**Experiment 9: Sampling Theory**

**Objective**
The objective of this Lab is to understand concepts and observe the effects of periodically sampling a continuous signal at different sampling rates, changing the sampling rate of a sampled signal, aliasing, and anti-aliasing filters.

**Introduction,**
A typical time-dependent signal, for example AC voltage, is continuous with respect to magnitude and time. Such signals are called analog signals. Using a normal (analog) oscilloscope we get an analog representation of such a signal.

Today mainly digital equipment is used for electrical measurements. The original analog signal is converted to a digital signal. A digital signal is discrete with respect to the magnitude as well as to the time. Therefore, conversion of an analog to a digital signal means the value of the analog signal function $F(t)$ is measured at discrete times.

**Sampling**
In order to store, transmit or process analog signals using digital hardware, we must first convert them into discrete-time signals by sampling.

The processed discrete-time signal is usually converted back to analog form by interpolation, resulting in a reconstructed analog signal $x_r(t)$.

The sampler reads the values of the analog signal $x_a(t)$ at equally spaced sampling instants. The time interval $Ts$ between adjacent samples is known as the sampling period (or sampling interval). The sampling rate, measured in samples per second, is $fs = 1/Ts$.

$$x[n] = x_a(nTs) \quad n = ..., -1, 0, 1, 2, ....$$

Also it possible to reconstruct $x_a(t)$ from its samples: $x_a(t) = x[tFs]$.

![Figure 4.1: Sampling and Reconstruction process](attachment://image.png)
Sampling Theorem

The uniform sampling theorem states that a bandlimited signal having no spectral components above \( f_m \) can be determined uniquely by values sampled at uniform intervals of:

\[
T_s \leq \frac{1}{2f_m}
\]

The upper limit on \( T_s \) can be expressed in terms of sampling rate, denoted \( f_s = 1/T_s \). The restriction, stated in term of the sampling rate, is known as the Nyquist criterion. The statement is:

\[
f_s \geq 2f_m
\]

The sampling rate \( f_s = 2f_m \) is also called Nyquist rate. The allow Nyquist criterion is a theoretically sufficient condition to allow an analog signal to be reconstructed completely from a set of a uniformly spaced discrete-time samples.

Impulse Sampling

Assume an analog waveform \( x(t) \), as shown in figure 4.2(a), with a Fourier transform, \( X(f) \) which is zero outside the interval \( (-f_m < f < f_m) \), as shown in figure 4.2(b). The sampling of \( x(t) \) can be viewed as the product of \( x(t) \) with a periodic train of unit impulse function \( x_\delta(t) \), shown in figure 4.2(c) and defined as

\[
x_\delta(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)
\]

where \( T_s \) is the sampling period and \( \delta(t) \) is the unit impulse or Dirac delta function. Let us choose \( T_s = 1/2f_m \), so that the Nyquist criterion is just satisfied. The sifting property of the impulse function states that

\[
x(t)\delta(t - t_o) = x(t_o)\delta(t - t_o)
\]

Using this property, we can see that \( x_\delta(t) \), the sampled version of \( x(t) \) shown in figure 4.2(e), is given by
Using the *frequency convolution property* of the Fourier transform we can transform the time-domain product \( x(t)x_\delta(t) \) to the frequency-domain convolution \( (f) \ast X_\delta(f) \),

\[
x_\delta(f) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \delta(f - nf_s)
\]

where \( X_\delta(f) \) is the Fourier transform of the impulse train \( x_\delta(t) \). Notice that the Fourier transform of an impulse train is another impulse train; the values of the periods of the two trains are reciprocally related to one another. Figure 4.2(c) and (d) illustrate the impulse train \( x_\delta(t) \) and its Fourier transform \( X_\delta(f) \), respectively.

We can solve for the transform \( X_s(f) \) of the sampled waveform:

\[
X_s(f) = X(f) \ast X_\delta(f) = X(f) \ast \left[ \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \delta(f - nf_s) \right]
\]

\[
X_s(f) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(f - nf_s)
\]
We therefore conclude that within the original bandwidth, the spectrum $X_S(f)$ is, to within a constant factor $(1/T_s)$, exactly the same as that of $x(t)$ in addition, the spectrum repeats itself periodically in frequency every $f_s$ hertz.
Aliasing

Aliasing in Frequency Domain

If $f_s$ does not satisfy the Nyquist rate, $f_s < 2f_m$, the different components of $X_S(f)$ overlap and will not be able to recover $x(t)$ exactly as shown in figure 8.3(b). This is referred to as **aliasing in frequency domain**.

![Figure 8.3: Spectra for various sampling rates. (a) Sampled spectrum $f_s > 2f_m$, (b) Sampled spectrum $f_s < 2f_m$](image)

**Sampling theory:**

$x(t)$: analoge signal  
$x_s(nT_s) = x(n)$: discrete signal  
$F$: analoge frequency Hz  
$f$: discrete frequency Hz  
$\Omega$: analoge frequency rad/sec  
$w$: discrete frequency rad/sec  
$T_s$: sampling period  

$$x(n) = x_s(nT_s)$$

Let the analoge signal $x(t) = \cos(2\pi Ft + \theta)$  

**sampling** $\Rightarrow x_s(nT_s) = \cos(2\pi F n T_s + \theta)$ $\Rightarrow x(n) = \cos(2\pi fn + \theta)$

Digital frequency (discrete frequency) = analoge frequency x sampling period  

$$f = FT_s$$
Figure 1 shows a simple example. The solid line describes a 0.5 Hz continuous-time sinusoidal signal and the dash-dot line describes a 1.5 Hz continuous-time sinusoidal signal. When both signals are sampled at the rate of $F_s = 2$ samples/sec, their samples coincide, as indicated by the circles. This means that $x_1[nT_s]$ is equal to $x_2[nT_s]$ and there is no way to distinguish the two signals apart from their sampled versions. This phenomenon is known as aliasing.

![Figure 1: Two sinusoidal signals are indistinguishable from their sampled versions](image)

$$x_1(nT_s) = \cos(2\pi F_1 n T_s) = \cos \left( 2\pi \times \frac{1}{2} \times n \times \frac{1}{2} \right) = \cos(0.5\pi n)$$

$$x_2(nT_s) = \cos(2\pi F_2 n T_s)$$

$$= \cos \left( 2\pi \times \frac{3}{2} \times n \times \frac{1}{2} \right) = \cos(1.5\pi n) = \cos(1.5\pi n - 2\pi n)$$

$$= \cos(-0.5\pi n) = \cos(-0.5\pi n) = \cos(0.5\pi n) \Rightarrow x_1(nT_s) = x_2(nT_s)$$
Practical Parts

\[ f_0 = 1000; \quad \text{%Frequency of sin} \]
\[ f_{s1} = 10000; \quad \text{%Sampling Frequency Fs>2Fm} \]
\[ f_{s2} = 1500; \quad \text{%Sampling Frequency Fs<2Fm} \]
\[ n = 0:1:50; \]
\[ x = \cos(2\pi f_0 n / f_{s1}); \]
\[ x_1 = \cos(2\pi f_0 n / f_{s2}); \]

```
figure (1)
subplot(2,1,1)
plot(n,x)
subplot(2,1,2)
hold on
plot(n,x)
stem(n,x,'r')
plot(n,x1,'g')
legend('Original function','Sampling With Fs>2Fm','Sampling With Fs<2Fm')
```
Do you now that the human voice bandwidth lay between 0-4.5K Hz, Try different Fs in the following code and see the effect of aliasing.

```matlab
% Record your voice for 5 seconds.
fs=1000;
recObj = audiorecorder(fs,16,1);
disp('Start speaking.')
recordblocking(recObj, 5);
disp('End of Recording.');
% Play back the recording.
play(recObj);
% Store data in double-precision array.
myRecording = getaudiodata(recObj);
% Plot the waveform.
plot(myRecording)
```