Design Via Frequency Response

This chapter covers the design of a feedback control system using frequency response techniques.

We will consider 4 cases:

1- Design by gain adjustment.
2- Design of a lead compensator.
3- Design of lag compensator.
4- Design of a lead-lag compensator.

Some Notes needed for the design:

1- Phase margin is related to OS%.
2- Bandwidth is related to both $\zeta$ (damping ratio) and Ts (settling time) or Tp (peak time).

\[
\Phi_M = \tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^4}}}
\]

\[
\omega_{BW} = \frac{\pi}{T_p \sqrt{1 - \zeta^2}} \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}
\]

3- We need to reshape the open loop transfer function’s frequency response to meet both the $\Phi_M$ requirements (OS%) & BW requirements (TS & Tp).
4- An open loop stable system is stable in closed-loop if the open loop magnitude frequency response has a gain of less than 0db at the frequency where the phase frequency response is 180°.

5- OS% is reduced by increasing the phase margin.

6- Speed of the response is increased by increasing the bandwidth.

7- Steady-state error is improved by increasing the low frequency magnitude response.

1- Design by Gain Adjustment:

Example (1)

For a unity feedback system with a forward transfer function

\[ G(s) = \frac{K}{s(s + 50)(s + 120)} \]

Use frequency response techniques to find the value of gain K, to yield a closed-loop step response with 20% overshoot.

```plaintext
n=[1];
d=[1,170,6000,0];
sys=tf(n,d);
sys=tf(n,d);
bode(sys)
```
In order to start at magnitude 0 db, we need to increase the magnitude plot by 75.6 db

20 log k = 75.6

K=6025.6

We need to start with K=6025.6

n=[6025.6];
d=[1,170,6000,0];
sys=tf(n,d);
bode(sys)
from the following equations we find damping ratio and phase margin.

\[
\%OS = e^{-\left(\frac{\zeta \pi}{\sqrt{1-\zeta^2}}\right)} \times 100, \quad \Phi_M = \tan^{-1}\left(\frac{2\zeta}{\sqrt{-2r^2 + \sqrt{1 + 4r^4}}}ight)
\]

\[\zeta = 0.456\]
\[\Phi_M = 48.152^\circ\]
\[180-\theta = 48.152\]
\[\Theta = -131.8^\circ\]
At phase -131.8° the magnitude is -30db with a frequency 27.3 rad/sec. we need to adjust the gain to force the magnitude curve to go through 0 db at this frequency. So the additional gain is $20\log k = 30$db so $k = 31.62$. However we started with a gain $k = 6025.2$ so the overall gain is $K = 31.62 \times 6025.2 = 190546.2$.

So the gain–adjusted open-loop transfer function is

$$G(s) = \frac{190546.2}{s(s + 50)(s + 120)}$$

The bode plot of the new system is
To insure that our design meets the required specifications we shall draw the step response of the closed loop system.

\[
\text{Sysc} = \text{feedback(sys,1)} \\
\text{step(sysc)}
\]

As you can see the overshooting is as required so our design is correct.

2-Design of a lead compensator:

Lead compensators are used to improve transient response of a system. In designing lead compensators via Bode plots, we want to change the phase diagram, increasing the phase margin to reduce the percent overshoot, and increasing the gain crossover to realize a faster transient response. The lead compensator is given by

\[
G_c(s) = \frac{1}{\beta} \frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}}
\]

Where \(\beta < 1\).
The new gain cross over frequency is given by

$$\omega_{max} = \frac{1}{T \sqrt{\beta}}$$

At this frequency the magnitude of the lead compensator will be

$$|Gc(j\omega_{max})| = \frac{1}{\sqrt{\beta}}$$

And the maximum phase shift of the compensator will be

$$\phi_{max} = \sin^{-1} \frac{1-\beta}{1+\beta}$$

**Example (2)**

Design a lead compensator for the system in example one to meet the following specifications: OS%= 20%, Ts=0.2 s, $K_p = 50$

1) First of all we need to find the closed loop bandwidth to meet the transient response requirement (OS%=20% & Ts=.2 s)

For 20% OS%

$$\omega_{BW} = \frac{4}{T \zeta \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}}$$

$$\omega_{BW} = 57.89 \text{ rad/sec}$$

2) We need to find the value of $K$ for the uncompensated system so that this value will satisfy the steady state error requirements.

$$K_p = \lim_{s \to 0} s G(s) \quad k=300000$$

3) Draw the bode plot for the uncompensated system

```matlab
% Uncompensated system
n=[300000];
d=[1,170,6000,0];
sys=tf(n,d);
bode(sys)
margin(sys)
```
From the plot we find that the phase margin of the uncompensated system will be 35.2° and the required phase margin is 48.15. So
The maximum phase shift of the compensator is $\phi_{max} = 48.15° - 35.2° + 10° = 23°$
Where 10° is a correction factor.

And

$$\beta = \frac{1 - \sin \phi_{max}}{1 + \sin \phi_{max}}$$

$\beta = .438$

The compensator magnitude is: Mag $= \frac{1}{\sqrt{\beta}} = 1.509$

Compensator magnitude = 20 log 1.509 = 3.575 db

At the gain crossover frequency the magnitude of the compensator is 3.575 db, however the magnitude of the compensated system should be 0 db at this point so the magnitude of the uncompensated system at this point should be -3.575 db. We find that the gain crossover frequency is $\omega_{max} = 49.4$ rad/sec).

Now we will find the zero and pole of the compensator. Note that the compensator should have unity gain in order to keep the steady state requirements as required.
So the compensator will be

\[ G_c(s) = \frac{1}{T \sqrt{\beta}} \left( s + \frac{1}{T} \right) \left( s + \frac{1}{\beta T} \right) \]

So the compensator will be

\[ G_c(s) = 2.27 \frac{s+32.73}{s+74.55} \]

Now we will draw the bode plot of the compensated system

```matlab
nc=[1,32.73];
dc=[1,74.55];
compensator=2.27*tf(nc,dc)

Transfer function:
2.27s + 74.3
----------------
s + 74.55

systot=sys*compensator
bode(systot)
margin(systot)
```
Note at -7 db the frequency is 85.6 rad/sec which is greater than the required bandwidth so we expect our design to be correct.
Finally we will plot the step response of the closed-loop system and make sure our design is correct.
3-Design of cascade lag compensator

Lag compensators are for the design of steady-state error without affecting transient response. Lag compensation changes the magnitude curve to go through 0 db at the desired phase margin. So the lag compensator will reduce the gain cross over frequency. The compensator is given by

\[ G_c(s) = \frac{1}{\alpha} \frac{s + \frac{1}{\alpha T}}{s + \frac{1}{\alpha T}} \]

Where \( \alpha > 1 \).

Example (3)

Design a lag compensator for the system in example one that will improve the steady-state error tenfold, will still operating with 20% overshoot.

\[ G(s) = \frac{190546.2}{s(s + 50)(s + 120)} \]

\[ K_V = \lim_{s \to 0} s G(s) \]

\[ K_{V,old} = \frac{190546.2}{50 \times 120} = 31.7577 \]

\[ K_{V,new} = 10 \times K_{V,old} = 317.577 \]

1) Find the gain \( K \) to meet the new steady-state requirement \( K_V = 317.577 \)

\[ K_V = \lim_{s \to 0} s G(s) = K \frac{190546.2}{50 \times 120} = 317.577 \]

\( K = 1905462 \)

So the system will be

\[ G(s) = \frac{1905462}{s(s + 50)(s + 120)} \]

2) Draw the bode plot of the uncompensated system

```matlab
n=[1905462];
d=[1,170,6000,0];
sys=tf(n,d);
bode(sys)
margin(sys)
```
The phase margin of the uncompensated system is \(-15.2^\circ\) for 20% overshoot \(z = 0.456\) and \(\phi_M = 48.152^\circ\). We add a 10° correction factor, so the required phase margin is \(\phi_M = 48.152^\circ + 10^\circ = 58.152^\circ\). So, \(180 - \theta = 58.152^\circ\). 

\[\theta = -121.848^\circ\]
From the bode plot we see that at theta= -122 (desired phase margin) the frequency (gain crossover frequency) is 20.4 rad/sec and the magnitude is 23 db. However at this frequency the magnitude should be 0 db. So we need a lag compensator to have a magnitude of -23db at this frequency.

The attenuation can be found as follow

\[ 20 \log \alpha = 23 \text{ db} \]

\[ \alpha = 14.125 \]

now we choose the zero of the compensator to be a decade below the gain crossover frequency (20.4 rad/sec).

so the compensator will be

\[ Gc(s) = \frac{1}{\alpha} \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}} \]

\[ Gc(s) = .071 \frac{s + 2.06}{s + .145} \]

The bode plot of the compensated system will be
And the closed loop step response is
Exercises:

1) Use frequency response methods to determine the value of gain $K$, to yield a step response with a 20% overshoot if

$$G(s) = \frac{K}{s(s+6)(s+12)}$$

2) Design a lag compensator so that the system with unity feedback where

$$G(s) = \frac{K}{(s+2)(s+6)(s+8)}$$

Operates with a 45° phase margin and static error constant of 100.

3) Problem 14 in text book.