Experiment 7: Fourier Series

Theory

A Fourier series is an infinite sum of harmonic functions (sines and cosines) with every term in the series having a frequency which is an integral multiple of some “principal” frequency and an amplitude that varies inversely with its frequency. The usefulness of such series is that any periodic function $f$ with period $T$ can be written as a Fourier series in the following way:

$$f(t) = a_0 + \sum_{k=1}^{\infty} \left[ a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right]$$

where $\omega_0$ is the fundamental frequency of the function; that is,

$$\omega_0 = \frac{2\pi}{T}$$

Where $T$ is the fundamental period of the function.

The Fourier coefficients, the $a$’s and $b$’s in the equation, may be computed by the following set of integrals:

$$a_0 = \frac{1}{T} \int_0^T f(t) dt$$

$$a_k = \frac{2}{T} \int_0^T f(t) \cos(k\omega_0 t) dt$$

$$b_k = \frac{2}{T} \int_0^T f(t) \sin(k\omega_0 t) dt$$

Now that we have a way to represent the function, we can use the Fourier series just as if it were the function itself and investigate its behavior in electronic systems.
Why sine and cosine ????!

**Vectors**

We are now going to discuss some formalism of three-dimensional vectors expressed in Cartesian coordinates, for the purpose of making comparisons to Fourier series. In Cartesian space, any vector can be written as a linear combination of the mutually perpendicular basis vectors \( \hat{x}, \hat{y}, \hat{z} \) in the following way:

\[
\vec{V} = a_x \hat{x} + a_y \hat{y} + a_z \hat{z}
\]

where the coefficients are given by

\[
a_x = \vec{V} \cdot \hat{x}, \quad a_y = \vec{V} \cdot \hat{y}, \quad a_z = \vec{V} \cdot \hat{z}
\]

The above expressions for the coefficients can be easily derived from the following perpendicularity (or, more generally, orthogonality) relations:

\[
\hat{x} \cdot \hat{y} = \hat{y} \cdot \hat{z} = \hat{z} \cdot \hat{x} = 0
\]
\[
\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1
\]

Any vector can be decomposed into a set of appropriately weighted orthonormal basis vectors

**Example:**

\[
\hat{r} = a_x \hat{x} + a_y \hat{y}
\]

\[
a_x = ??, \quad a_y = ??
\]
Perfume inner product with the basis vectors

\[ \hat{r} = a_x \hat{x} + a_y \hat{y} \]

\[ a_x = \hat{r} \cdot \hat{x}, \quad a_y = \hat{r} \cdot \hat{y} \]

\[ a_x = 1.3 \cdot 1 + 0.75 \cdot 0 = 1.3 \]

\[ a_y = 1.3 \cdot 0 + 0.75 \cdot 1 = 0.75 \]

The orthogonality conditions simply express that the basis vectors \( \hat{x}, \hat{y}, \hat{z} \) are linearly independent. Although this section about vectors is elementary and may appear unnecessary, we will see presently (and you may have figured it out by now) there are similarities between this vector formalism and that of the Fourier series.

**Comparison between Vectors and Fourier Series**

The two previous sections were written suggestively, to make comparisons between the formalism for the Fourier series and for vectors. The similarities between the two can provide us with some insight about Fourier series (for those with knowledge of linear algebra, these similarities arise since we can create inner product sources for both three-dimension a vectors and for periodic functions of a given period). It's should be clear that the harmonic functions making up a periodic function are analogous to the unit vectors making up a vector, and the coefficients \( a_n, b_n \) in a Fourier series are analogous to the components \( V_i \) of a vector:

\[ \cos, \sin \leftrightarrow \hat{x}, \hat{y}, \hat{z} \]

\[ a_n, b_n \leftrightarrow a_x, a_y, a_z \]

So, the harmonic functions are the elements that go into making a certain periodic function (they will be the same for all functions with the same period), and the coefficients are the amount of each harmonic we need to make the particular function. This way of thinking about Fourier series is extremely powerful and will serve you well if you learn it now. So, if you have any doubts that you fully understand the idea, reread the previous section and talk about it with others until you do understand.
Compare Fourier Series to vector decomposition:

**Vector Decomp.**

\[ \hat{r} = a_x \hat{x} + a_y \hat{y} \]

\[ a_x = \hat{r} \cdot \hat{x}, \]

\[ a_y = \hat{r} \cdot \hat{y} \]

**Fourier Series**

\[ x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n \omega_0 t) + b_n \sin(n \omega_0 t) \]

\[ a_0 = \frac{1}{T} \int_{t_o}^{t_o+T} x(t) dt, \]

\[ a_n = \frac{2}{T} \int_{t_o}^{t_o+T} x(t) \cos(n \omega_0 t) dt, \]

\[ b_n = \frac{2}{T} \int_{t_o}^{t_o+T} x(t) \sin(n \omega_0 t) dt \]
1:

\[ a_n = \frac{2}{n\pi} \sin \left( \frac{n\pi}{2} \right) \]

\[ b_n = 0 \]

\[ w_0 = 1 \]

\[ a_0 = .5 \]

\[ x(t) = .5 + \frac{2}{\pi} \left( \cos t - \frac{1}{3} \cos 3t + \frac{1}{5} \cos 5t - \frac{1}{7} \cos 7t + \ldots \right) \]

```matlab
\[ t=\text{linspace}(-2*\pi,2*\pi,1000); \]
\[ x=\text{square}(t+\pi/2,50) \]
\[ \text{plot}(t,x) \]
\[ \text{axis}([-2*\pi,2*\pi,-.2, 1.2]); \]
```

```matlab
% Compute the series terms
\[ \text{sumterms} = \text{zeros}(16,\text{length}(t)); \]
\[ \text{sumterms}(1,:) = .5; \]
\[ \text{for } n=1:\text{size(sumterms,1)}-1 \]
\[ \quad \text{sumterms}(n+1,:) = \frac{2}{(n\pi)} \sin(\frac{n\pi}{2}) \cos(nt); \]
\[ \text{end} \]

```matlab
% Cumulative sum
\[ x_N=\text{cumsum(sumterms)}; \]
```

```matlab
\[ \text{ind}=0; \]
\[ \text{for } N=[0,1:2:\text{size(sumterms,1)}-1] \]
\[ \quad \text{ind}=\text{ind}+1; \]
\[ \quad \text{subplot}(3,3,\text{ind}) \]
\[ \quad \text{plot}(t,x_N(N+1,:),'k',t,x,'r--') \]
```

```matlab
\[ \text{axis}([-2*\pi,2*\pi,-.2, 1.2]); \]
\[ \text{xlabel('t')} \]
\[ \text{ylabel(['x_{', num2str(N), '}'}(t))} \]
```
```
The compact trigonometric Fourier series

\[
f(t) = C_o + \sum_{n=1}^{\infty} C_n \cos(nw_0 t + \theta_n)
\]

and so the coefficients given by

| C is the amplitude or coefficient | \[ C_n = \sqrt{a_n^2 + b_n^2}, \quad C_o = a_o \] |
| n is the harmonic | \[ \theta_n = \tan^{-1} \frac{-b_n}{a_n} \] |
| \theta is the phase | | |
| w is the radian frequency | | |
| T the period | | |

**FREQUENCY SPECTRA**

**Amplitude spectrum** - The plot of amplitude \( C_n \) versus the (radian) frequency \( \omega \) (use the function

\[
C_n = \sqrt{a_n^2 + b_n^2}
\]

\[ \theta_n = \tan^{-1} \frac{-b_n}{a_n} \]

**Phase spectrum** - The plot of the phase \[ \theta_n = \tan^{-1} \frac{-b_n}{a_n} \] versus the (radian) frequency \( \omega \).

These two plots convey all of the information that the plot of \( f(t) \) as a function of \( t \) does. The frequency spectra of a signal constitute the **frequency-domain** description of \( f(t) \) whereas \( f(t) \) as a function of \( t \) is the **time-domain** description.

**Bandwidth**
The difference (in radians) of the highest and lowest frequencies in the phase spectrum is the bandwidth.
Time Vs Frequency Domain
Figure 4-1. Signal Formed by Adding Three Frequency Components.
Fourier Series with GUI

Gibbs Phenomena

Explanation

We begin this discussion by taking a signal with a finite number of discontinuities (like a square pulse) and finding its Fourier Series representation. We then attempt to reconstruct it from these Fourier coefficients. What we find is that the more coefficients we use, the more the signal begins to resemble the original. However, around the discontinuities, we observe rippling that does not seem to subside. As we consider even more coefficients, we notice that the ripples narrow, but do not shorten. As we approach an infinite number of coefficients, this rippling still does not go away. This is when we apply the idea of almost everywhere. While these ripples remain (never dropping below 9% of the pulse height), the area inside them tends to zero, meaning that the energy of this ripple goes to zero. This means that their width is approaching zero and we can assert that the reconstruction is exactly the original except at the points of discontinuity.

Below we will use the square wave, along with its Fourier Series representation, and show several figures that reveal this phenomenon more mathematically.
Figure 1 shows several Fourier series approximations of the square wave using a varied number of terms, denoted by $K$:

When comparing the square wave to its Fourier series representation in Figure 1, it is not clear that the two are equal. The fact that the square wave's Fourier series requires more terms for a given representation accuracy is not important. However, close inspection of Figure 1 does reveal a potential issue: Does the Fourier series really equal the square wave at all values of $t$? In particular, at each step-change in the square wave, the Fourier series exhibits a peak followed by rapid oscillations. As more terms are added to the series, the oscillations seem to become more rapid and smaller, but the peaks are not decreasing. Consider this mathematical question intuitively: Can a discontinuous function, like the square wave, be expressed as a sum, even an infinite one, of continuous ones? One should at least be suspicious, and in fact, it can't be thus
expressed. This issue brought Fourier much criticism from the French Academy of Science (Laplace, Legendre, and Lagrange comprised the review committee) for several years after its presentation on 1807. It was not resolved for also a century, and its resolution is interesting and important to understand from a practical viewpoint.

The extraneous peaks in the square wave's Fourier series never disappear; they are termed Gibb's phenomenon after the American physicist Josiah Willard Gibbs. They occur whenever the signal is discontinuous, and will always be present whenever the signal has jumps.
Examples on Fourier Series

1. Compute and plot the Fourier coefficients (the spectrum) for the following periodic signal.

\[ c_n = \frac{4}{n\pi} \sin\left(\frac{n\pi}{4}\right) \sin\left(\frac{n\pi}{2}\right) e^{-1^{n+\frac{1}{2}}}. \]
To plot the Fourier series to check your answers:

```matlab
T=6;
w0 = 2*pi/T;
t = -1.5*T:T/1000:1.5*T;
N = input('Number of harmonics ');% dc component
c0 = 0;
x = c0*ones(1,length(t));
for n=1:N,
    cn = -4*j/n/pi*sin(pi*n/6)*sin(n*pi/2)*exp(-j*n*pi/3);
    c_n = conj(cn);
    x = x + cn*exp(j*n*w0*t) + c_n*exp(-j*n*w0*t);
end
plot(t,x)
title(['N = ',num2str(N)])
```
2. Repeat problem 1 for the following signal:

\[ c_n = \cos \left( \frac{\pi}{2} n \omega_0 \right) \cdot \frac{1}{5(1-(n\omega_0)^2)} \]

```matlab
>> n=-10:10;
>> cn=cos(pi/2*n*w0)/5./(1-(n*w0).^2);
>> subplot(221), stem(n,abs(cn))
>> title('|c_n|')
>> subplot(222), stem(n,angle(cn))
>> title('angle(c_n) in rad')
```

will give first plots shown below. To check your answer, you can plot the truncated series and see if it converges correctly.

```matlab
T=10;
w0 = 2*pi/T;
t = -1.5:T/1000:1.5*T;
N = input('Number of harmonics');
c0 = 1/5;
x = c0*ones(1,length(t)); % dc component
for n=1:N,
    cn = cos(pi/2*n*w0)/5/(1-(n*w0)^2);
    c_n = cn;
    x = x + cn*exp(j*n*w0*t) + c_n*exp(-j*n*w0*t);
end
plot(t,x)
title(['N = ',num2str(N)])
```
3- Compute and plot the Fourier coefficients (the spectrum) for the following periodic signal.

\[ C_n = \frac{1}{n \pi} \left( 1 - \frac{1}{2} (-1)^n \right) = \frac{1}{n \pi} e^{-\frac{i \pi n}{2}} \]

\[ C_0 = \frac{3}{4} \]

% truncated Fourier series for a staircase signal
T = 4;
w0 = 2*pi/T;
t = 0:T/1000:3*T;
N = input('input N')
nneg = -N:-1;
npos = 1:N;
c0 = 3/4;
cneg = 1/j/pi./nneg.*(-1.^nneg-.5*exp(-j*nneg*pi/2));
cpos = 1/j/pi./npos.*(-1.^npos-.5*exp(-j*npos*pi/2));
n = [nneg 0 npos];
c_n = [cneg c0 cpos];
x = c_n*exp(j*w0*n'*t);
x = real(x); % strips off negligible imaginary parts
plot(t,x)
xlabel('Time (sec)')
title(['x',num2str(N),'.(t)'])

The truncated series is plotted from top to bottom for \( N = 3, N = 10, \) and \( N = 40. \)
4- Compute and plot the Fourier coefficients (the spectrum) for the following periodic signal.

\[ C_n = \frac{2}{5 n^2 \omega_0} \left[ \cos(n \omega_0) - 2 \cos(2n \omega_0) \right], \quad C_0 = \frac{3}{5} \]

\[ x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j n \frac{\pi}{5} t} \]

% computes exponential Fourier series for trapezoidal wave
T = 5;
w = 2*pi/5;
t = -T:T/1000:2*T;
N = input('Number of harmonics ');
c0 = 3/5;
w0 = pi;
xN = c0*ones(1,length(t)); % dc component
for n=1:N,
cn = 2/5./n.^2/w/w*(cos(n*w)-cos(2*n*w));
c_n = cn;
xN = xN + cn*exp(j*n*w*t) + c_n*exp(-j*n*w*t);
end
plot(t,xN)
title(['' N = ',num2str(N)])

Running this program for N = 3, N = 10 and N = 40 gives: