)
Fatigue Strength

\[(S'_f)_N = \sigma'_F (2N)^b\]  \hspace{1cm} (6–9)

\[f = \frac{\sigma'_F}{S_{ut}} (2 \cdot 10^3)^b\]  \hspace{1cm} (6–10)

\[\sigma'_F = S_{ut} + 50 \text{ kpsi}\]

\[\sigma'_F = S_{ut} + 345 \text{ MPa}\]

\[b = -\frac{\log \left( \frac{\sigma'_F}{S'_e} \right)}{\log (2N_e)}\]  \hspace{1cm} (6–12)

Substitute Eqs. 6–11 & 6–12 into Eqs. 6–9 and 6–10 to obtain expressions for \(S'_f\) and \(f\)

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Fatigue Strength Fraction $f$

- Plot Eq. (6–10) for the fatigue strength fraction $f$ of $S_{ut}$ at $10^3$ cycles
- Use $f$ from plot for $S'_f = f S_{ut}$ at $10^3$ cycles on S-N diagram
- Assume $S_e = S'_e = 0.5 S_{ut}$ at $10^6$ cycles

Fig. 6–18
Equations for S-N Diagram

Fig. 6–10
Equations for $S$-$N$ Diagram

- Write equation for $S$-$N$ line from $10^3$ to $10^6$ cycles
- Two known points
  - At $N=10^3$ cycles, $S_f = f S_{ut}$
  - At $N=10^6$ cycles, $S_f = S_e$
- Equations for line:
  \[
  S_f = a \ N^b \tag{6-13}
  \]
  \[
  a = \frac{(f S_{ut})^2}{S_e} \tag{6-14}
  \]
  \[
  b = -\frac{1}{3} \log \left( \frac{f S_{ut}}{S_e} \right) \tag{6-15}
  \]
Equations for S-N Diagram

- If a completely reversed stress $\sigma_{\text{rev}}$ is given, setting $S_f = \sigma_{\text{rev}}$ in Eq. (6–13) and solving for $N$ gives,

  $$N = \left(\frac{\sigma_{\text{rev}}}{a}\right)^{1/b} \tag{6-16}$$

- Typical S-N diagram is only applicable for completely reversed stresses

- For other stress situations, a completely reversed stress with the same life expectancy must be used on the S-N diagram
Low-cycle Fatigue

- $1 \leq N \leq 10^3$
- On the idealized $S-N$ diagram on a log-log scale, failure is predicted by a straight line between two points:

$(10^3, f_S) \text{ and } (1, S_{ut})$

$$S_f \geq S_{ut} N^{(\log f)/3} \quad 1 \leq N \leq 10^3$$

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Example 6-2

Given a 1050 HR steel, estimate

a. the rotating-beam endurance limit at $10^6$ cycles.

b. the endurance strength of a polished rotating-beam specimen corresponding to $10^4$ cycles to failure

c. the expected life of a polished rotating-beam specimen under a completely reversed stress of 55 kpsi.

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Endurance Limit Modifying Factors

- Endurance limit $S'_e$ is for carefully prepared and tested specimen
- If warranted, $S_e$ is obtained from testing of actual parts
- When testing of actual parts is not practical, a set of factors are used to adjust the endurance limit

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Endurance Limit Modifying Factors

\[ S_e = k_a k_b k_c k_d k_e k_f S'_e \]

- \( k_a \) = surface condition factor
- \( k_b \) = size factor
- \( k_c \) = load factor
- \( k_d \) = temperature factor
- \( k_e \) = reliability factor
- \( k_f \) = miscellaneous-effects factor

\( S'_e \) = rotary-beam test specimen endurance limit
\( S_e \) = endurance limit at the critical location of a machine part in the geometry and condition of use
Surface Factor $k_a$

- $k_a$ is a function of ultimate strength
- Higher strengths more sensitive to rough surfaces

$$k_a = a S_{ut}^b$$  \(6-19\)

- **Table 6–2**

<table>
<thead>
<tr>
<th>Surface Finish</th>
<th>$S_{ut}$, kpsi</th>
<th>$S_{ut}$, MPa</th>
<th>Exponent $b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ground</td>
<td>1.34</td>
<td>1.58</td>
<td>$-0.085$</td>
</tr>
<tr>
<td>Machined or cold-drawn</td>
<td>2.70</td>
<td>4.51</td>
<td>$-0.265$</td>
</tr>
<tr>
<td>Hot-rolled</td>
<td>14.4</td>
<td>57.7</td>
<td>$-0.718$</td>
</tr>
<tr>
<td>As-forged</td>
<td>39.9</td>
<td>272</td>
<td>$-0.995$</td>
</tr>
</tbody>
</table>

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Example 6-3

A steel has a min ultimate strength of 520 MPa and a machined surface. Estimate $k_a$. 

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Size Factor $k_b$ rotating & Round

- Larger parts have greater surface area at high stress levels
- Likelihood of crack initiation is higher For bending and torsion loads,

$$k_b = \begin{cases} 
(d/0.3)^{-0.107} = 0.879d^{-0.107} & 0.11 \leq d \leq 2 \text{ in} \\
0.91d^{-0.157} & 2 < d \leq 10 \text{ in} \\
(d/7.62)^{-0.107} = 1.24d^{-0.107} & 2.79 \leq d \leq 51 \text{ mm} \\
1.51d^{-0.157} & 51 < d \leq 254 \text{ mm} 
\end{cases}$$

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Size Factor $k_b$ rotating & Round

- Applies only for round, rotating diameter
- For axial load, there is no size effect,

$$k_b = 1$$
Size Factor $k_b$ not round & rotating

- An equivalent round rotating diameter is obtained.
- Volume of material stressed at and above 95% of max stress = same volume in rotating-beam specimen.
- Lengths cancel, so equate areas
Size Factor $k_b$ not round & rotating

- For a rotating round section, the 95% stress area is the area of a ring,

$$A_{0.95\sigma} = \frac{\pi}{4}[d^2 - (0.95d)^2] = 0.0766d^2 \quad (6-22)$$

- Equate 95% stress area for other conditions to Eq. (6-22) and solve for $d$ as the equivalent round rotating diameter

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Size Factor $k_b$ round & not rotating

- For non-rotating round,

$$A_{0.95\sigma} = 0.01046d^2$$ \hspace{1cm} (6-23)

- Equating to Eq. (6-22) and solving for equivalent diameter,

$$d_e = 0.370d$$ \hspace{1cm} (6-24)
Size Factor $k_b$ not round & not rotating

- For rectangular section $h \times b$, $A_{95\sigma} = 0.05 hb$. Equating to Eq. (6–22),

$$d_e = 0.808(hb)^{1/2} \quad (6–25)$$
Size Factor $k_b$

Table 6–3: $A_{0.95\sigma}$ for common non-rotating structural shapes undergoing bending

$$A_{0.95\sigma} = 0.01046d^2$$
$$d_e = 0.370d$$

$$A_{0.95\sigma} = 0.05hb$$
$$d_e = 0.808\sqrt{hb}$$
Size Factor $k_b$

Table 6–3: $A_{95\sigma}$ for common non-rotating structural shapes undergoing bending

\[ A_{0.95\sigma} = \begin{cases} 
0.10at_f & \text{axis 1-1} \\
0.05ba & \text{axis 1-1} \\
0.05ab & \text{axis 1-1} \\
0.052xa + 0.1t_f(b - x) & \text{axis 2-2} \\
\end{cases} \]

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Example 6-4

A steel shaft loaded in bending is 32 mm in diameter, abutting a filleted shoulder 38 mm in diameter. The shaft material has a mean ultimate tensile strength of 690 MPa. Estimate the Marin size factor $k_b$ if the shaft is used in

a. A rotating mode.

b. A nonrotating mode.

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Loading Factor $k_c$

- Accounts for changes in endurance limit for different types of fatigue loading.
- Only to be used for **single load types**.
- Use **Combination Loading method** (Sec. 6–14) when more than one load type is present.

$$k_c = \begin{cases} 
1 & \text{bending} \\
0.85 & \text{axial} \\
0.59 & \text{torsion}^{17}
\end{cases}$$

(6–26)
Temperature Factor $k_d$

- Endurance limit appears to maintain same relation to ultimate strength for elevated temperatures as at RT

- **Table 6-4**: Effect of Operating Temperature on Tensile Strength of Steel.
  
  1. $S_T = $ tensile strength at operating temperature
  2. $S_{RT} = $ tensile strength at room temperature
# Temperature Factor $k_d$

## Table 6–4

<table>
<thead>
<tr>
<th>Temperature, °C</th>
<th>$S_t/S_{RT}$</th>
<th>Temperature, °F</th>
<th>$S_t/S_{RT}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>1.000</td>
<td>70</td>
<td>1.000</td>
</tr>
<tr>
<td>50</td>
<td>1.010</td>
<td>100</td>
<td>1.008</td>
</tr>
<tr>
<td>100</td>
<td>1.020</td>
<td>200</td>
<td>1.020</td>
</tr>
<tr>
<td>150</td>
<td>1.025</td>
<td>300</td>
<td>1.024</td>
</tr>
<tr>
<td>200</td>
<td>1.020</td>
<td>400</td>
<td>1.018</td>
</tr>
<tr>
<td>250</td>
<td>1.000</td>
<td>500</td>
<td>0.995</td>
</tr>
<tr>
<td>300</td>
<td>0.975</td>
<td>600</td>
<td>0.963</td>
</tr>
<tr>
<td>350</td>
<td>0.943</td>
<td>700</td>
<td>0.927</td>
</tr>
<tr>
<td>400</td>
<td>0.900</td>
<td>800</td>
<td>0.872</td>
</tr>
<tr>
<td>450</td>
<td>0.843</td>
<td>900</td>
<td>0.797</td>
</tr>
<tr>
<td>500</td>
<td>0.768</td>
<td>1000</td>
<td>0.698</td>
</tr>
<tr>
<td>550</td>
<td>0.672</td>
<td>1100</td>
<td>0.567</td>
</tr>
<tr>
<td>600</td>
<td>0.549</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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Temperature Factor $k_d$

- If ultimate strength is known for OT, then just use that strength. Let $k_d = 1$.
- If ultimate strength is known only at RT, use Table 6–4 to estimate ultimate strength at OT. With that strength, let $k_d = 1$.
- Use ultimate strength at RT and apply $k_d$ from Table 6–4 to the endurance limit.

$$k_d = \frac{S_T}{S_{RT}}$$  \hspace{1cm} (6–28)
Temperature Factor $k_d$

- A fourth-order polynomial curve fit of the data of Table 6–4 can be used in place of the table,

$$k_d = 0.975 + 0.432(10^{-3})T_F - 0.115(10^{-5})T_F^2$$
$$+ 0.104(10^{-8})T_F^3 - 0.595(10^{-12})T_F^4$$

(6–27)
Example 6-5

A 1035 steel has a tensile strength of 70 kpsi and is to be used for a part that sees 450°F in service. Estimate the Marin temperature modification factor and \((S_e)_{450°}\) if

a. RT endurance limit by test is \((S’_e)_{70°} = 39.0\) kpsi

b. Only the tensile strength at RT is known.
Reliability Factor $k_e$

- Fig. 6–17, $S'_e = 0.5 S_{ut}$ is typical of the data and represents 50% reliability.
- Reliability factor adjusts to other reliabilities.

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Reliability Factor $k_e$

Fig. 6–17

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## Reliability Factor $k_e$

### Table 6–5

<table>
<thead>
<tr>
<th>Reliability, %</th>
<th>Transformation Variate $z_a$</th>
<th>Reliability Factor $k_e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>0</td>
<td>1.000</td>
</tr>
<tr>
<td>90</td>
<td>1.288</td>
<td>0.897</td>
</tr>
<tr>
<td>95</td>
<td>1.645</td>
<td>0.868</td>
</tr>
<tr>
<td>99</td>
<td>2.326</td>
<td>0.814</td>
</tr>
<tr>
<td>99.9</td>
<td>3.091</td>
<td>0.753</td>
</tr>
<tr>
<td>99.99</td>
<td>3.719</td>
<td>0.702</td>
</tr>
<tr>
<td>99.999</td>
<td>4.265</td>
<td>0.659</td>
</tr>
<tr>
<td>99.9999</td>
<td>4.753</td>
<td>0.620</td>
</tr>
</tbody>
</table>

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Miscellaneous-Effects Factor $k_f$

- Consider other possible factors:
  - Residual stresses
  - Directional characteristics from cold working
  - Case hardening
  - Corrosion
  - Surface conditioning
- Limited data is available.
- May require research or testing.

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Stress Concentration and Notch Sensitivity

- Obtain $K_t$ as usual (Appendix A–15)
- $K_f = \text{fatigue stress-concentration factor}$
- $q = \text{notch sensitivity, ranging from 0 (not sensitive) to 1 (fully sensitive)}$

$$q = \frac{K_f - 1}{K_t - 1} \quad \text{or} \quad q_{\text{shear}} = \frac{K_{fs} - 1}{K_{ts} - 1} \quad (6-31)$$

- For $q = 0$, $K_f = 1$
- For $q = 1$, $K_f = K_t$

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Notch Sensitivity

**Fig. 6–20:** $q$ for bending or axial loading

\[ K_f = 1 + q(K_t - 1) \]
Notch Sensitivity

**Fig. 6-21:** $q_s$ for torsional loading

\[
K_{fs} = 1 + q_s (K_{ts} - 1)
\]
Notch Sensitivity

Use curve fit equations for Figs. 6–20 & 6–21 to get notch sensitivity, or go directly to $K_f$.

\[ q = \frac{1}{1 + \sqrt{\frac{a}{r}}} \]  \hspace{1cm} (6–34)

\[ K_f = 1 + \frac{K_t - 1}{1 + \sqrt{a/r}} \]  \hspace{1cm} (6–33)

Bending or axial:

\[ \sqrt{a} = 0.246 - 3.08(10^{-3})S_{ut} + 1.51(10^{-5})S^2_{ut} - 2.67(10^{-8})S^3_{ut} \]

Torsion:

\[ \sqrt{a} = 0.190 - 2.51(10^{-3})S_{ut} + 1.35(10^{-5})S^2_{ut} - 2.67(10^{-8})S^3_{ut} \]
Notch Sensitivity for Cast Irons

- Cast irons are already full of discontinuities, which are included in the strengths.
- Additional notches do not add much additional harm.
- Recommended to use $q = 0.2$ for cast irons.
Example 6-6

A steel shaft in bending has an ultimate strength of 690 MPa and a shoulder with a fillet radius of 3 mm connecting a 32-mm diameter with a 38-mm diameter. Estimate $K_f$ using:

a. Figure 6–20
b. Equations (6–33) and (6–35)
Application of Fatigue Stress Concentration Factor

- Use $K_f$ as a multiplier to increase the nominal stress.
- Some designers apply $1/K_f$ as a Marin factor to reduce $S_e$.
- For infinite life, either method is equivalent, since
  \[ n_f = \frac{S_e}{K_f \sigma} = \frac{1/K_f}{\sigma} S_e \]
- For finite life, increasing stress is more conservative. Decreasing $S_e$ applies more to high cycle than low cycle.

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Example 6-7

For the step-shaft of Ex. 6–6, it is determined that the fully corrected endurance limit is $S_e = 280$ MPa. Consider the shaft undergoes a fully reversing nominal stress in the fillet of $(\sigma_{rev})_{nom} = 260$ MPa. Estimate the number of cycles to failure.

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Example 6-8

A 1015 hot-rolled steel bar has been machined to a diameter of 1 in. It is to be placed in reversed axial loading for 70,000 cycles to failure in an operating environment of 550°F. Using ASTM minimum properties, and a reliability of 99%, estimate the endurance limit and fatigue strength at 70,000 cycles.

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Example 6-9

Figure 6–22a shows a rotating shaft simply supported in ball bearings at A and D and loaded by a non-rotating force $F$ of 6.8 kN. Using ASTM “minimum” strengths, estimate the life of the part.

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Characterizing Fluctuating Stresses

- The S-N diagram is applicable for completely reversed stresses
- Other fluctuating stresses exist
- Sinusoidal loading patterns are common, but not necessary
Fluctuating Stresses

Figure 6–23

fluctuating stress with high frequency ripple

non-sinusoidal fluctuating stress

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Fluctuating Stresses

Figure 6–23

-- General Fluctuating

-- Completely Reversed

Repeated

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Characterizing Fluctuating Stresses

Stress ratio

\[ R = \frac{\sigma_{\text{min}}}{\sigma_{\text{max}}} \]

Amplitude ratio

\[ A = \frac{\sigma_a}{\sigma_m} \]

\[ \sigma_m = \frac{\sigma_{\text{max}} + \sigma_{\text{min}}}{2} \]

\[ \sigma_a = \frac{\sigma_{\text{max}} - \sigma_{\text{min}}}{2} \]

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Application of $K_f$ for Fluctuating Stresses

- For fluctuating loads at points with stress concentration, the best approach is to design to avoid all localized plastic strain.
- In this case, $K_f$ should be applied to both alternating and midrange stress components.

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Application of $K_f$ for Fluctuating Stresses

- When localized strain does occur, some methods (nominal mean stress method and residual stress method) recommend only applying $K_f$ to the alternating stress.
Application of $K_f$ for Fluctuating Stresses

- **Dowling method** recommends applying $K_f$ to the alternating stress and $K_{fm}$ to the mid-range stress, where $K_{fm}$ is

\[
K_{fm} = K_f \quad K_f |\sigma_{\text{max},o}| < S_y \\
K_{fm} = \frac{S_y - K_f \sigma_{ao}}{|\sigma_{mo}|} \quad K_f |\sigma_{\text{max},o}| > S_y \\
K_{fm} = 0 \quad K_f |\sigma_{\text{max},o} - \sigma_{\text{min},o}| > 2S_y
\]

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Plot of Alternating vs Midrange Stress

- Most common and simple to use
- Goodman or Modified Goodman diagram

Mohammad Suliman Abuhaiba, Ph.D., PE
Commonly Used Failure Criteria

Fig. 6–27

Mohammad Suliman Abuhaiba, Ph.D., PE
Equations for Commonly Used Failure Criteria

\[
\text{Soderberg} \quad \frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_y} = \frac{1}{n} \quad (6-45)
\]

\[
\text{mod-Goodman} \quad \frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_{ut}} = \frac{1}{n} \quad (6-46)
\]

\[
\text{Gerber} \quad \frac{n \sigma_a}{S_e} + \left(\frac{n \sigma_m}{S_{ut}}\right)^2 = 1 \quad (6-47)
\]

\[
\text{ASME-elliptic} \quad \left(\frac{n \sigma_a}{S_e}\right)^2 + \left(\frac{n \sigma_m}{S_y}\right)^2 = 1 \quad (6-48)
\]

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Summarizing Tables for Failure Criteria

- Tables 6–6 to 6–8: equations for Modified Goodman, Gerber, ASME-elliptic, and Langer failure criteria
  - 1st row: fatigue criterion
  - 2nd row: yield criterion
  - 3rd row: intersection of static and fatigue criteria
  - 4th row: equation for fatigue factor of safety
  - 1st column: intersecting equations
  - 2nd column: coordinates of the intersection

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Table 6–6: Modified Goodman and Langer Failure Criteria (1st Quadrant)

<table>
<thead>
<tr>
<th>Intersecting Equations</th>
<th>Intersection Coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{S_a}{S_e} + \frac{S_m}{S_{ut}} = 1 )</td>
<td>( S_a = \frac{r S_e S_{ut}}{r S_{ut} + S_e} )</td>
</tr>
<tr>
<td>( S_m = \frac{S_a}{r} )</td>
<td></td>
</tr>
<tr>
<td>( \frac{S_a}{S_y} + \frac{S_m}{S_y} = 1 )</td>
<td>( S_a = \frac{r S_y}{1 + r} )</td>
</tr>
<tr>
<td>( S_m = \frac{S_y}{1 + r} )</td>
<td></td>
</tr>
<tr>
<td>( \frac{S_a}{S_e} + \frac{S_m}{S_{ut}} = 1 )</td>
<td>( S_m = \frac{(S_y - S_e) S_{ut}}{S_{ut} - S_e} )</td>
</tr>
<tr>
<td>( S_a = S_y - S_m, r_{crit} = \frac{S_a}{S_m} )</td>
<td></td>
</tr>
<tr>
<td>( \frac{S_a}{S_y} + \frac{S_m}{S_y} = 1 )</td>
<td></td>
</tr>
</tbody>
</table>

Fatigue factor of safety

\[
n_f = \frac{1}{\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_{ut}}}\]

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### Table 6–7: Gerber & Langer Failure Criteria (1st Quadrant)

<table>
<thead>
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<tr>
<td>( \frac{S_a}{S_e} + \left( \frac{S_m}{S_{ut}} \right)^2 = 1 )</td>
<td>( S_a = \frac{r^2 S_{ut}^2}{2S_e} \left[ -1 + \sqrt{1 + \left( \frac{2S_e}{rS_{ut}} \right)^2} \right] )</td>
</tr>
<tr>
<td>Load line ( r = \frac{S_a}{S_m} )</td>
<td>( S_m = \frac{S_a}{r} )</td>
</tr>
<tr>
<td>( \frac{S_a}{S_y} + \frac{S_m}{S_y} = 1 )</td>
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<tr>
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<td>( S_m = \frac{S_{ut}^2}{2S_e} \left[ 1 - \sqrt{1 + \left( \frac{2S_e}{S_{ut}} \right)^2 \left( 1 - \frac{S_y}{S_e} \right)} \right] )</td>
</tr>
<tr>
<td>( \frac{S_a}{S_y} + \frac{S_m}{S_y} = 1 )</td>
<td>( S_a = S_y - S_m, r_{\text{crit}} = \frac{S_a}{S_m} )</td>
</tr>
</tbody>
</table>

Fatigue factor of safety

\[
n_f = \frac{1}{2} \left( \frac{S_{ut}}{\sigma_m} \right)^2 \frac{\sigma_a}{S_e} \left[ -1 + \sqrt{1 + \left( \frac{2\sigma_m S_e}{S_{ut} \sigma_a} \right)^2} \right] \quad \sigma_m > 0
\]

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</tr>
</tbody>
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Load line \( r = \frac{S_a}{S_m} \)

Fatigue factor of safety

\[
 n_f = \sqrt{\frac{1}{\left( \frac{\sigma_a}{S_e} \right)^2 + \left( \frac{\sigma_m}{S_y} \right)^2}}
\]
Example 6-10

A 1.5-in-diameter bar has been machined from an AISI 1050 cold-drawn bar. This part is to withstand a fluctuating tensile load varying from 0 to 16 kip. Because of the ends, and the fillet radius, a fatigue stress-concentration factor $K_f$ is 1.85 for $10^6$ or larger life. Find $S_a$ and $S_m$ and the factor of safety guarding against fatigue and first-cycle yielding, using

a. Gerber fatigue line

b. ASME-elliptic fatigue line.

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Example 6-10
Example 6-10

[Diagram of stress amplitude vs. midrange stress]

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Example 6-11

A flat-leaf spring is used to retain an oscillating flat-faced follower in contact with a plate cam. The follower range of motion is 2 in and fixed, so the alternating component of force, bending moment, and stress is fixed, too. The spring is preloaded to adjust to various cam speeds. The preload must be increased to prevent follower float or jump. For lower speeds the preload should be decreased to obtain longer life of cam and follower surfaces. The spring is a steel cantilever 32 in long, 2 in wide, and 1/4 in thick, as seen in Fig. 6–30a. The spring strengths are $S_{ut} = 150$ kpsi, $S_y = 127$ kpsi, and $S_e = 28$ kpsi fully corrected. The total cam motion is 2 in. The designer wishes to preload the spring by deflecting it 2 in for low speed and 5 in for high speed.
Example 6-11

a. Plot Gerber-Langer failure lines with the load line.

b. What are the strength factors of safety corresponding to 2 in and 5 in preload?
Example 6-11

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Example 6-12

A steel bar undergoes cyclic loading such that $\sigma_{\text{max}} = 60$ kpsi and $\sigma_{\text{min}} = -20$ kpsi. For the material, $S_{ut} = 80$ kpsi, $S_y = 65$ kpsi, a fully corrected endurance limit of $S_e = 40$ kpsi, and $f = 0.9$. Estimate the number of cycles to a fatigue failure using:

a. Modified Goodman criterion.
b. Gerber criterion.
Fatigue Criteria for Brittle Materials

- First quadrant fatigue failure criteria follows a concave upward Smith-Dolan locus,
  \[
  \frac{S_a}{S_e} = \frac{1 - S_m/S_{ut}}{1 + S_m/S_{ut}}
  \]  
  \[\text{(6-50)}\]

- Or as a design equation,
  \[
  \frac{n\sigma_a}{S_e} = \frac{1 - n\sigma_m/S_{ut}}{1 + n\sigma_m/S_{ut}}
  \]  
  \[\text{(6-51)}\]
Fatigue Criteria for Brittle Materials

- For a radial load line of slope $r$, the intersection point is

$$S_a = \frac{r S_{ut} + S_e}{2} \left[ -1 + \sqrt{1 + \frac{4r S_{ut}S_e}{(r S_{ut} + S_e)^2}} \right]$$  \hspace{1cm} (6-52)

- In the second quadrant,

$$S_a = S_e + \left( \frac{S_e}{S_{ut}} - 1 \right) S_m \quad -S_{ut} \leq S_m \leq 0 \quad \text{ (for cast iron)}$$  \hspace{1cm} (6-53)

- **Table A-24:** properties of gray cast iron, including endurance limit

- Endurance limit already includes $k_a$ and $k_b$

- Average $k_c$ for axial and torsional is 0.9
Example 6-13

A grade 30 gray cast iron is subjected to a load $F$ applied to a 1 by 3/8-in cross-section link with a 1/4-in-diameter hole drilled in the center as depicted in Fig. 6–31a. The surfaces are machined. In the neighborhood of the hole, what is the factor of safety guarding against failure under the following conditions:

a. The load $F = 1000$ lbf tensile, steady.
b. The load is 1000 lbf repeatedly applied.
c. The load fluctuates between $-1000$ lbf and 300 lbf without column action.

Use the Smith-Dolan fatigue locus.

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Example 6-13
Example 6-13

![Graph showing alternating stress, $\sigma_a$, and midrange stress, $\sigma_m$, with key points labeled: $S_{ut}$, $S_a = 18.5$ kpsi, $S_a = 7.63$ kpsi, and $r$ values at different points.]
Torsional Fatigue Strength

- **Testing:** steady-stress component has no effect on the endurance limit for torsional loading if the material is:
  - ductile, polished, notch-free, and cylindrical.
- For less than perfect surfaces, the modified Goodman line is more reasonable.
- For pure torsion cases, use $k_c = 0.59$ to convert normal endurance strength to shear endurance strength.
- For shear ultimate strength, recommended to use

$$S_{su} = 0.67 S_{ut} \quad (6-54)$$

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Combinations of Loading Modes

- For combined loading, use **Distortion Energy theory** to combine them.
- Obtain Von Mises stresses for both midrange and alternating components.
- Apply appropriate $K_f$ to each type of stress.
- For load factor, use $k_c = 1$. The **torsional load factor** ($k_c = 0.59$) is inherently included in the von Mises equations.
Combinations of Loading Modes

- If needed, axial load factor can be divided into the axial stress.

\[
\sigma_a' = \left\{ \left[ (K_f)_{\text{bending}}(\sigma_a)_{\text{bending}} + (K_f)_{\text{axial}} \frac{(\sigma_a)_{\text{axial}}}{0.85} \right]^2 + 3 \left[ (K_f)_{\text{torsion}}(\tau_a)_{\text{torsion}} \right]^2 \right\}^{1/2}
\]

\[
(6-55)
\]

\[
\sigma_m' = \left\{ \left[ (K_f)_{\text{bending}}(\sigma_m)_{\text{bending}} + (K_f)_{\text{axial}}(\sigma_m)_{\text{axial}} \right]^2 + 3 \left[ (K_f)_{\text{torsion}}(\tau_m)_{\text{torsion}} \right]^2 \right\}^{1/2}
\]

\[
(6-56)
\]

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Static Check for Combination Loading

- Distortion Energy theory still applies for check of static yielding
- Obtain Von Mises stress for maximum stresses
- Stress concentration factors are not necessary to check for yielding at first cycle

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Static Check for Combination Loading

- Alternate simple check is to obtain conservative estimate of $\sigma'_\text{max}$ by summing $\sigma'_a$ and $\sigma'_m$

$$\sigma'_\text{max} = \left[\left(\sigma_a + \sigma_m\right)^2 + 3\left(\tau_a + \tau_m\right)^2\right]^{1/2}$$

$$n_y = \frac{S_y}{\sigma'_\text{max}}$$

- $\sigma'_\text{max} \approx \sigma'_a + \sigma'_m$

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Example 6-14

A rotating shaft is made of 42×4 mm AISI 1018 CD steel tubing and has a 6-mm-diameter hole drilled transversely through it. Estimate the factor of safety guarding against fatigue and static failures using the Gerber and Langer failure criteria for the following loading conditions:

a. The shaft is subjected to a completely reversed torque of 120 N.m in phase with a completely reversed bending moment of 150 N.m.

b. The shaft is subjected to a pulsating torque fluctuating from 20 to 160 N.m and a steady bending moment of 150 N.m.
Example 6-14

![Graph showing Von Mises amplitude stress component and steady stress component](image)

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Varying Fluctuating Stresses
Cumulative Fatigue Damage

- A common situation is to load at \( \sigma_1 \) for \( n_1 \) cycles, then at \( \sigma_2 \) for \( n_2 \) cycles, etc.
- The cycles at each stress level contributes to the fatigue damage.
- Accumulation of damage is represented by the Palmgren-Miner cycle-ratio summation rule, also known as Miner’s rule

\[
\sum \frac{n_i}{N_i} = c
\]

(6–57)
Cumulative Fatigue Damage

- $n_i =$ number of cycles at stress level $\sigma_i$
- $N_i =$ number of cycles to failure at stress level $\sigma_i$
- $c =$ experimentally found to be in the range $0.7 < c < 2.2$, with an average value near unity
- $D =$ the accumulated damage,

$$D = \sum \frac{n_i}{N_i}$$  \hspace{1cm} (6-58)
Example 6-15

Given a part with $S_{ut} = 151 \text{ kpsi}$ and at the critical location of the part, $S_e = 67.5 \text{ kpsi}$. For the loading of Fig. 6–33, estimate the number of repetitions of the stress-time block in Fig. 6–33 that can be made before failure.
Illustration of Miner’s Rule

- **Figure 6–34:** effect of Miner’s rule on endurance limit and fatigue failure line.
- Damaged material line is predicted to be parallel to original material line.

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Weaknesses of Miner’s Rule

Miner’s rule fails to agree with experimental results in two ways

1. It predicts the static strength $S_{ut}$ is damaged.
2. It does not account for the order in which the stresses are applied
Manson’s Method

- Manson’s method overcomes deficiencies of Miner’s rule.

- All fatigue lines on $S-N$ diagram converge to a common point at $0.9S_{ut}$ at $10^3$ cycles.

- It requires each line to be constructed in the same historical order in which the stresses occur.

Fig. 6–35
Surface Fatigue Strength

- When two surfaces roll or roll and slide against one another, a pitting failure may occur after a certain number of cycles.
- The surface fatigue mechanism is complex and not definitively understood.
- Factors include Hertz stresses, number of cycles, surface finish, hardness, lubrication, and temperature.

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Surface Fatigue Strength

- From Eqs. (3–73) and (3–74), the pressure in contacting cylinders,

\[ b = \sqrt{\frac{2F}{\pi l} \frac{(1 - v_1^2)/E_1 + (1 - v_2^2)/E_2}{(1/d_1) + (1/d_2)}} \]

(6–59)

\[ p_{max} = \frac{2F}{\pi bl} \]

(6–60)

\[ b^2 = \frac{4F}{\pi w} \frac{(1 - v_1^2)/E_1 + (1 - v_2^2)/E_2}{1/r_1 + 1/r_2} \]

(6–61)

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Surface Fatigue Strength

- Converting to radius $r$ and width $w$ instead of length $l$,

$$p_{\text{max}} = \frac{2F}{\pi bw}$$  \hspace{1cm} (6–62)

$$S_C = \frac{2F}{\pi bw}$$  \hspace{1cm} (6–63)

- $p_{\text{max}} = \textit{surface endurance strength}$ (contact strength, contact fatigue strength, or Hertzian endurance strength)
Surface Fatigue Strength

- Combining Eqs. (6–61) and (6–63),

\[ \frac{F}{w} \left( \frac{1}{r_1} + \frac{1}{r_2} \right) = \pi S_C^2 \left[ \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} \right] = K_1 \quad (6–64) \]

- \( K_1 = \) Buckingham’s load-stress factor, or wear factor

- In gear studies, a similar factor is used,

\[ K_g = \frac{K_1}{4} \sin \phi \quad (6–65) \]
Surface Fatigue Strength

- From Eq. (6–64), with material property terms incorporated into an elastic coefficient $C_P$

$$S_C = C_P \sqrt{\frac{F}{w} \left( \frac{1}{r_1} + \frac{1}{r_2} \right)}$$  \hspace{1cm} (6–66)
Surface Fatigue Strength

- Experiments show the following relationships

\[ K_1 = \alpha_1 N^{\beta_1} \quad K_g = \alpha N^b \quad S_C = \alpha N^\beta \]

\[
\beta_1 = \frac{\log(K_1/K_2)}{\log(N_1/N_2)} \quad b = \frac{\log(K_g/K_g)}{\log(N_1/N_2)} \quad \beta = \frac{\log(S_C/S_C)}{\log(N_1/N_2)}
\] (6–67)

- Data on induction-hardened steel give \((S_C)_{10^7} = 271 \text{ kpsi}\) and \((S_C)_{10^8} = 239 \text{ kpsi}\), so \(\beta\), from Eq. (6–67), is

\[
\beta = \frac{\log(271/239)}{\log(10^7/10^8)} = -0.055
\]
Surface Fatigue Strength

- A long standing correlation in steels between $S_C$ and $H_B$ at $10^8$ cycles is

\[
(S_C)_{10^8} = \begin{cases} 
0.4H_B - 10 \text{ kpsi} \\
2.76H_B - 70 \text{ MPa}
\end{cases}
\]  

(6-68)

- AGMA uses

\[
0.99(S_C)_{10^7} = 0.327H_B + 26 \text{ kpsi}
\]  

(6-69)
Surface Fatigue Strength

- Incorporating design factor into Eq. (6–66),

\[ \sigma_C = C_P \sqrt{\frac{F}{wn_d}} \left( \frac{1}{r_1} + \frac{1}{r_2} \right) = \frac{C_P}{\sqrt{n_d}} \sqrt{\frac{F}{w}} \left( \frac{1}{r_1} + \frac{1}{r_2} \right) = \frac{S_C}{\sqrt{n_d}} \]
Surface Fatigue Strength

- Since this is nonlinear in its stress-load transformation, the definition of $n_d$ depends on whether load or stress is the primary consideration for failure.
- If loss of function is focused on the load,
  \[ n_d = \left( \frac{S_C}{\sigma_C} \right)^2 \]
- If loss of function is focused on the stress,
  \[ n_d = \frac{S_C}{\sigma_C} \]

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