Chapter 3

Load and Stress Analysis
Shear Force and Bending Moments in Beams

(a) Positive bending

(b) Negative bending

Positive shear

Negative shear

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Distributed Load on Beam

Distributed load \( q(x) \): load intensity

\[
V = \frac{dM}{dx}
\]

\[
\int_{V_A}^{V_B} dV = V_B - V_A = \int_{x_A}^{x_B} q \, dx
\]

\[
\frac{dV}{dx} = \frac{d^2M}{dx^2} = q
\]

\[
\int_{M_A}^{M_B} dM = M_B - M_A = \int_{x_A}^{x_B} V \, dx
\]

- Change in shear force from \( A \) to \( B \) = area of load diagram between \( x_A \) & \( x_B \).
- Change in moment from \( A \) to \( B \) = area of shear-force diagram between \( x_A \) & \( x_B \).
Shear-Moment Diagrams

Fig. 3-5

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Stress element

Shear stress is often resolved into perpendicular components
- First subscript indicates direction of surface normal
- Second subscript indicates direction of shear stress

In most cases, “cross shears” are equal
\[
\tau_{yx} = \tau_{xy} \quad \tau_{zy} = \tau_{yz} \quad \tau_{xz} = \tau_{zx}
\]  

Plane stress occurs when stresses on one surface are zero

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Cutting plane stress element at an arbitrary angle and balancing stresses gives plane-stress transformation equations

\[
\sigma = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\phi + \tau_{xy} \sin 2\phi
\]

\[
\tau = -\frac{\sigma_x - \sigma_y}{2} \sin 2\phi + \tau_{xy} \cos 2\phi
\]
Principal Stresses for Plane Stress

• Differentiating Eq. (3-8) with respect to $\phi$ and setting equal to zero maximizes $\sigma$ and gives

$$\tan 2\phi_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$ \hspace{1cm} (3–10)

• The two values of $2\phi_p$ are the principal directions.
• Stresses in the principal directions are the principal stresses.

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$ \hspace{1cm} (3–13)

• The principal direction surfaces have zero shear stresses.
• There is a third principal stress, equal to zero for plane stress.

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Extreme-value Shear Stresses for Plane Stress

- Performing similar procedure with shear stress in Eq. (3-9), max shear stresses are found to be on surfaces that are ±45º from principal directions.
- The two extreme-value shear stresses are

\[ \tau_1, \tau_2 = \pm \sqrt{\left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} \]  

(3–14)

- Three extreme-value shear stresses.

\[ \tau_{1/2} = \frac{\sigma_1 - \sigma_2}{2}, \quad \tau_{2/3} = \frac{\sigma_2 - \sigma_3}{2}, \quad \tau_{1/3} = \frac{\sigma_1 - \sigma_3}{2} \]  

(3–16)

- If principal stresses are ordered so that \( \sigma_1 > \sigma_2 > \sigma_3 \), then \( \tau_{\text{max}} = \tau_{1/3} \)
Mohr’s Circle Diagram

Fig. 3–10

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Example 3-4

A stress element has $\sigma_x = 80$ MPa and $\tau_{xy} = 50$ MPa cw, as shown in Fig. 3–11a.

(a) Using Mohr’s circle, find the principal stresses and directions, and show these on a stress element correctly aligned with respect to the $xy$ coordinates. Draw another stress element to show $\tau_1$ and $\tau_2$, find the corresponding normal stresses, and label the drawing completely.

(b) Repeat part $a$ using the transformation equations only.
Example 3-4

Principal stress orientation

Max shear orientation

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Example 3-4

Substituting $\phi_p = 64.3^\circ$ into Eq. (3–9) again yields $\tau = 0$, indicating that $-24.03$ MPa is also a principal stress. Once the principal stresses are calculated they can be ordered such that $\sigma_1 \geq \sigma_2$. Thus, $\sigma_1 = 104.03$ MPa and $\sigma_2 = -24.03$ MPa.

Since for $\sigma_1 = 104.03$ MPa, $\phi_p = -25.7^\circ$, and since $\phi$ is defined positive ccw in the transformation equations, we rotate clockwise $25.7^\circ$ for the surface containing $\sigma_1$. We see in Fig. 3–11c that this totally agrees with the semigraphical method.

To determine $\tau_1$ and $\tau_2$, we first use Eq. (3–11) to calculate $\phi_s$:

$$\phi_s = \frac{1}{2} \tan^{-1} \left( -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \right) = \frac{1}{2} \tan^{-1} \left( -\frac{80}{2(-50)} \right) = 19.3^\circ, 109.3^\circ$$

For $\phi_s = 19.3^\circ$, Eqs. (3–8) and (3–9) yield

$$\sigma = \frac{80 + 0}{2} + \frac{80 - 0}{2} \cos[2(19.3)] + (-50) \sin[2(19.3)] = 40.0 \text{ MPa}$$

$$\tau = -\frac{80 - 0}{2} \sin[2(19.3)] + (-50) \cos[2(19.3)] = -64.0 \text{ MPa}$$

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Example 3-4

(b) The transformation equations are programmable. From Eq. (3-10),

\[ \phi_p = \frac{1}{2} \tan^{-1} \left( \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \right) = \frac{1}{2} \tan^{-1} \left( \frac{2(-50)}{80} \right) = -25.7^\circ, 64.3^\circ \]

From Eq. (3-8), for the first angle \( \phi_p = -25.7^\circ \),

\[ \sigma = \frac{80 + 0}{2} + \frac{80 - 0}{2} \cos[2(-25.7)] + (-50) \sin[2(-25.7)] = 104.03 \text{ MPa} \]

The shear on this surface is obtained from Eq. (3-9) as

\[ \tau = -\frac{80 - 0}{2} \sin[2(-25.7)] + (-50) \cos[2(-25.7)] = 0 \text{ MPa} \]

which confirms that 104.03 MPa is a principal stress. From Eq. (3-8), for \( \phi_p = 64.3^\circ \),

\[ \sigma = \frac{80 + 0}{2} + \frac{80 - 0}{2} \cos[2(64.3)] + (-50) \sin[2(64.3)] = -24.03 \text{ MPa} \]
General Three-Dimensional Stress

\[ \sigma^3 - (\sigma_x + \sigma_y + \sigma_z)\sigma^2 + (\sigma_x \sigma_y + \sigma_x \sigma_z + \sigma_y \sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)\sigma \]
\[- (\sigma_x \sigma_y \sigma_z + 2\tau_{xy} \tau_{yz} \tau_{zx} - \sigma_x \tau_{yz}^2 - \sigma_y \tau_{zx}^2 - \sigma_z \tau_{xy}^2) = 0 \] (3–15)

\[ \tau_{1/2} = \frac{\sigma_1 - \sigma_2}{2} \quad \tau_{2/3} = \frac{\sigma_2 - \sigma_3}{2} \quad \tau_{1/3} = \frac{\sigma_1 - \sigma_3}{2} \] (3–16)

Principal stresses are ordered such that \( \sigma_1 > \sigma_2 > \sigma_3 \), in which case \( \tau_{\text{max}} = \tau_{1/3} \)

Fig. 3–12

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Elastic Strain

- **Hooke’s law**

\[ \sigma = E \epsilon \]  \hspace{1cm} (3-17)

- **For axial stress in x direction,**

\[ \epsilon_x = \frac{\sigma_x}{E} \hspace{1cm} \epsilon_y = \epsilon_z = -\nu \frac{\sigma_x}{E} \]  \hspace{1cm} (3-18)

- **For a stress element undergoing \( \sigma_x, \sigma_y, \) and \( \sigma_z, \)** simultaneously,

\[ \epsilon_x = \frac{1}{E} \left[ \sigma_x - \nu (\sigma_y + \sigma_z) \right] \]
\[ \epsilon_y = \frac{1}{E} \left[ \sigma_y - \nu (\sigma_x + \sigma_z) \right] \]  \hspace{1cm} (3-19)
\[ \epsilon_z = \frac{1}{E} \left[ \sigma_z - \nu (\sigma_x + \sigma_y) \right] \]
Elastic Strain

- Hooke’s law for shear:
  \[ \tau = G\gamma \]  
  \( (3-20) \)

- *Shear strain* \( \gamma \) is the change in a right angle of a stress element when subjected to pure shear stress.

- \( G \) is the *shear modulus of elasticity* or *modulus of rigidity*.

- For a linear, isotropic, homogeneous material,
  \[ E = 2G(1 + \nu) \]  
  \( (3-21) \)
Uniformly Distributed Stresses

- For tension and compression,
  \[ \sigma = \frac{F}{A} \]  \hspace{1cm} (3–22)

- For direct shear (no bending present),
  \[ \tau = \frac{V}{A} \]  \hspace{1cm} (3–23)
Normal Stresses for Beams in Bending

- Straight beam in positive bending
- $x$ axis is *neutral axis*
- $xz$ plane is *neutral plane*
- *Neutral axis* is coincident with the *centroidal axis* of cross section

\[
\sigma_x = -\frac{My}{I}
\]

\[
I = \int y^2 \, dA
\]

\[
\sigma_{\text{max}} = \frac{Mc}{I}
\]
Example 3-5

A beam having a T section with the dimensions shown in Fig. 3–15 is subjected to a bending moment of 1600 N \cdot m, about the negative z axis, that causes tension at the top surface. Locate the neutral axis and find the maximum tensile and compressive bending stresses.

Fig. 3–15

Dimensions in mm

\[ A = 12(75) + 12(88) = 1956 \text{ mm}^2 \]

\[ 1956c_1 = 12(75)(6) + 12(88)(56) \]

\[ c_2 = 100 - 32.99 = 67.01 \text{ mm} \]

\[ I_1 = \frac{1}{12}bh^3 = \frac{1}{12}(75)12^3 = 1.080 \times 10^4 \text{ mm}^4 \]

\[ I_2 = \frac{1}{12}(12)88^3 = 6.815 \times 10^5 \text{ mm}^4 \]

\[ \sigma = \frac{Mc_1}{I} \]

\[ \sigma = -\frac{Mc_2}{I} \]

\[ I = [1.080 \times 10^4 + 12(75)26.99^2] + [6.815 \times 10^5 + 12(88)23.01^2] \]

\[ = 1.907 \times 10^6 \text{ mm}^4 \]

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Transverse Shear Stress

Transverse shear stress is always accompanied with bending stress

\[ I = \frac{Ac^2}{3} \]

\[ Q = \int_{y_1}^{c} y \, dA = b \int_{y_1}^{c} y \, dy = \frac{by^2}{2} \bigg|_{y_1}^{c} = \frac{b}{2} (c^2 - y_1^2) \]

\[ \tau = \frac{VQ}{Ib} = \frac{V}{2I} (c^2 - y_1^2) = \frac{3V}{2A} \left(1 - \frac{y_1^2}{c^2}\right) \quad (3-31) \]
# Maximum Values of Transverse Shear Stress

<table>
<thead>
<tr>
<th>Beam Shape</th>
<th>Formula</th>
<th>Beam Shape</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular</td>
<td>$\tau_{\text{ave}} = \frac{V}{A}$</td>
<td>Hollow, thin-walled round</td>
<td>$\tau_{\text{ave}} = \frac{V}{A}$</td>
</tr>
<tr>
<td></td>
<td>$\tau_{\text{max}} = \frac{3V}{2A}$</td>
<td></td>
<td>$\tau_{\text{max}} = \frac{2V}{A}$</td>
</tr>
<tr>
<td>Circular</td>
<td>$\tau_{\text{ave}} = \frac{V}{A}$</td>
<td>Structural I beam (thin-walled)</td>
<td>$\tau_{\text{max}} = \frac{V}{A_{\text{web}}}$</td>
</tr>
<tr>
<td></td>
<td>$\tau_{\text{max}} = \frac{4V}{3A}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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Example 3-7

A beam 12 in long is to support a load of 488 lbf acting 3 in from the left support, as shown in Fig. 3–20a. The beam is an I beam with the cross-sectional dimensions shown. To simplify the calculations, assume a cross section with square corners, as shown in Fig. 3–20c. Points of interest are labeled (a, b, c, and d) at distances y from the neutral axis of 0 in, 1.240\(^-\) in, 1.240\(^+\) in, and 1.5 in (Fig. 3–20c). At the critical axial location along the beam, find the following information.

(a) Determine the profile of the distribution of the transverse shear stress, obtaining values at each of the points of interest.

(b) Determine the bending stresses at the points of interest.

(c) Determine the maximum shear stresses at the points of interest, and compare them.

\[ R_1 = 366 \text{ lbf} \quad R_2 = 122 \text{ lbf} \]
Example 3-7

\[ I = \frac{(2.33)(3.00)^3}{12} - 2 \left[ \frac{(1.08)(2.48)^3}{12} \right] = 2.50 \text{ in}^4 \]

Finding \( Q \) at each point of interest using Eq. (3–30) gives

\[ Q_a = \left( 1.24 + \frac{0.260}{2} \right) [(2.33)(0.260)] + \left( \frac{1.24}{2} \right) [(1.24)(0.170)] = 0.961 \text{ in}^3 \]

\[ Q_b = Q_c = \left( 1.24 + \frac{0.260}{2} \right) [(2.33)(0.260)] = 0.830 \text{ in}^3 \]

\[ Q_d = (1.5)(0) = 0 \text{ in}^3 \]

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Example 3-7

\[
\tau_a = \frac{VQ_a}{Ib_a} = \frac{(366)(0.961)}{(2.50)(0.170)} = 828 \text{ psi}
\]

\[
\tau_b = \frac{VQ_b}{Ib_b} = \frac{(366)(0.830)}{(2.50)(0.170)} = 715 \text{ psi}
\]

\[
\tau_c = \frac{VQ_c}{Ib_c} = \frac{(366)(0.830)}{(2.50)(2.33)} = 52.2 \text{ psi}
\]

\[
\tau_d = \frac{VQ_d}{Ib_d} = \frac{(366)(0)}{(2.50)(2.33)} = 0 \text{ psi}
\]

\[
\sigma_a = \frac{My_a}{I} = \frac{(1098)(0)}{2.50} = 0 \text{ psi}
\]

\[
\sigma_b = \sigma_c = -\frac{My_b}{I} = -\frac{(1098)(1.24)}{2.50} = -545 \text{ psi}
\]

\[
\sigma_d = -\frac{My_d}{I} = -\frac{(1098)(1.50)}{2.50} = -659 \text{ psi}
\]

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(c) Now at each point of interest, consider a stress element that includes the bending stress and the transverse shear stress. The maximum shear stress for each stress element can be determined by Mohr’s circle, or analytically by Eq. (3–14) with \( \sigma_y = 0 \),

\[
\tau_{\text{max}} = \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}
\]

Thus, at each point

\[
\tau_{\text{max},a} = \sqrt{0 + (828)^2} = 828 \text{ psi}
\]

\[
\tau_{\text{max},b} = \sqrt{\left(\frac{-545}{2}\right)^2 + (715)^2} = 765 \text{ psi}
\]

\[
\tau_{\text{max},c} = \sqrt{\left(\frac{-545}{2}\right)^2 + (52.2)^2} = 277 \text{ psi}
\]

\[
\tau_{\text{max},d} = \sqrt{\left(\frac{-659}{2}\right)^2 + 0} = 330 \text{ psi}
\]
Torsion

\[ \theta = \frac{T l}{G J} \]

\[ \tau = \frac{T \rho}{J} \]

\[ \tau_{\text{max}} = \frac{T r}{J} \]

Fig. 3–21

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Example 3-9

The 1.5-in-diameter solid steel shaft shown in Fig. 3–24a is simply supported at the ends. Two pulleys are keyed to the shaft where pulley B is of diameter 4.0 in and pulley C is of diameter 8.0 in. Considering bending and torsional stresses only, determine the locations and magnitudes of the greatest tensile, compressive, and shear stresses in the shaft.

Fig. 3–24
Example 3-9
Example 3-9

The net moment on a section is the vector sum of the components.

\[ M = \sqrt{M_y^2 + M_z^2} \]

At point B,

\[ M_B = \sqrt{2000^2 + 8000^2} = 8246 \text{ lbf} \cdot \text{in} \]

At point C,

\[ M_C = \sqrt{4000^2 + 4000^2} = 5657 \text{ lbf} \cdot \text{in} \]

Thus the maximum bending moment is 8246 lbf \cdot \text{in} and the maximum bending stress at pulley B is

\[ \sigma = \frac{M d/2}{\pi d^4/64} = \frac{32M}{\pi d^3} = \frac{32(8246)}{\pi (1.5^3)} = 24,890 \text{ psi} = 24.89 \text{ kpsi} \]

The maximum torsional shear stress occurs between B and C and is

\[ \tau = \frac{T d/2}{\pi d^4/32} = \frac{16T}{\pi d^3} = \frac{16(1600)}{\pi (1.5^3)} = 2414 \text{ psi} = 2.414 \text{ kpsi} \]
Example 3-9

The maximum bending and torsional shear stresses occur just to the right of pulley B at points E and F as shown in Fig. 3–24e. At point E, the maximum tensile stress will be $\sigma_1$ given by

$$\sigma_1 = \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \frac{24.89}{2} + \sqrt{\left(\frac{24.89}{2}\right)^2 + 2.414^2} = 25.12 \text{ kpsi}$$

Location: at B ($x = 10^+$)

Max. compression and shear

$$\beta = \tan^{-1} \frac{8000}{2000} = 76^\circ$$
Example 3-9

At point $F$, the maximum compressive stress will be $\sigma_2$ given by

$$\sigma_2 = \frac{-\sigma}{2} - \sqrt{\left(\frac{-\sigma}{2}\right)^2 + \tau^2} = \frac{-24.89}{2} - \sqrt{\left(\frac{-24.89}{2}\right)^2 + 2.414^2} = -25.12 \text{ kpsi}$$

The extreme shear stress also occurs at $E$ and $F$ and is

$$\tau_1 = \sqrt{\left(\frac{\pm\sigma}{2}\right)^2 + \tau^2} = \sqrt{\left(\frac{\pm24.89}{2}\right)^2 + 2.414^2} = 12.68 \text{ kpsi}$$
Stress Concentration

- Localized increase of stress near discontinuities
- $K_t$ is Theoretical (Geometric) Stress Concentration Factor

\[ K_t = \frac{\sigma_{\text{max}}}{\sigma_0} \quad K_{ts} = \frac{\tau_{\text{max}}}{\tau_0} \]
Theoretical Stress Concentration Factor

Figure A-15-1
Bar in tension or simple compression with a transverse hole. \( \sigma_0 = \frac{F}{A} \), where \( A = (w - d)t \) and \( t \) is the thickness.
Theoretical Stress Concentration Factor

**Figure A-15-9**

Round shaft with shoulder fillet in bending. $\alpha_0 = Mc/I$, where $c = d/2$ and $I = \pi d^4/64$. 

$K_t$ as a function of $r/d$ and $D/d$.

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Stress Concentration for Static and Ductile Conditions

With static loads and ductile materials

- Highest stressed fibers yield (cold work)
- Load is shared with next fibers
- Cold working is localized
- Overall part does not see damage unless ultimate strength is exceeded
- Stress concentration effect is commonly ignored for static loads on ductile materials
Example 3-13

The 2-mm-thick bar shown in Fig. 3–30 is loaded axially with a constant force of 10 kN. The bar material has been heat treated and quenched to raise its strength, but as a consequence it has lost most of its ductility. It is desired to drill a hole through the center of the 40-mm face of the plate to allow a cable to pass through it. A 4-mm hole is sufficient for the cable to fit, but an 8-mm drill is readily available. Will a crack be more likely to initiate at the larger hole, the smaller hole, or at the fillet?

Fig. 3–30
Example 3-13

Since material is brittle, the effect of stress concentrations near discontinuities must be considered. Dealing with the hole first, for a 4-mm hole, nominal stress is

\[ \sigma_0 = \frac{F}{A} = \frac{F}{(w - d)t} = \frac{10000}{(40 - 4)2} = 139 \text{ MPa} \]

The theoretical stress concentration factor, from Fig. A–15–1, with \( d/w = 4/40 = 0.1 \), is \( K_t = 2.7 \). The maximum stress is

\[ \sigma_{\text{max}} = K_t \sigma_0 = 2.7(139) = 380 \text{ MPa} \]

Fig. A–15–1
Example 3-13

Similarly, for an 8-mm hole,

\[
\sigma_0 = \frac{F}{A} = \frac{10\,000}{(40-8)2} = 156 \text{ MPa}
\]

With \( d/w = 8/40 = 0.2 \), then \( Kt = 2.5 \), and the maximum stress is

\[
\sigma_{\text{max}} = Kt\sigma_0 = 2.5(156) = 390 \text{ MPa}
\]

Though stress concentration is higher with the 4-mm hole, increased nominal stress with the 8-mm hole has more effect on maximum stress. For the fillet,

\[
\sigma_0 = \frac{F}{A} = \frac{10\,000}{(34)2} = 147 \text{ MPa}
\]

From Table A–15–5, \( D/d = 40/34 = 1.18 \), and \( r/d = 1/34 = 0.026 \). Then \( Kt = 2.5 \).

\[
\sigma_{\text{max}} = Kt\sigma_0 = 2.5(147) = 368 \text{ MPa}
\]

crack will most likely occur with 8-mm hole, next the 4-mm hole, and least likely at fillet

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Chapter 4

Deflection and Stiffness

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Spring Rate

- Relation between force and deflection, \( F = F(y) \)
  \[
  k(y) = \lim_{\Delta y \to 0} \frac{\Delta F}{\Delta y} = \frac{dF}{dy}
  \]  
  (4-1)

- **Spring rate**
  \[
  k = \frac{F}{y}
  \]  
  (4-2)

- For linear springs, \( k \) is constant, called *spring constant*
Axially-Loaded Stiffness

- Total extension or contraction of a uniform bar in tension or compression
  \[ \delta = \frac{Fl}{AE} \]  \hspace{1cm} \text{(4-3)}

- Spring constant, with \( k = \frac{F}{\delta} \)
  \[ k = \frac{AE}{l} \]  \hspace{1cm} \text{(4-4)}
Torsionally-Loaded Stiffness

- Angular deflection (in radians) of a uniform solid or hollow round bar subjected to a twisting moment $T$

$$\theta = \frac{Tl}{GJ}$$  \hspace{1cm} (4-5)

- Torsional

$$k = \frac{T}{\theta} = \frac{GJ}{l}$$  \hspace{1cm} (4-7)
Deflection Due to Bending

\[ q \frac{d^4 y}{dx^4} = \frac{EI}{l} \]

\[ \frac{V}{EI} = \frac{d^3 y}{dx^3} \]

\[ \frac{M}{EI} = \frac{d^2 y}{dx^2} \]

\[ \theta = \frac{dy}{dx} \]

\[ y = f(x) \]

\[ y_0 = y_l = 0 \]
Example 4-1

For the beam in Fig. 4–2, the bending moment equation, for $0 \leq x \leq l$, is

$$M = \frac{wl}{2}x - \frac{w}{2}x^2$$

Using Eq. (4–12), determine the equations for the slope and deflection of the beam, the slopes at the ends, and the maximum deflection.

Fig. 4–2

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The boundary conditions for the simply supported beam are \( y = 0 \) at \( x = 0 \) and \( l \). Applying the first condition, \( y = 0 \) at \( x = 0 \), to Eq. (2) results in \( C_2 = 0 \). Applying the second condition to Eq. (2) with \( C_2 = 0 \),
**Strain Energy**

- External work done on elastic member in deforming it is transformed into *strain energy*, or *potential energy*.
- Strain energy equals product of average force and deflection.

\[
U = \frac{F}{2} y = \frac{F^2}{2k} \tag{4-15}
\]

- For axial loading, applying \( k = \frac{AE}{l} \) from Eq. (4-4),

\[
U = \frac{F^2 l}{2AE} \begin{cases} \text{tension and compression} \\ U = \int \frac{F^2}{2AE} \, dx \end{cases} \tag{4-16}
\]
Some Common Strain Energy Formulas

For torsional loading, applying \( k = GJ/l \) from Eq. (4-7),

\[
U = \frac{T^2l}{2GJ} \quad \text{torsion} \tag{4-18}
\]

\[
U = \frac{1}{2GJ} \int T^2 \, dx \quad \tag{4-19}
\]

\[
U = \frac{F^2l}{2AG} \quad \text{direct shear} \tag{4-20}
\]

\[
U = \frac{1}{2AG} \int F^2 \, dx \quad \tag{4-21}
\]

\[
U = \frac{M^2l}{2EI} \quad \text{bending} \tag{4-22}
\]

\[
U = \frac{1}{2EI} \int M^2 \, dx \quad \tag{4-23}
\]
Some Common Strain Energy Formulas

For transverse shear loading,

\[
U = \frac{CV^2l}{2AG} \left\{ \text{transverse shear} \right\} \quad (4-24)
\]

\[
U = \int \frac{CV^2}{2AG} \, dx \quad (4-25)
\]

C is a modifier dependent on the cross sectional shape.

### Table 4-1

<table>
<thead>
<tr>
<th>Strain-Energy Correction Factors for Transverse Shear</th>
<th>Beam Cross-Sectional Shape</th>
<th>Factor C</th>
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<tbody>
<tr>
<td></td>
<td>Circular</td>
<td>1.11</td>
</tr>
<tr>
<td></td>
<td>Thin-walled tubular, round</td>
<td>2.00</td>
</tr>
<tr>
<td></td>
<td>Box sections†</td>
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</tr>
<tr>
<td></td>
<td>Structural sections†</td>
<td>1.00</td>
</tr>
</tbody>
</table>

†Use area of web only.

Dr. Mohammad Suliman Abuhaiba, PE
Example 4-8

A cantilever beam with a round cross section has a concentrated load $F$ at the end, as shown in Fig. 4–9a. Find the strain energy in the beam.

$V = -F$, and the bending moment is $M = -Fx$

$U_{\text{shear}} = \frac{CV^2l}{2AG} = \frac{1.11F^2l}{2AG}$

$U_{\text{bend}} = \int \frac{M^2}{2EI} \, dx = \frac{1}{2EI} \int_0^l (-Fx)^2 \, dx = \frac{F^2l^3}{6EI}$

$U = U_{\text{bend}} + U_{\text{shear}} = \frac{F^2l^3}{6EI} + \frac{1.11F^2l}{2AG}$
Castigliano’s Theorem

- When forces act on elastic systems subject to small displacements, displacement corresponding to any force, in the direction of force, is equal to the partial derivative of total strain energy with respect to that force.

\[ \delta_i = \frac{\partial U}{\partial F_i} \]  

(4–26)

- For rotational displacement, in radians,

\[ \theta_i = \frac{\partial U}{\partial M_i} \]  

(4–27)
Example 4-9

The cantilever of Ex. 4–8 is a carbon steel bar 10 in long with a 1-in diameter and is loaded by a force \( F = 100 \text{ lbf} \).

(a) Find the maximum deflection using Castigliano’s theorem, including that due to shear.

(b) What error is introduced if shear is neglected?

\[
U = \frac{F^2l^3}{6EI} + \frac{1.11F^2l}{2AG}
\]

\[
y_{\text{max}} = \frac{\partial U}{\partial F} = \frac{Fl^3}{3EI} + \frac{1.11Fl}{AG}
\]

\[
I = \frac{\pi d^4}{64} = \frac{\pi (1)^4}{64} = 0.0491 \text{ in}^4
\]

\[
A = \frac{\pi d^2}{4} = \frac{\pi (1)^2}{4} = 0.7854 \text{ in}^2
\]

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Utilizing a Fictitious Force

- Castigliano’s method can be used to find a deflection at a point even if there is no force applied at that point.
- Apply a fictitious force $Q$ at the point, and in the direction, of the desired deflection.
- Set up the equation for total strain energy including energy due to $Q$.
- Take derivative of total strain energy with respect to $Q$.
- Once derivative is taken, $Q$ is no longer needed and can be set to zero.

$$\delta = \frac{\partial U}{\partial Q} \bigg|_{Q=0}$$  \hspace{1cm} (4–28)
Common Deflection Equations

\[
\delta_i = \frac{\partial U}{\partial F_i} = \int \frac{1}{AE} \left( F \frac{\partial F}{\partial F_i} \right) \, dx
\]

\[
\theta_i = \frac{\partial U}{\partial M_i} = \int \frac{1}{GJ} \left( T \frac{\partial T}{\partial M_i} \right) \, dx
\]

\[
\delta_i = \frac{\partial U}{\partial F_i} = \int \frac{1}{EI} \left( M \frac{\partial M}{\partial F_i} \right) \, dx
\]

tension and compression  \hspace{1cm} (4-29)

torsion  \hspace{1cm} (4-30)
bending  \hspace{1cm} (4-31)

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Example 4-10

Using Castigliano’s method, determine the deflections of points A and B due to the force $F$ applied at the end of the step shaft shown in Fig. 4–10. The second area moments for sections $AB$ and $BC$ are $I_1$ and $2I_1$, respectively.
Example 4-10

To avoid the need to determine the ground reaction forces, define the origin of $x$ at the left end of the beam as shown. For $0 \leq x \leq l$, the bending moment is

$$M = -Fx$$

(1)

Since $F$ is at $A$ and in the direction of the desired deflection, the deflection at $A$ from Eq. (4–31) is

$$\delta_A = \frac{\partial U}{\partial F} = \int_0^l \frac{1}{EI} \left( M \frac{\partial M}{\partial F} \right) dx$$

(2)

Substituting Eq. (1) into Eq. (2), noting that $I = I_1$ for $0 \leq x \leq l/2$, and $I = 2I_1$ for $l/2 \leq x \leq l$, we get

$$\delta_A = \frac{1}{E} \left[ \int_0^{l/2} \frac{1}{I_1} (-Fx)(-x) \, dx + \int_{l/2}^l \frac{1}{2I_1} (-Fx)(-x) \, dx \right]$$

$$= \frac{1}{E} \left[ \frac{FI_1^3}{24I_1} + \frac{7FI_1^3}{48I_1} \right] = \frac{3}{16} \frac{FI_1^3}{EI_1}$$

which is positive, as it is in the direction of $F$. 
Example 4-10

For $B$, a fictitious force $Q$ is necessary at the point. Assuming $Q$ acts down at $B$, and $x$ is as before, the moment equation is

$$M = -Fx \quad 0 \leq x \leq l/2$$

$$M = -Fx - Q \left(x - \frac{l}{2}\right) \quad l/2 \leq x \leq l$$

(3)

For Eq. (4–31), we need $\partial M/\partial Q$. From Eq. (3),

$$\frac{\partial M}{\partial Q} = 0 \quad 0 \leq x \leq l/2$$

$$\frac{\partial M}{\partial Q} = - \left(x - \frac{l}{2}\right) \quad l/2 \leq x \leq l$$

(4)
Example 4-10

Once the derivative is taken, \( Q \) can be set to zero, so Eq. (4–31) becomes

\[
\delta_B = \left[ \int_0^l \frac{1}{EI_1} \left( M \frac{\partial M}{\partial Q} \right) dx \right]_{Q=0}
\]

\[
= \frac{1}{EI_1} \int_0^{l/2} (-Fx)(0)dx + \frac{1}{E(2I_1)} \int_{l/2}^l (-Fx) \left[-\left(x - \frac{l}{2}\right)\right] dx
\]

Evaluating the last integral gives

\[
\delta_B = \frac{F}{2EI_1} \left( \frac{x^3}{3} - \frac{lx^2}{4} \right) \bigg|_{l/2}^l = \frac{5}{96} \frac{Fl^3}{EI_1}
\]

which again is positive, in the direction of \( Q \).
Compression Members

- **Column** – A member loaded in compression such that either its length or eccentric loading causes it to experience more than pure compression

- Four categories of columns
  - Long columns with central loading
  - Intermediate-length columns with central loading
  - Columns with eccentric loading
  - Struts or short columns with eccentric loading
Long Columns with Central Loading

- When $P$ reaches critical load, column becomes unstable and bending develops rapidly.
- Critical load depends on end conditions.

Fig. 4–18
Euler Column Formula

- For pin-ended column, critical load is given by \textit{Euler column formula},

\[ P_{cr} = \frac{\pi^2 EI}{l^2} \quad (4-42) \]

- Applies to other end conditions with addition of constant \( C \) for each end condition

\[ P_{cr} = \frac{C \pi^2 EI}{l^2} \quad (4-43) \]
# Recommended Values for End Condition Constant

## Table 4–2

<table>
<thead>
<tr>
<th>Column End Conditions</th>
<th>End-Condition Constant C</th>
<th>Theoretical Value</th>
<th>Conservative Value</th>
<th>Recommended Value*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed-free</td>
<td></td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>Rounded-rounded</td>
<td></td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Fixed-rounded</td>
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<td>2</td>
<td>1</td>
<td>1.2</td>
</tr>
<tr>
<td>Fixed-fixed</td>
<td></td>
<td>4</td>
<td>1</td>
<td>1.2</td>
</tr>
</tbody>
</table>

*To be used only with liberal factors of safety when the column load is accurately known.

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Long Columns with Central Loading

- Using $I = Ak^2$, where $A$ is the area and $k$ is the radius of gyration, Euler column formula can be expressed as
  \[
  \frac{P_{cr}}{A} = \frac{C\pi^2 E}{(l/k)^2}
  \]  
  \((4-44)\)

- $l/k$ is slenderness ratio, used to classify columns according to length categories.

- $P_{cr}/A$ is critical unit load, load per unit area necessary to place column in a condition of unstable equilibrium.

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Euler Curve

Plotting $P_{cr}/A$ vs $l/k$, with $C = 1$ gives curve $PQR$

$$\frac{P_{cr}}{A} = \frac{C \pi^2 E}{(l/k)^2}$$  

(4–44)

- vulnerability to failure near point $Q$
- Since buckling is sudden and catastrophic, a conservative approach near $Q$ is desired
- Point $T$ is usually defined such that $P_{cr}/A = S_y/2$, giving

$$\left(\frac{l}{k}\right)_T = \left(\frac{2\pi^2 C E}{S_y}\right)^{1/2}$$

- For $(l/k) > (l/k)_1$, use Euler equation
- For $(l/k) \leq (l/k)_1$, use a parabolic curve between $S_y$ and $T$

Fig. 4–19

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Intermediate-Length Columns with Central Loading

- For intermediate-length columns, where \((l/k) \leq (l/k)_1\), use a parabolic curve between \(S_y\) and \(T\)

- General form of parabola
  \[
  \frac{P_{cr}}{A} = a - b \left( \frac{l}{k} \right)^2
  \]

- If parabola starts at \(S_y\), then \(a = S_y\)

- If parabola fits tangent to Euler curve at \(T\), then
  \[
  b = \left( \frac{S_y}{2 \pi} \right)^2 \frac{1}{CE
  \]

- Results in parabolic formula, also known as J.B. Johnson formula
  \[
  \frac{P_{cr}}{A} = S_y - \left( \frac{S_y}{2 \pi} \frac{l}{k} \right)^2 \frac{1}{CE} \quad \frac{l}{k} \leq \left( \frac{l}{k} \right)_1
  \]

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Columns with Eccentric Loading

- For eccentrically loaded column with eccentricity \( e \),
  \[
  M = -P(e+y)
  \]

- Substituting into \( \frac{d^2y}{dx^2} = \frac{M}{EI} \),
  \[
  \frac{d^2y}{dx^2} + \frac{P}{EI} y = -\frac{Pe}{EI}
  \]

- Solving with boundary conditions
  \( y = 0 \) at \( x = 0 \) and at \( x = l \)

\[
y = e\left[\tan\left(\frac{l}{2}\sqrt{\frac{P}{EI}}\right)\sin\left(\sqrt{\frac{P}{EI}}x\right) + \cos\left(\sqrt{\frac{P}{EI}}x\right) - 1\right]
\]

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Columns with Eccentric Loading

At midspan where $x = l/2$

$$
\delta = e \left[ \sec \left( \sqrt{\frac{P}{EI}} \frac{l}{2} \right) - 1 \right]
$$

$$
M_{\text{max}} = P(e + \delta) = Pe \sec \left( \frac{l}{2} \sqrt{\frac{P}{EI}} \right)
$$

Max compressive stress includes axial and bending

$$
\sigma_c = \frac{P}{A} \left[ 1 + \frac{ec}{k^2} \sec \left( \frac{l}{2k} \sqrt{\frac{P}{EA}} \right) \right]
$$

- Substituting $M_{\text{max}}$ from Eq. (4-48)

$$
\frac{P}{A} = \frac{S_{yc}}{1 + (ec/k^2) \sec[(l/2k)\sqrt{P/AE}]}
$$

Using $S_{yc}$ as max value of $\sigma_c$, and solving for $P/A$, we obtain secant column formula

$$
\sigma_c = \frac{P}{A} + \frac{Mc}{I} = \frac{P}{A} + \frac{Mc}{Ak^2}
$$
Secant Column Formula

- Secant Column Formula

\[ \frac{P}{A} = \frac{S_{yc}}{1 + (ec/k^2) \sec[(l/2k)\sqrt{P/\text{AE}}]} \]

- \( ec/k^2 \) is eccentricity ratio

- Design charts of secant column formula for various eccentricity ratio can be prepared for a given material strength
Example 4-16

Develop specific Euler equations for the sizes of columns having
(a) Round cross sections
(b) Rectangular cross sections

Solution

(a) Using $A = \pi d^2/4$ and $k = \sqrt{I/A} = [(\pi d^4/64)/(\pi d^2/4)]^{1/2} = d/4$ with Eq. (4–44) gives

$$d = \left(\frac{64P_{cr}l^2}{\pi^3 CE}\right)^{1/4}$$

(4–51)
Example 4-16

(b) For the rectangular column, we specify a cross section $h \times b$ with the restriction that $h \leq b$. If the end conditions are the same for buckling in both directions, then buckling will occur in the direction of the least thickness. Therefore

$$I = \frac{bh^3}{12} \quad A = bh \quad k^2 = \frac{I}{A} = \frac{h^2}{12}$$

Substituting these in Eq. (4–44) gives

$$b = \frac{12P_{cr}l^2}{\pi^2 C Eh^3} \quad h \leq b \quad (4–52)$$

Note, however, that rectangular columns do not generally have the same end conditions in both directions.
Chapter 5

Failures Resulting from Static Loading
Maximum Shear Stress Theory (MSS)

- Yielding begins when *max shear stress* in a stress element exceeds max shear stress in a tension test specimen of the same material when that specimen begins to yield.

- For a tension test specimen, max shear stress is $\sigma_1/2$.

- At yielding, when $\sigma_1 = S_y$, max shear stress is $S_y/2$.

- For any stress element, use Mohr’s circle to find max shear stress. Compare max shear stress to $S_y/2$.

- Ordering principal stresses such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$,

$$
\tau_{\text{max}} = \frac{\sigma_1 - \sigma_3}{2} \geq \frac{S_y}{2} \quad \text{or} \quad \sigma_1 - \sigma_3 \geq S_y
$$

$$
\tau_{\text{max}} = \frac{S_y}{2n} \quad \text{or} \quad \sigma_1 - \sigma_3 = \frac{S_y}{n}
$$

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Maximum Shear Stress Theory (MSS)

- consider a plane stress state

- Let $\sigma_A$ & $\sigma_B$ represent two non-zero principal stresses, then order them with the zero principal stress such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$

- Assuming $\sigma_A \geq \sigma_B$ there are three cases to consider

- Case 1: $\sigma_A \geq \sigma_B \geq 0$, $\sigma_1 = \sigma_A$ and $\sigma_3 = 0$
  ✓ Eq. (5–1) reduces to $\sigma_A \geq S_y$

- Case 2: $\sigma_A \geq 0 \geq \sigma_B$, $\sigma_1 = \sigma_A$ and $\sigma_3 = \sigma_B$
  ✓ Eq. (5–1) reduces to $\sigma_A - \sigma_B \geq S_y$

- Case 3: $0 \geq \sigma_A \geq \sigma_B$, $\sigma_1 = 0$ and $\sigma_3 = \sigma_B$
  ✓ Eq. (5–1) reduces to $\sigma_B \leq -S_y$

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Maximum Shear Stress Theory (MSS)

- Plot three cases on principal stress axes
- **Case 1**: $\sigma_A \geq \sigma_B \geq 0$
  - $\sigma_A \geq S_y$
- **Case 2**: $\sigma_A \geq 0 \geq \sigma_B$
  - $\sigma_A - \sigma_B \geq S_y$
- **Case 3**: $0 \geq \sigma_A \geq \sigma_B$
  - $\sigma_B \leq -S_y$
- Other lines are symmetric cases
- Inside envelope is predicted safe zone

Fig. 5–7

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Maximum Shear Stress Theory (MSS)

- Comparison to experimental data
- Conservative in all quadrants
Distortion Energy (DE) Failure Theory

- Also known as: Octahedral Shear Stress, Shear Energy, Von Mises
- Ductile materials stressed hydrostatically (equal principal stresses) exhibited yield strengths greatly in excess of expected values.
- If strain energy is divided into hydrostatic volume changing energy and angular distortion energy, yielding is primarily affected by distortion energy.
- Yielding occurs when distortion strain energy per unit volume reaches distortion strain energy per unit volume for yield in simple tension or compression of the same material.

\[
\begin{align*}
\sigma_2 & \quad \sigma_1 > \sigma_2 > \sigma_3 \\
\sigma_3 & = \sigma_{av} \\
\sigma_1 - \sigma_{av} & + (c) \text{ Distortional component}
\end{align*}
\]

(a) Triaxial stresses  (b) Hydrostatic component  (c) Distortional component
Distortion Energy (DE) Failure Theory

- Hydrostatic stress is average of principal stresses
  \[ \sigma_{av} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \]  

- Strain energy per unit volume,
  \[ u = \frac{1}{2}[\epsilon_1 \sigma_1 + \epsilon_2 \sigma_2 + \epsilon_3 \sigma_3] \]

\[ \epsilon_x = \frac{1}{E} \left[ \sigma_x - \nu(\sigma_y + \sigma_z) \right] \]
\[ \epsilon_y = \frac{1}{E} \left[ \sigma_y - \nu(\sigma_x + \sigma_z) \right] \]  

\[ \epsilon_z = \frac{1}{E} \left[ \sigma_z - \nu(\sigma_x + \sigma_y) \right] \]

\[ u = \frac{1}{2E} \left[ \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1) \right] \]

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Distortion Energy (DE) Failure Theory

- Strain energy for producing only volume change is obtained by substituting $\sigma_{av}$ for $\sigma_1$, $\sigma_2$, and $\sigma_3$

$$u_v = \frac{1 - 2\nu}{6E} \left( \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_1\sigma_2 + 2\sigma_2\sigma_3 + 2\sigma_3\sigma_1 \right)$$  \hspace{1cm} (5-7)

- Obtain distortion energy by subtracting volume changing energy, Eq. (5–7), from total strain energy, Eq. (b)

$$u_d = u - u_v = \frac{1 + \nu}{3E} \left[ \frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2} \right]$$  \hspace{1cm} (5-8)

- Tension test specimen at yield has $\sigma_1 = S_y$ and $\sigma_2 = \sigma_3 = 0$

$$u_d = \frac{1 + \nu}{3E} S_y^2$$

- DE theory predicts failure when distortion energy exceeds distortion energy of tension test specimen

$$\left[ \frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2} \right]^{1/2} \geq S_y$$  \hspace{1cm} (5-10)
Von Mises Stress

\[ \sigma' = \left( \frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2} \right)^{1/2} \]  

(5–12)

- For plane stress, simplifies to

\[ \sigma' = \left( \sigma_A^2 - \sigma_A \sigma_B + \sigma_B^2 \right)^{1/2} \]  

(5–13)

- In terms of \( xyz \) components, in three dimensions

\[ \sigma' = \frac{1}{\sqrt{2}} \left[ (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \right]^{1/2} \]  

(5–14)

- In terms of \( xyz \) components, for plane stress

\[ \sigma' = \left( \sigma_x^2 - \sigma_x \sigma_y + \sigma_y^2 + 3\tau_{xy}^2 \right)^{1/2} \]  

(5–15)

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Distortion Energy Theory With Von Mises Stress

- Von Mises Stress: a single, equivalent, or effective stress for the entire general state of stress in a stress element.
- DET simply compares von Mises stress to yield strength.

\[ \sigma' \geq S_y \]  \hspace{1cm} (5-11)

- Introducing a design factor,

\[ \sigma' = \frac{S_y}{n} \]  \hspace{1cm} (5-19)

- Expressing as factor of safety,

\[ n = \frac{S_y}{\sigma'} \]

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Octahedral Stresses

- Octahedral stresses are identical on 8 surfaces symmetric to the principal stress directions.
- Octahedral stresses allow representation of any stress situation with a set of normal and shear stresses.

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Octahedral Stresses

- Octahedral normal stresses are normal to the octahedral surfaces, and are equal to the average of the principal stresses.

- Octahedral shear stresses lie on the octahedral surfaces.

- Yielding begins when octahedral shear stress in a stress element exceeds octahedral shear stress in a tension test specimen at yield.

- For a tension test specimen at yielding, $\sigma_1 = S_y$, $\sigma_2 = \sigma_3 = 0$.

\[ \tau_{\text{oct}} = \frac{\sqrt{2}}{3} S_y \]  \hfill (5-17)

\[ \tau_{\text{oct}} = \frac{1}{3} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2} \]  \hfill (5-16)
Octahedral Shear Stress Failure Theory

\[
\left[ \frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2} \right]^{1/2} \geq S_y
\]

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Shear Strength Predictions

DE theory predicts shear strength as

\[ S_{sy} = 0.577 S_y \]

- For pure shear loading, Mohr’s circle shows that \( \sigma_A = -\sigma_B = \tau \)
- Plotting this equation on principal stress axes gives load line for pure shear case
- Intersection of pure shear load line with failure curve indicates shear strength has been reached
- Each failure theory predicts shear strength to be some fraction of normal strength
- For MSS theory, intersecting pure shear load line with failure line \([\text{Eq. (5–5)}]\) results in
  \[ S_{sy} = 0.5 S_y \]
- For DE theory, intersection pure shear load line with failure curve \([\text{Eq. (5–11)}]\) gives
  \[ \left(3\tau_{xy}^2\right)^{1/2} = S_y \]
Example 5-1

A hot-rolled steel has a yield strength of $S_{yt} = S_{yc} = 100$ kpsi and a true strain at fracture of $\varepsilon_f = 0.55$. Estimate the factor of safety for the following principal stress states:

(a) $\sigma_x = 70$ kpsi, $\sigma_y = 70$ kpsi, $\tau_{xy} = 0$ kpsi
(b) $\sigma_x = 60$ kpsi, $\sigma_y = 40$ kpsi, $\tau_{xy} = -15$ kpsi
(c) $\sigma_x = 0$ kpsi, $\sigma_y = 40$ kpsi, $\tau_{xy} = 45$ kpsi
(d) $\sigma_x = -40$ kpsi, $\sigma_y = -60$ kpsi, $\tau_{xy} = 15$ kpsi
(e) $\sigma_1 = 30$ kpsi, $\sigma_2 = 30$ kpsi, $\sigma_3 = 30$ kpsi

Solution

Since $\varepsilon_f > 0.05$ and $S_{yt}$ and $S_{yc}$ are equal, the material is ductile and both the distortion-energy (DE) theory and maximum-shear-stress (MSS) theory apply. Both will be used for comparison. Note that cases $a$ to $d$ are plane stress states.
Example 5-1

A tabular summary of the factors of safety is included for comparisons.

<table>
<thead>
<tr>
<th></th>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
<th>(d)</th>
<th>(e)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DE</td>
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<td>1.70</td>
<td>1.14</td>
<td>1.70</td>
<td>∞</td>
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<tr>
<td>MSS</td>
<td>1.43</td>
<td>1.47</td>
<td>1.02</td>
<td>1.47</td>
<td>∞</td>
</tr>
</tbody>
</table>

Fig. 5–11

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Mohr Theory

- Some materials have compressive strengths different from tensile strengths
- *Mohr theory* is based on three simple tests: tension, compression, and shear
- Plotting Mohr’s circle for each, bounding curve defines failure envelope

Fig. 5–12

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Coulomb-Mohr Theory

- For ductile material, use tensile and compressive yield strengths
- For brittle material, use tensile & compressive ultimate strengths

Curved failure curve is difficult to determine analytically

*Coulomb-Mohr* theory simplifies to linear failure envelope using only tension and compression tests (dashed circles)

\[
\frac{\sigma_1}{S_t} = \frac{\sigma_3}{S_c} = \frac{1}{n}
\]

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To plot on principal stress axes, consider three cases

- **Case 1:** \( \sigma_A \geq \sigma_B \geq 0 \)  
  For this case, \( \sigma_1 = \sigma_A \) and \( \sigma_3 = 0 \)
  - Eq. (5–22) reduces to
    \[
    \frac{\sigma_A}{S_t} - \frac{\sigma_B}{S_c} = 1
    \]  
  
- **Case 2:** \( \sigma_A \geq 0 \geq \sigma_B \)  
  For this case, \( \sigma_1 = \sigma_A \) and \( \sigma_3 = \sigma_B \)
  - Eq. (5-22) reduces to
    \[
    \frac{\sigma_A}{S_t} - \frac{\sigma_B}{S_c} \geq 1
    \]  

- **Case 3:** \( 0 \geq \sigma_A \geq \sigma_B \)  
  For this case, \( \sigma_1 = 0 \) and \( \sigma_3 = \sigma_B \)
  - Eq. (5–22) reduces to
    \[
    \sigma_B \leq -S_c
    \]
Coulomb-Mohr Theory

- Plot three cases on principal stress axes
- Similar to MSS theory, except with different strengths for compression and tension

- Intersect the pure shear load line with the failure line to determine the shear strength
- Since failure line is a function of tensile and compressive strengths, shear strength is also a function of these terms.

\[ S_{sy} = \frac{S_{yt} S_{yc}}{S_{yt} + S_{yc}} \]

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Example 5-2

A 25-mm-diameter shaft is statically torqued to 230 N · m. It is made of cast 195-T6 aluminum, with a yield strength in tension of 160 MPa and a yield strength in compression of 170 MPa. It is machined to final diameter. Estimate the factor of safety of the shaft.

Solution

The maximum shear stress is given by

\[ \tau = \frac{16T}{\pi d^3} = \frac{16(230)}{\pi \left[25 \left(10^{-3}\right)\right]^3} = 75 \left(10^6\right) \text{N/m}^2 = 75 \text{ MPa} \]

The two nonzero principal stresses are 75 and −75 MPa, making the ordered principal stresses \( \sigma_1 = 75 \), \( \sigma_2 = 0 \), and \( \sigma_3 = -75 \) MPa. From Eq. (5–26), for yield,

\[ n = \frac{1}{\sigma_1/S_{yt} - \sigma_3/S_{yc}} = \frac{1}{75/160 - (-75)/170} = 1.10 \]
Example 5-2

Alternatively, from Eq. (5–27),

\[ S_{sy} = \frac{S_{yt} S_{yc}}{S_{yt} + S_{yc}} = \frac{160(170)}{160 + 170} = 82.4 \text{ MPa} \]

and \( \tau_{\text{max}} = 75 \text{ MPa} \). Thus,

\[ n = \frac{S_{sy}}{\tau_{\text{max}}} = \frac{82.4}{75} = 1.10 \]
Experimental data indicates some differences in failure for brittle materials.

Failure criteria is generally ultimate fracture rather than yielding.

Compressive strengths are usually larger than tensile strengths.
# Brittle Coulomb-Mohr

## Failure criteria

<table>
<thead>
<tr>
<th>Quadrant condition</th>
<th>Failure criteria</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_A \geq \sigma_B \geq 0 )</td>
<td>( \sigma_A = \frac{S_{ut}}{n} ) (5–31a)</td>
</tr>
<tr>
<td>( \sigma_A \geq 0 \geq \sigma_B )</td>
<td>( \frac{\sigma_A}{S_{ut}} - \frac{\sigma_B}{S_{uc}} = \frac{1}{n} ) (5–31b)</td>
</tr>
<tr>
<td>( 0 \geq \sigma_A \geq \sigma_B )</td>
<td>( \sigma_B = -\frac{S_{uc}}{n} ) (5–31c)</td>
</tr>
</tbody>
</table>

![Diagram of Brittle Coulomb-Mohr](image)

**Fig. 5–14**

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Brittle Failure Experimental Data

- Coulomb-Mohr is conservative in the 4th quadrant
- Modified Mohr criteria adjusts to better fit the data in the 4th quadrant

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Modified-Mohr

Quadrant condition

| $\sigma_A \geq \sigma_B$ | $\geq 0$ |
| $\sigma_A \geq 0 \geq \sigma_B$ | and $\left| \frac{\sigma_B}{\sigma_A} \right| \leq 1$ |
| $\sigma_A \geq 0 \geq \sigma_B$ | and $\left| \frac{\sigma_B}{\sigma_A} \right| > 1$ |
| $0 \geq \sigma_A \geq \sigma_B$ |

Failure criteria

$$\sigma_A = \frac{S_{ut}}{n} \quad (5-32a)$$
$$\sigma_A = \frac{S_{ul}}{n} \quad (5-32a)$$
$$\frac{(S_{uc} - S_{ut})}{S_{uc} S_{ut}} \sigma_A - \frac{\sigma_B}{S_{uc}} = \frac{1}{n} \quad (5-32b)$$
$$\sigma_B = -\frac{S_{uc}}{n} \quad (5-32c)$$

---

Gray cast-iron data
Example 5-5

Consider the wrench in Ex. 5–3, Fig. 5–16, as made of cast iron, machined to dimension. The force $F$ required to fracture this part can be regarded as the strength of the component part. If the material is ASTM grade 30 cast iron, find the force $F$ with

(a) Coulomb-Mohr failure model.

(b) Modified Mohr failure model.

Fig. 5–16
Example 5-5

We assume that the lever $DC$ is strong enough, and not part of the problem. Since grade 30 cast iron is a brittle material and cast iron, the stress-concentration factors $K_t$ and $K_{ts}$ are set to unity. From Table A–24, the tensile ultimate strength is 31 kpsi and the compressive ultimate strength is 109 kpsi. The stress element at $A$ on the top surface will be subjected to a tensile bending stress and a torsional stress. This location, on the 1-in.-diameter section fillet, is the weakest location, and it governs the strength of the assembly. The normal stress $\sigma_x$ and the shear stress at $A$ are given by

$$ \sigma_x = K_t \frac{M}{I/c} = K_t \frac{32M}{\pi d^3} = (1) \frac{32(14F)}{\pi (1)^3} = 142.6F $$

$$ \tau_{xy} = K_{ts} \frac{T_r}{J} = K_{ts} \frac{16T}{\pi d^3} = (1) \frac{16(15F)}{\pi (1)^3} = 76.4F $$

From Eq. (3–13) the nonzero principal stresses $\sigma_A$ and $\sigma_B$ are

$$ \sigma_A, \sigma_B = \frac{142.6F + 0}{2} \pm \sqrt{\left(\frac{142.6F - 0}{2}\right)^2 + (76.4F)^2} = 175.8F, -33.2F $$

This puts us in the fourth-quadrant of the $\sigma_A, \sigma_B$ plane.
Example 5-5

(a) For BCM, Eq. (5–31b) applies with $n = 1$ for failure.

\[
\frac{\sigma_A}{S_{ut}} - \frac{\sigma_B}{S_{uc}} = \frac{175.8F}{31(10^3)} - \frac{(-33.2F)}{109(10^3)} = 1
\]

Solving for $F$ yields

\[
F = 167 \text{ lbf}
\]

(b) For MM, the slope of the load line is $|\sigma_B/\sigma_A| = 33.2/175.8 = 0.189 < 1$. Obviously, Eq. (5–32a) applies.

\[
\frac{\sigma_A}{S_{ut}} = \frac{175.8F}{31(10^3)} = 1
\]

\[
F = 176 \text{ lbf}
\]

As one would expect from inspection of Fig. 5–19, Coulomb-Mohr is more conservative.

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Selection of Failure Criteria in Flowchart Form

Brittle behavior

< $\varepsilon_f$ ≤ 0.05

Yes

No

Conservative?

Mod. Mohr (MM) Eq. (5–32)

Brittle Coulomb-Mohr (BCM) Eq. (5–31)

Ductile Coulomb-Mohr (DCM) Eq. (5–26)

< $S_{yt}$ ≤ $S_{yc}$?

Yes

No

Conservative?

Distortion-energy (DE) Eqs. (5–15) and (5–19)

Maximum shear stress (MSS) Eq. (5–3)

Ductile behavior

≥ 0.05

No

Yes

Fig. 5–21

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Chapter 6

Fatigue Failure Resulting from Variable Loading

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Stages of Fatigue Failure

- **Stage I** – Initiation of micro-crack due to cyclic plastic deformation
- **Stage II** – Progresses to macro-crack that repeatedly opens and closes, creating bands called *beach marks*
- **Stage III** – Crack has propagated far enough that remaining material is insufficient to carry the load, and fails by simple ultimate failure

*Fig. 6–1*
Fatigue-Life Methods

- Three major fatigue life models
- Methods predict life in number of cycles to failure, $N$, for a specific level of loading

1. Stress-life method
   - Least accurate, particularly for low cycle applications
   - Most traditional, easiest to implement

2. Strain-life method
   - Detailed analysis of plastic deformation at localized regions
   - Several idealizations are compounded, leading to uncertainties in results

3. Linear-elastic fracture mechanics method
   - Assumes crack exists
   - Predicts crack growth with respect to stress intensity
Stress-Life Method

- Test specimens are subjected to repeated stress while counting cycles to failure
- Specimen subjected to pure bending with no transverse shear
- As specimen rotates, stress fluctuates between equal magnitudes of tension and compression, completely reversed stress cycling

Strength corresponding to the knee is called endurance limit $S_e$

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S-N Diagram for Nonferrous Metals

- Nonferrous metals often do not have an endurance limit.
- Fatigue strength $S_f$ is reported at a specific number of cycles.
- Figure 6–11 shows typical S-N diagram for aluminums.

- Stress levels below $S_e$ predict infinite life.
- Between $10^3$ & $10^6$ cycles, finite life is predicted.
- Below $10^3$ cycles is known as low cycle, and is often considered quasi-static. Yielding usually occurs before fatigue in this zone.

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Strain-Life Method

- Fatigue failure begins at a local discontinuity
- When stress at discontinuity exceeds elastic limit, plastic strain occurs
- Cyclic plastic strain can change elastic limit, leading to fatigue
- Fig. 6–12 shows true stress-true strain hysteresis loops of first five stress reversals
Relation of Fatigue Life to Strain

- Figure 6–13: relationship of fatigue life to true-strain amplitude
- *Fatigue ductility coefficient* $\varepsilon_F'$: true strain corresponding to fracture in one reversal (point A in Fig. 6–12)
- *Fatigue strength coefficient* $\sigma_F'$: true stress corresponding to fracture in one reversal (point A in Fig. 6–12)

- **Fatigue ductility exponent** $c$: slope of plastic-strain line, and is power to which life $2N$ must be raised to be proportional to true plastic-strain amplitude. $2N$ stress reversals corresponds to $N$ cycles.
- **Fatigue strength exponent** $b$: slope of elastic-strain line, and is power to which life $2N$ must be raised to be proportional to true-stress amplitude.

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Relation of Fatigue Life to Strain

- Total strain amplitude is half the total strain range
  \[ \frac{\Delta \varepsilon}{2} = \frac{\Delta \varepsilon_e}{2} + \frac{\Delta \varepsilon_p}{2} \]  

- Equation of the plastic-strain line in Fig. 6–13
  \[ \frac{\Delta \varepsilon_p}{2} = \varepsilon'_F (2N)^c \]  

- Equation of the elastic strain line in Fig. 6–13
  \[ \frac{\Delta \varepsilon_e}{2} = \frac{\sigma'_F}{E} (2N)^b \]  

- Applying Eq. (a), the total-strain amplitude is
  \[ \frac{\Delta \varepsilon}{2} = \frac{\sigma'_F}{E} (2N)^b + \varepsilon'_F (2N)^c \]  

- Known as Manson-Coffin relationship between fatigue life and total strain
- Some values of coefficients and exponents given in Table A–23
The Endurance Limit

Simplified estimate of endurance limit for steels for the rotating-beam specimen, $S'_e$

\[
S'_e = \begin{cases} 
  0.5 S_{ul} & S_{ul} \leq 200 \text{ kpsi (1400 MPa)} \\
  100 \text{ kpsi} & S_{ul} > 200 \text{ kpsi} \\
  700 \text{ MPa} & S_{ul} > 1400 \text{ MPa}
\end{cases}
\]

Fig. 6–17

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Fatigue Strength

- For design, an approximation of the idealized S-N diagram is desirable.
- To estimate the fatigue strength at $10^3$ cycles, start with Eq. (6-2)

$$\frac{\Delta \varepsilon_e}{2} = \frac{\sigma_F'}{E} (2N)^b$$

(6–2)

- Define the specimen fatigue strength at a specific number of cycles as

$$(S_F')_N = E \Delta \varepsilon_e / 2$$

- Combine with Eq. (6–2),

$$(S_F')_N = \sigma_F' (2N)^b$$

(6–9)
Fatigue Strength

\[(S'_f)_N = \sigma'_F (2N)^b\]  \hspace{1cm} (6–9)

- At 10^3 cycles, \[(S'_f)_{10^3} = \sigma'_F (2 \cdot 10^3)^b = f S_{ut}\]
- \(f\) is the fraction of \(S_{ut}\) represented by \((S'_f)_{10^3}\)
- Solving for \(f\),
\[f = \frac{\sigma'_F}{S_{ut}} (2 \cdot 10^3)^b\]  \hspace{1cm} (6–10)

- The SAE approximation for steels with \(H_B \leq 500\) may be used.
\[\sigma'_F = S_{ut} + 50 \text{ kpsi} \quad \text{or} \quad \sigma'_F = S_{ut} + 345 \text{ MPa} \]  \hspace{1cm} (6–11)
- To find \(b\), substitute the endurance strength and corresponding cycles into Eq. (6–9) and solve for \(b\)
\[b = -\frac{\log (\sigma'_F / S'_e)}{\log (2N_e)}\]  \hspace{1cm} (6–12)

- Eqs. (6–11) and (6–12) can be substituted into Eqs. (6–9) and (6–10) to obtain expressions for \(S'_f\) and \(f\)

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Fatigue Strength Fraction $f$

- Plot Eq. (6–10) for fatigue strength fraction $f$ of $S_{ut}$ at $10^3$ cycles
- Use $f$ from plot for $S'_{f} = fS_{ut}$ at $10^3$ cycles on $S$-$N$ diagram
- Assumes $S_e = S'_{e} = 0.5S_{ut}$ at $10^6$ cycles

![Graph showing the relationship between $f$ and $S_{ut}$]
Equations for S-N Diagram

- Write equation for S-N line from $10^3$ to $10^6$ cycles
- Two known points
  - At $N = 10^3$ cycles, $S_f = f S_{ut}$
  - At $N = 10^6$ cycles, $S_f = S_e$
- Equations for line:

\[ S_f = a \ N^b \]

\[ a = \frac{(f S_{ut})^2}{S_e} \]

\[ b = -\frac{1}{3} \log \left( \frac{f S_{ut}}{S_e} \right) \]

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Equations for S-N Diagram

- If a completely reversed stress $\sigma_{\text{rev}}$ is given, setting $S_f = \sigma_{\text{rev}}$ in Eq. (6–13) and solving for $N$ gives,

$$N = \left( \frac{\sigma_{\text{rev}}}{a} \right)^{1/b} \quad (6-16)$$

- Note that the typical S-N diagram is only applicable for completely reversed stresses.

- For other stress situations, a completely reversed stress with the same life expectancy must be used on the S-N diagram.
Low - cycle Fatigue

- Low-cycle fatigue is defined for fatigue failures in the range $1 \leq N \leq 10^3$
- On the idealized S-N diagram on a log-log scale, failure is predicted by a straight line between two points $(10^3, f_{S_{ut}})$ and $(1, S_{ut})$

$$S_f \geq S_{ut} N^{(\log f)/3} \quad 1 \leq N \leq 10^3$$

(6-17)
Example 6-2

Given a 1050 HR steel, estimate:
(a) the rotating-beam endurance limit at $10^6$ cycles.
(b) the endurance strength of a polished rotating-beam specimen corresponding to $10^4$ cycles to failure.
(c) the expected life of a polished rotating-beam specimen under a completely reversed stress of 55 kpsi.

Solution
(a) From Table A–20, $S_{ut} = 90$ kpsi. From Eq. (6–8),

$$S'_e = 0.5(90) = 45 \text{ kpsi}$$

(b) From Fig. 6–18, for $S_{ut} = 90$ kpsi, $f = 0.86$. From Eq. (6–14),

$$a = \frac{[0.86(90)]^2}{45} = 133.1 \text{ kpsi}$$

From Eq. (6–15),

$$b = -\frac{1}{3} \log \left[ \frac{0.86(90)}{45} \right] = -0.0785$$

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Example 6-2

Thus, Eq. (6–13) is

\[ S'_f = 133.1 \, N^{-0.0785} \]

For $10^4$ cycles to failure, \( S'_f = 133.1(10^4)^{-0.0785} = 64.6 \text{ kpsi} \)

(c) From Eq. (6–16), with \( \sigma_{\text{rev}} = 55 \text{ kpsi} \),

\[ N = \left( \frac{55}{133.1} \right)^{1/0.0785} = 77500 = 7.75(10^4) \text{ cycles} \]

Keep in mind that these are only estimates. So expressing the answers using three-place accuracy is a little misleading.
Endurance Limit Modifying Factors

- Endurance limit $S'_e$ is for carefully prepared and tested specimen
- If warranted, $S_e$ is obtained from testing of actual parts
- A set of *Marin factors* are used to adjust the endurance limit

$$S_e = k_a k_b k_c k_d k_e k_f S'_e$$

$k_a = $ surface condition modification factor  
$k_b = $ size modification factor  
$k_c = $ load modification factor  
$k_d = $ temperature modification factor  
$k_e = $ reliability factor$^{13}$  
$k_f = $ miscellaneous-effects modification factor  
$S'_e = $ rotary-beam test specimen endurance limit  
$S_e = $ endurance limit at the critical location of a machine part in the geometry and condition of use

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Surface Factor $k_a$

- Stresses tend to be high at the surface
- Surface finish has an impact on initiation of cracks at localized stress concentrations
- Higher strengths are more sensitive to rough surfaces.

$$k_a = a S_{ut}^b$$  \hspace{1cm} (6-19)

<table>
<thead>
<tr>
<th>Surface Finish</th>
<th>$S_{ut}$, kpsi</th>
<th>$S_{ut}$, MPa</th>
<th>Exponent $b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ground</td>
<td>1.34</td>
<td>1.58</td>
<td>-0.085</td>
</tr>
<tr>
<td>Machined or cold-drawn</td>
<td>2.70</td>
<td>4.51</td>
<td>-0.265</td>
</tr>
<tr>
<td>Hot-rolled</td>
<td>14.4</td>
<td>57.7</td>
<td>-0.718</td>
</tr>
<tr>
<td>As-forged</td>
<td>39.9</td>
<td>272.</td>
<td>-0.995</td>
</tr>
</tbody>
</table>

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Size Factor $k_b$

- Larger parts have greater surface area at high stress levels
- Likelihood of crack initiation is higher
- For bending & torsion loads, trend of the size factor data is given by

$$k_b = \begin{cases} 
(d/0.3)^{-0.107} = 0.879d^{-0.107} & 0.11 \leq d \leq 2 \text{ in} \\
0.91d^{-0.157} & 2 < d \leq 10 \text{ in} \\
(d/7.62)^{-0.107} = 1.24d^{-0.107} & 2.79 \leq d \leq 51 \text{ mm} \\
1.51d^{-0.157} & 51 < d \leq 254 \text{ mm} 
\end{cases} \quad (6-20)$$

- Applies only for round, rotating diameter
- For axial load, there is no size effect, so $k_b = 1$
Size Factor $k_b$

- For parts that are not round and rotating, an equivalent round rotating diameter is obtained.
- Equate the volume of material stressed at and above 95% of max stress to the same volume in the rotating-beam specimen.
- Lengths cancel, so equate areas.
- For a rotating round section, the 95% stress area is the area of a ring,
  \[ A_{0.95\sigma} = \frac{\pi}{4} [d^2 - (0.95d)^2] = 0.0766d^2 \]  (6–22)
- Equate 95% stress area for other conditions to Eq. (6–22) and solve for $d$ as the equivalent round rotating diameter.
- For non-rotating round,
  \[ A_{0.95\sigma} = 0.01046d^2 \]  (6–23)
- Equating to Eq. (6-22) and solving for equivalent diameter,
  \[ d_e = 0.370d \]  (6–24)
## Size Factor $k_b$

### Table 6–3

$A_{0.95\sigma}$ for common non-rotating structural shapes

<table>
<thead>
<tr>
<th>Shape</th>
<th>Axis</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Round</td>
<td>1-1</td>
<td>$A_{0.95\sigma} = 0.01046d^2$</td>
</tr>
<tr>
<td></td>
<td>2-2</td>
<td>$d_e = 0.370d$</td>
</tr>
<tr>
<td>Flange</td>
<td>1-1</td>
<td>$A_{0.95\sigma} = 0.10at_f$</td>
</tr>
<tr>
<td></td>
<td>2-2</td>
<td>$0.05ba$, $t_f &gt; 0.025a$</td>
</tr>
<tr>
<td>Beam</td>
<td>1-1</td>
<td>$A_{0.95\sigma} = 0.05ab$</td>
</tr>
<tr>
<td></td>
<td>2-2</td>
<td>$0.052xa + 0.1t_f(b - x)$</td>
</tr>
</tbody>
</table>

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September 28, 2013
Example 6-4

A steel shaft loaded in bending is 32 mm in diameter, abutting a filleted shoulder 38 mm in diameter. The shaft material has a mean ultimate tensile strength of 690 MPa. Estimate the Marin size factor $k_b$ if the shaft is used in

(a) A rotating mode.
(b) A nonrotating mode.

Solution

(a) From Eq. (6–20)

$$k_b = \left( \frac{d}{7.62} \right)^{-0.107} = \left( \frac{32}{7.62} \right)^{-0.107} = 0.858$$

Answer

(b) From Table 6–3,

$$d_e = 0.37d = 0.37(32) = 11.84 \text{ mm}$$

From Eq. (6–20),

Answer

$$k_b = \left( \frac{11.84}{7.62} \right)^{-0.107} = 0.954$$
Loading Factor $k_c$

- Accounts for changes in endurance limit for different types of fatigue loading.
- Only to be used for single load types. Use Combination Loading method (Sec. 6–14) when more than one load type is present.

$$k_c = \begin{cases} 
1 & \text{bending} \\
0.85 & \text{axial} \\
0.59 & \text{torsion}^{17}
\end{cases} \quad (6–26)$$
Temperature Factor $k_d$

- Endurance limit appears to maintain same relation to ultimate strength for elevated temperatures as at room temperature.
- This relation is summarized in Table 6–4.

### Table 6–4

<table>
<thead>
<tr>
<th>Temperature, °C</th>
<th>$S_T/S_{RT}$</th>
<th>Temperature, °F</th>
<th>$S_T/S_{RT}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>1.000</td>
<td>70</td>
<td>1.000</td>
</tr>
<tr>
<td>50</td>
<td>1.010</td>
<td>100</td>
<td>1.008</td>
</tr>
<tr>
<td>100</td>
<td>1.020</td>
<td>200</td>
<td>1.020</td>
</tr>
<tr>
<td>150</td>
<td>1.025</td>
<td>300</td>
<td>1.024</td>
</tr>
<tr>
<td>200</td>
<td>1.020</td>
<td>400</td>
<td>1.018</td>
</tr>
<tr>
<td>250</td>
<td>1.000</td>
<td>500</td>
<td>0.995</td>
</tr>
<tr>
<td>300</td>
<td>0.975</td>
<td>600</td>
<td>0.963</td>
</tr>
<tr>
<td>350</td>
<td>0.943</td>
<td>700</td>
<td>0.927</td>
</tr>
<tr>
<td>400</td>
<td>0.900</td>
<td>800</td>
<td>0.872</td>
</tr>
<tr>
<td>450</td>
<td>0.843</td>
<td>900</td>
<td>0.797</td>
</tr>
<tr>
<td>500</td>
<td>0.768</td>
<td>1000</td>
<td>0.698</td>
</tr>
<tr>
<td>550</td>
<td>0.672</td>
<td>1100</td>
<td>0.567</td>
</tr>
<tr>
<td>600</td>
<td>0.549</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Data source: Fig. 2–9.
**Temperature Factor** \( k_d \)

- If ultimate strength is known for operating temperature, then just use that strength. Let \( k_d = 1 \) and proceed as usual.

- If ultimate strength is known only at room temperature, then use Table 6–4 to estimate ultimate strength at operating temperature. With that strength, let \( k_d = 1 \) and proceed as usual.

- Alternatively, use ultimate strength at room temperature and apply temperature factor from Table 6–4 to the endurance limit.

\[
k_d = \frac{S_T}{S_{RT}} \quad (6-28)
\]

- A fourth-order polynomial curve fit of the underlying data of Table 6–4 can be used in place of the table, if desired.

\[
k_d = 0.975 + 0.432(10^{-3})T_F - 0.115(10^{-5})T_F^2
\]
\[
+ 0.104(10^{-8})T_F^3 - 0.595(10^{-12})T_F^4 \quad (6-27)
\]
Example 6-5

A 1035 steel has a tensile strength of 70 kpsi and is to be used for a part that sees 450°F in service. Estimate the Marin temperature modification factor and \((S_e)_{450°}\) if
(a) The room-temperature endurance limit by test is \((S'_{e})_{70°} = 39.0\) kpsi.
(b) Only the tensile strength at room temperature is known.

Solution

(a) First, from Eq. (6–27),

\[
k_d = 0.975 + 0.432(10^{-3})(450) - 0.115(10^{-5})(450^2)
+ 0.104(10^{-8})(450^3) - 0.595(10^{-12})(450^4) = 1.007
\]

Thus,

Answer

\[(S_e)_{450°} = k_d(S'_{e})_{70°} = 1.007(39.0) = 39.3\) kpsi
Example 6-5

(b) Interpolating from Table 6–4 gives

\[
(S_T/S_{RT})_{450^\circ} = 1.018 + (0.995 - 1.018) \frac{450 - 400}{500 - 400} = 1.007
\]

Thus, the tensile strength at 450°F is estimated as

\[
(S_{ut})_{450^\circ} = (S_T/S_{RT})_{450^\circ} (S_{ut})_{70^\circ} = 1.007(70) = 70.5 \text{ kpsi}
\]

From Eq. (6–8) then,

\[
(S_e)_{450^\circ} = 0.5 (S_{ut})_{450^\circ} = 0.5(70.5) = 35.2 \text{ kpsi}
\]

Part a gives the better estimate due to actual testing of the particular material.
Reliability Factor $k_e$

- From Fig. 6–17, $S'_e = 0.5 S_{ut}$ is typical of the data and represents 50% reliability.
- Reliability factor adjusts to other reliabilities.

A reliability of $R = 0.90$ means that there is a 90% chance that the part will perform its proper function without failure.
# Reliability Factor $k_e$

## Table 6–5

<table>
<thead>
<tr>
<th>Reliability, %</th>
<th>Transformation Variate $z_\alpha$</th>
<th>Reliability Factor $k_e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>0</td>
<td>1.000</td>
</tr>
<tr>
<td>90</td>
<td>1.288</td>
<td>0.897</td>
</tr>
<tr>
<td>95</td>
<td>1.645</td>
<td>0.868</td>
</tr>
<tr>
<td>99</td>
<td>2.326</td>
<td>0.814</td>
</tr>
<tr>
<td>99.9</td>
<td>3.091</td>
<td>0.753</td>
</tr>
<tr>
<td>99.99</td>
<td>3.719</td>
<td>0.702</td>
</tr>
<tr>
<td>99.999</td>
<td>4.265</td>
<td>0.659</td>
</tr>
<tr>
<td>99.9999</td>
<td>4.753</td>
<td>0.620</td>
</tr>
</tbody>
</table>

Dr. Mohammad Suliman Abuhaiba, PE
Miscellaneous-Effects Factor $k_f$

- Consider other possible factors.
  - Residual stresses
  - Directional characteristics from cold working
  - Case hardening
  - Corrosion
  - Surface conditioning, e.g. electrolytic plating and metal spraying
  - Cyclic Frequency
  - Frettage Corrosion

- Limited data is available.

- May require research or testing.
Stress Concentration and Notch Sensitivity

- For dynamic loading, stress concentration effects must be applied.
- Obtain $K_t$ as usual (Appendix A–15)
- For fatigue, some materials are not fully sensitive to $K_t$ so a reduced value can be used.
- Define $K_f$ as the fatigue stress-concentration factor.
- Define $q$ as notch sensitivity, ranging from 0 (not sensitive) to 1 (fully sensitive).

\[ q = \frac{K_f - 1}{K_t - 1} \quad \text{or} \quad q_{\text{shear}} = \frac{K_{fs} - 1}{K_{ts} - 1} \]  \hspace{0.5cm} (6–31)

- For $q = 0$, $K_f = 1$
- For $q = 1$, $K_f = K_t$
Notch Sensitivity

- Obtain $q$ for bending or axial loading from Fig. 6–20.
- Then get $K_f$ from Eq. (6–32):

$$K_f = 1 + q(K_t - 1)$$
Notch Sensitivity

- Obtain $q_s$ for torsional loading from Fig. 6–21.
- Then get $K_{fs}$ from Eq. (6–32): $K_{fs} = 1 + q_s(K_{ts} - 1)$
Notch Sensitivity

Alternatively, can use curve fit equations for Figs. 6–20 and 6–21 to get notch sensitivity, or go directly to $K_f$.

$$q = \frac{1}{1 + \frac{\sqrt{a}}{\sqrt{r}}} \quad (6-34)$$

$$K_f = 1 + \frac{K_t - 1}{1 + \sqrt{a/r}} \quad (6-33)$$

Bending or axial:

$$\sqrt{a} = 0.246 - 3.08(10^{-3})S_{ut} + 1.51(10^{-5})S_{ut}^2 - 2.67(10^{-8})S_{ut}^3 \quad (6-35a)$$

Torsion:

$$\sqrt{a} = 0.190 - 2.51(10^{-3})S_{ut} + 1.35(10^{-5})S_{ut}^2 - 2.67(10^{-8})S_{ut}^3 \quad (6-35b)$$
Notch Sensitivity for Cast Irons

- Cast irons are already full of discontinuities, which are included in the strengths.
- Additional notches do not add much additional harm.
- Recommended to use $q = 0.2$ for cast irons.

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Example 6-6

A steel shaft in bending has an ultimate strength of 690 MPa and a shoulder with a fillet radius of 3 mm connecting a 32-mm diameter with a 38-mm diameter. Estimate $K_f$ using:
(a) Figure 6–20.
(b) Equations (6–33) and (6–35).

Solution

From Fig. A–15–9, using $D/d = 38/32 = 1.1875$, $r/d = 3/32 = 0.09375$, we read the graph to find $K_t \doteq 1.65$.

(a) From Fig. 6–20, for $S_{ut} = 690$ MPa and $r = 3$ mm, $q \doteq 0.84$. Thus, from Eq. (6–32)

$$K_f = 1 + q(K_t - 1) \doteq 1 + 0.84(1.65 - 1) = 1.55$$

(b) From Eq. (6–35a) with $S_{ut} = 690$ MPa = 100 kpsi, $\sqrt{a} = 0.0622\sqrt{\text{in}} = 0.313\sqrt{\text{mm}}$. Substituting this into Eq. (6–33) with $r = 3$ mm gives

$$K_f = 1 + \frac{K_t - 1}{1 + \sqrt{a/r}} \doteq 1 + \frac{1.65 - 1}{1 + \frac{0.313}{\sqrt{3}}} = 1.55$$

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Application of Fatigue Stress Concentration Factor

- Use $K_f$ as a multiplier to increase the nominal stress.
- Some designers sometimes applied $1/ K_f$ as a Marin factor to reduce $S_e$.
- For infinite life, either method is equivalent, since

$$ n_f = \frac{S_e}{K_f \sigma} = \left( \frac{1}{K_f} \right) S_e $$

- For finite life, increasing stress is more conservative. Decreasing $S_e$ applies more to high cycle than low cycle.
Example 6-8

A 1015 hot-rolled steel bar has been machined to a diameter of 1 in. It is to be placed in reversed axial loading for 70,000 cycles to failure in an operating environment of 550°F. Using ASTM minimum properties, and a reliability of 99 percent, estimate the endurance limit and fatigue strength at 70,000 cycles.

Solution

From Table A–20, $S_{ut} = 50$ kpsi at 70°F. Since the rotating-beam specimen endurance limit is not known at room temperature, we determine the ultimate strength at the elevated temperature first, using Table 6–4. From Table 6–4,

$$\left( \frac{S_T}{S_{RT}} \right)_{550°F} = \frac{0.995 + 0.963}{2} = 0.979$$

The ultimate strength at 550°F is then

$$(S_{ut})_{550°F} = \left( \frac{S_T}{S_{RT}} \right)_{550°F} (S_{ut})_{70°F} = 0.979(50) = 49.0$ kpsi

The rotating-beam specimen endurance limit at 550°F is then estimated from Eq. (6–8) as

$$S'_e = 0.5(49) = 24.5$$ kpsi
Example 6-8

Next, we determine the Marin factors. For the machined surface, Eq. (6–19) with Table 6–2 gives

\[ k_a = a S_{ut}^b = 2.70(49^{-0.265}) = 0.963 \]

For axial loading, from Eq. (6–21), the size factor \( k_b = 1 \), and from Eq. (6–26) the loading factor is \( k_c = 0.85 \). The temperature factor \( k_d = 1 \), since we accounted for the temperature in modifying the ultimate strength and consequently the endurance limit. For 99 percent reliability, from Table 6–5, \( k_e = 0.814 \). Finally, since no other conditions were given, the miscellaneous factor is \( k_f = 1 \). The endurance limit for the part is estimated by Eq. (6–18) as

\[
S_e = k_a k_b k_c k_d k_e k_f S_e' \\
= 0.963(1)(0.85)(1)(0.814)(1)24.5 = 16.3 \text{ kpsi}
\]
Example 6-8

For the fatigue strength at 70,000 cycles we need to construct the S-N equation. From p. 285, since $S_{ut} = 49 < 70 \text{ kpsi}$, then $f = 0.9$. From Eq. (6–14)

$$a = \frac{(f S_{ut})^2}{S_e} = \frac{[0.9(49)]^2}{16.3} = 119.3 \text{ kpsi}$$

and Eq. (6–15)

$$b = -\frac{1}{3} \log \left( \frac{f S_{ut}}{S_e} \right) = -\frac{1}{3} \log \left[ \frac{0.9(49)}{16.3} \right] = -0.1441$$

Finally, for the fatigue strength at 70,000 cycles, Eq. (6–13) gives

$$S_f = a \ N^b = 119.3(70,000)^{-0.1441} = 23.9 \text{ kpsi}$$
Fluctuating Stresses

- Define $\sigma_m$ as midrange steady component of stress (mean stress) and $\sigma_a$ as amplitude of alternating component of stress.

\[
\sigma_m = \frac{\sigma_{\text{max}} + \sigma_{\text{min}}}{2}
\]

\[
\sigma_a = \left| \frac{\sigma_{\text{max}} - \sigma_{\text{min}}}{2} \right|
\]

stress ratio \quad R = \frac{\sigma_{\text{min}}}{\sigma_{\text{max}}}

amplitude ratio \quad A = \frac{\sigma_a}{\sigma_m}

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Application of $K_f$ for Fluctuating Stresses

- For fluctuating loads at points with stress concentration, the best approach is to design to avoid all localized plastic strain.
- In this case, $K_f$ should be applied to both alternating and midrange stress components.
- When localized strain does occur, some methods (nominal mean stress method and residual stress method) recommend only applying $K_f$ to the alternating stress.
- Dowling method recommends applying $K_f$ to alternating stress and $K_{fm}$ to mid-range stress, where $K_{fm}$ is

\[
K_{fm} = K_f \\
K_{fm} = \frac{S_y - K_f \sigma_{ao}}{|\sigma_{mo}|} \\
K_{fm} = 0
\]

\[
K_f |\sigma_{max,o}| < S_y \\
K_f |\sigma_{max,o}| > S_y \\
K_f |\sigma_{max,o} - \sigma_{min,o}| > 2S_y
\]
Modified Goodman Diagram

- Midrange stress is plotted on abscissa
- All other components of stress are plotted on the ordinate
Master Fatigue Diagram

Displays four stress components as well as two stress ratios

Fig. 6–26
Plot of Alternating vs Midrange Stress

- Modified Goodman line from $S_e$ to $S_{ut}$ is one simple representation of the limiting boundary for infinite life.
Plot of Alternating vs Midrange Stress

- Experimental data on normalized plot of $\sigma_a$ vs $\sigma_m$
- Demonstrates little effect of negative midrange stress

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Commonly Used Failure Criteria

- Gerber passes through the data
- ASME-elliptic passes through data and incorporates rough yielding check

- Modified Goodman is linear, so simple to use for design. It is more conservative than Gerber.
- Soderberg provides a very conservative single check of both fatigue and yielding.

- Langer line represents standard yield check.
- It is equivalent to comparing max stress to yield strength.

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Equations for Commonly Used Failure Criteria

- Intersecting a constant slope load line with each failure criteria produces design equations.
- $n$ is the design factor or factor of safety for infinite fatigue life.

Soderberg
\[
\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_y} = \frac{1}{n} \tag{6-45}
\]

mod-Goodman
\[
\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_{ut}} = \frac{1}{n} \tag{6-46}
\]

Gerber
\[
\frac{n\sigma_a}{S_e} + \left( \frac{n\sigma_m}{S_{ut}} \right)^2 = 1 \tag{6-47}
\]

ASME-elliptic
\[
\left( \frac{n\sigma_a}{S_e} \right)^2 + \left( \frac{n\sigma_m}{S_y} \right)^2 = 1 \tag{6-48}
\]
Summarizing Tables for Failure Criteria

- Tables 6–6 to 6–8 summarize the pertinent equations for Modified Goodman, Gerber, ASME-elliptic, and Langer failure criteria.
- The first row gives fatigue criterion.
- The second row gives yield criterion.
- The third row gives the intersection of static and fatigue criteria.
- The fourth row gives the equation for fatigue factor of safety.
- The first column gives the intersecting equations.
- The second column gives the coordinates of the intersection.

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### Summarizing Table for Modified Goodman

**Table 6-6**

<table>
<thead>
<tr>
<th>Intersection Equations</th>
<th>Intersection Coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{S_a}{S_e} + \frac{S_m}{S_{ut}} = 1 )</td>
<td>( S_a = \frac{r S_e S_{ut}}{r S_{ut} + S_e} )</td>
</tr>
<tr>
<td>( \frac{S_a}{S_y} + \frac{S_m}{S_y} = 1 )</td>
<td>( S_a = \frac{r S_y}{1 + r} )</td>
</tr>
<tr>
<td>( \frac{S_a}{S_e} + \frac{S_m}{S_{ut}} = 1 )</td>
<td>( S_m = \frac{(S_y - S_e) S_{ut}}{S_{ut} - S_e} )</td>
</tr>
<tr>
<td>( \frac{S_a}{S_y} + \frac{S_m}{S_y} = 1 )</td>
<td>( S_a = S_y - S_m, r_{crit} = S_a / S_m )</td>
</tr>
</tbody>
</table>

Fatigue factor of safety

\[
n_f = \frac{1}{\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_{ut}}}\]

---

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## Summarizing Table for Gerber

### Table 6–7

<table>
<thead>
<tr>
<th>Intersecting Equations</th>
<th>Intersection Coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{S_a}{S_e} + \left( \frac{S_m}{S_{ut}} \right)^2 = 1$</td>
<td>$S_a = \frac{r^2 S_{ut}^2}{2 S_e} \left[ -1 + \sqrt{1 + \left( \frac{2 S_e}{r S_{ut}} \right)^2} \right]$</td>
</tr>
<tr>
<td>Load line $r = \frac{S_a}{S_m}$</td>
<td>$S_m = \frac{S_a}{r}$</td>
</tr>
<tr>
<td>$\frac{S_a}{S_y} + \frac{S_m}{S_y} = 1$</td>
<td>$S_a = \frac{r S_y}{1 + r}$</td>
</tr>
<tr>
<td>Load line $r = \frac{S_a}{S_m}$</td>
<td>$S_m = \frac{S_y}{1 + r}$</td>
</tr>
<tr>
<td>$\frac{S_a}{S_e} + \left( \frac{S_m}{S_{ut}} \right)^2 = 1$</td>
<td>$S_m = \frac{S_{ut}^2}{2 S_e} \left[ 1 - \sqrt{1 + \left( \frac{2 S_e}{S_{ut}} \right)^2 \left( 1 - \frac{S_y}{S_e} \right)} \right]$</td>
</tr>
<tr>
<td>$\frac{S_a}{S_y} + \frac{S_m}{S_y} = 1$</td>
<td>$S_a = S_y - S_m$, $r_{crit} = \frac{S_a}{S_m}$</td>
</tr>
</tbody>
</table>

**Fatigue factor of safety**

$$n_f = \frac{1}{2} \left( \frac{S_{ut}}{\sigma_m} \right)^2 \frac{\sigma_a}{S_e} \left[ -1 + \sqrt{1 + \left( \frac{2 \sigma_m S_e}{S_{ut} \sigma_a} \right)^2} \right]$$  \quad $\sigma_m > 0$
### Summarizing Table for ASME-Elliptic

#### Table 6–8

<table>
<thead>
<tr>
<th>Intersecting Equations</th>
<th>Intersection Coordinates</th>
</tr>
</thead>
</table>
| \[
\left( \frac{S_a}{S_e} \right)^2 + \left( \frac{S_m}{S_y} \right)^2 = 1
\] | \[
S_a = \sqrt{\frac{r^2 S_e^2 S_y^2}{S_e^2 + r^2 S_y^2}}
\]
| Load line \( r = \frac{S_a}{S_m} \) | \[
S_m = \frac{S_a}{r}
\] |
| \[
\frac{S_a}{S_y} + \frac{S_m}{S_y} = 1
\] | \[
S_a = \frac{r S_y}{1 + r}
\]
| Load line \( r = \frac{S_a}{S_m} \) | \[
S_m = \frac{S_y}{1 + r}
\] |
| \[
\left( \frac{S_a}{S_e} \right)^2 + \left( \frac{S_m}{S_y} \right)^2 = 1
\] | \[
S_a = 0, \quad \frac{2 S_y S_e^2}{S_e^2 + S_y^2}
\]
| \[
\frac{S_a}{S_y} + \frac{S_m}{S_y} = 1
\] | \[
S_m = S_y - S_a, \quad r_{\text{crit}} = S_a / S_m
\] |

Fatigue factor of safety

\[
n_f = \sqrt{\frac{1}{\left( \sigma_a / S_e \right)^2 + \left( \sigma_m / S_y \right)^2}}
\]

---

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**Example 6-10**

A 1.5-in-diameter bar has been machined from an AISI 1050 cold-drawn bar. This part is to withstand a fluctuating tensile load varying from 0 to 16 kip. Because of the ends, and the fillet radius, a fatigue stress-concentration factor $K_f$ is 1.85 for $10^6$ or larger life. Find $S_a$ and $S_m$ and the factor of safety guarding against fatigue and first-cycle yielding, using (a) the Gerber fatigue line and (b) the ASME-elliptic fatigue line.

**Solution**

We begin with some preliminaries. From Table A–20, $S_{ut} = 100$ kpsi and $S_y = 84$ kpsi. Note that $F_a = F_m = 8$ kip. The Marin factors are, deterministically,

$$k_a = 2.70(100)^{-0.265} = 0.797: \text{Eq. (6–19), Table 6–2, p. 288}$$

$$k_b = 1 \text{ (axial loading, see } k_c)$$

$$k_c = 0.85: \text{Eq. (6–26), p. 290}$$

$$k_d = k_e = k_f = 1$$

$$S_e = 0.797(1)0.850(1)(1)(1)0.5(100) = 33.9 \text{ kpsi: Eqs. (6–8), (6–18), p. 282, p. 287}$$

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Example 6-10

The nominal axial stress components $\sigma_{ao}$ and $\sigma_{mo}$ are

$$\sigma_{ao} = \frac{4F_a}{\pi d^2} = \frac{4(8)}{\pi 1.5^2} = 4.53 \text{ kpsi} \quad \sigma_{mo} = \frac{4F_m}{\pi d^2} = \frac{4(8)}{\pi 1.5^2} = 4.53 \text{ kpsi}$$

Applying $K_f$ to both components $\sigma_{ao}$ and $\sigma_{mo}$ constitutes a prescription of no notch yielding:

$$\sigma_a = K_f \sigma_{ao} = 1.85(4.53) = 8.38 \text{ kpsi} = \sigma_m$$

(a) Let us calculate the factors of safety first. From the bottom panel from Table 6–7 the factor of safety for fatigue is

$$n_f = \frac{1}{2} \left( \frac{100}{8.38} \right)^2 \left( \frac{8.38}{33.9} \right) \left\{ -1 + \sqrt{1 + \left[ \frac{2(8.38)33.9}{100(8.38)} \right]^2} \right\} = 3.66$$

From Eq. (6–49) the factor of safety guarding against first-cycle yield is

$$n_y = \frac{S_y}{\sigma_a + \sigma_m} = \frac{84}{8.38 + 8.38} = 5.01$$
Example 6-10

Thus, we see that fatigue will occur first and the factor of safety is 3.68. This can be seen in Fig. 6–28 where the load line intersects the Gerber fatigue curve first at point B. If the plots are created to true scale it would be seen that $n_f = OB/OA$. 

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Example 6-10

From the first panel of Table 6–7, $r = \sigma_a / \sigma_m = 1$,

$$S_a = \frac{(1)^2100^2}{2(33.9)} \left[ -1 + \sqrt{1 + \left[ \frac{2(33.9)}{(1)100} \right]^2} \right] = 30.7 \text{ kpsi}$$

$$S_m = \frac{S_a}{r} = \frac{30.7}{1} = 30.7 \text{ kpsi}$$

As a check on the previous result, $n_f = OB/OA = S_a/\sigma_a = S_m/\sigma_m = 30.7/8.38 = 3.66$ and we see total agreement.

Fig. 6–28
Example 6-10

We could have detected that fatigue failure would occur first without drawing Fig. 6–28 by calculating $r_{crit}$. From the third row third column panel of Table 6–7, the intersection point between fatigue and first-cycle yield is

$$S_m = \frac{100^2}{2(33.9)} \left[1 - \sqrt{1 + \left(\frac{2(33.9)}{100}\right)^2 \left(1 - \frac{84}{33.9}\right)}\right] = 64.0 \text{ kpsi}$$

$$S_a = S_y - S_m = 84 - 64 = 20 \text{ kpsi}$$

The critical slope is thus

$$r_{crit} = \frac{S_a}{S_m} = \frac{20}{64} = 0.312$$

which is less than the actual load line of $r = 1$. This indicates that fatigue occurs before first-cycle-yield.

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Example 6-10

(b) Repeating the same procedure for the ASME-elliptic line, for fatigue

\[ n_f = \sqrt{\frac{1}{(8.38/33.9)^2 + (8.38/84)^2}} = 3.75 \]

Again, this is less than \( n_y = 5.01 \) and fatigue is predicted to occur first. From the first row second column panel of Table 6–8, with \( r = 1 \), we obtain the coordinates \( S_a \) and \( S_m \) of point \( B \) in Fig. 6–29 as

\[ S_a = \sqrt{\frac{(1)^233.9^2(84)^2}{33.9^2 + (1)^284^2}} = 31.4 \text{ kpsi,} \]

\[ S_m = \frac{S_a}{r} = \frac{31.4}{1} = 31.4 \text{ kpsi} \]

To verify the fatigue factor of safety,

\[ n_f = \frac{S_a}{\sigma_a} = \frac{31.4}{8.38} = 3.75. \]
Example 6-10

As before, let us calculate $r_{\text{crit}}$. From the third row second column panel of Table 6–8.

$$S_a = \frac{2(84)33.9^2}{33.9^2 + 84^2} = 23.5 \text{ kpsi}, \quad S_m = S_y - S_a = 84 - 23.5 = 60.5 \text{ kpsi}$$

$$r_{\text{crit}} = \frac{S_a}{S_m} = \frac{23.5}{60.5} = 0.388$$

which again is less than $r = 1$, verifying that fatigue occurs first with $n_f = 3.75$.

The Gerber and the ASME-elliptic fatigue failure criteria are very close to each other and are used interchangeably. The ANSI/ASME Standard B106.1M–1985 uses ASME-elliptic for shafting.
Fatigue Criteria for Brittle Materials

- For many brittle materials, the first quadrant fatigue failure criteria follows a concave upward Smith-Dolan locus,
  \[
  \frac{S_a}{S_e} = \frac{1 - S_m/S_{ut}}{1 + S_m/S_{ut}}
  \]  
  \(6-50\)

- Or as a design equation,
  \[
  \frac{n\sigma_a}{S_e} = \frac{1 - n\sigma_m/S_{ut}}{1 + n\sigma_m/S_{ut}}
  \]  
  \(6-51\)

- For a radial load line of slope \(r\), the intersection point is
  \[
  S_a = \frac{rS_{ut} + S_e}{2} \left[-1 + \sqrt{1 + \frac{4rS_{ut}S_e}{(rS_{ut} + S_e)^2}}\right]
  \]  
  \(6-52\)

- In the second quadrant,
  \[
  S_a = S_e + \left(\frac{S_e}{S_{ut}} - 1\right)S_m \quad -S_{ut} \leq S_m \leq 0 \quad \text{(for cast iron)}
  \]  
  \(6-53\)
Torsional Fatigue Strength

- Testing has found that steady-stress component has no effect on endurance limit for torsional loading if the material is ductile, polished, notch-free, and cylindrical.
- However, for less than perfect surfaces, the modified Goodman line is more reasonable.
- For pure torsion cases, use $k_c = 0.59$ to convert normal endurance strength to shear endurance strength.
- For shear ultimate strength, recommended to use

$$S_{SU} = 0.67 S_{UT} \quad (6-54)$$

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Combinations of Loading Modes

- When more than one type of loading (bending, axial, torsion) exists, use the Distortion Energy theory to combine them.
- Obtain von Mises stresses for both midrange and alternating components.
- Apply appropriate $K_f$ to each type of stress.
- For load factor, use $k_c = 1$. The torsional load factor ($k_c = 0.59$) is inherently included in the von Mises equations.
- If needed, axial load factor can be divided into the axial stress.

\[
\sigma'_a = \left\{ \left[ (K_f)_{\text{bending}} (\sigma_a)_{\text{bending}} + (K_f)_{\text{axial}} \frac{(\sigma_a)_{\text{axial}}}{0.85} \right]^2 + 3 \left[ (K_f s)_{\text{tension}} (\tau_a)_{\text{tension}} \right]^2 \right\}^{1/2}
\]  

\[\sigma'_m = \left\{ \left[ (K_f)_{\text{bending}} (\sigma_m)_{\text{bending}} + (K_f)_{\text{axial}} (\sigma_m)_{\text{axial}} \right]^2 + 3 \left[ (K_f s)_{\text{tension}} (\tau_m)_{\text{tension}} \right]^2 \right\}^{1/2}\]

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Static Check for Combination Loading

- Distortion Energy theory still applies for check of static yielding
- Obtain von Mises stress for maximum stresses (sum of midrange and alternating)
- Stress concentration factors are not necessary to check for yielding at first cycle

\[ \sigma'_{\text{max}} = \left[ (\sigma_a + \sigma_m)^2 + 3(\tau_a + \tau_m)^2 \right]^{1/2} \]

\[ n_y = \frac{S_y}{\sigma'_{\text{max}}} \]

- Alternate simple check is to obtain conservative estimate of \( \sigma'_{\text{max}} \) by summing \( \sigma'_a \) and \( \sigma'_m \)

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