Confidence Intervals for Variance and Standard Deviation

The Chi-Square Distribution

The point estimate for \( \sigma^2 \) is \( \hat{s}^2 \), and the point estimate for \( \sigma \) is \( s \). \( s^2 \) is the most unbiased estimate for \( \sigma^2 \).

You can use the chi-square distribution to construct a confidence interval for the variance and standard deviation.

If the random variable \( X \) has a normal distribution, then the distribution of

\[ \chi^2 = \frac{(n-1)s^2}{\sigma^2} \]

forms a chi-square distribution for samples of any size \( n > 1 \).
The Chi-Square Distribution

Four properties of the chi-square distribution are as follows.

1. All chi-square values $\chi^2$ are greater than or equal to zero.

2. The chi-square distribution is a family of curves, each determined by the degrees of freedom. To form a confidence interval for $\sigma^2$, use the $\chi^2$-distribution with degrees of freedom. To form a confidence interval for $\sigma^2$, use the $\chi^2$-distribution with degrees of freedom equal to one less than the sample size.

3. The area under each curve of the chi-square distribution equals one.

4. Find the critical value $z_c$ that corresponds to the given level of confidence.

5. Chi-square distributions are positively skewed.

Critical Values for $X^2$

There are two critical values for each level of confidence. The value $X^2_R$ represents the right-tail critical value and $X^2_L$ represents the left-tail critical value.

Area to the right of $X^2_R$

Area to the right of $X^2_L$

The area between the left and right critical values is $c$. 
Critical Values for $X^2$

**Example:**
Find the critical values $X^2_R$ and $X^2_L$ for an 80% confidence when the sample size is 18.

Because the sample size is 18, there are
d.f. = $n - 1 = 18 - 1 = 17$ degrees of freedom,

Area to the right of $X^2_R = \frac{1 - c}{2} = \frac{1 - 0.8}{2} = 0.1$

Area to the right of $X^2_L = \frac{1 + c}{2} = \frac{1 + 0.8}{2} = 0.9$

Use the Chi-square distribution table to find the critical values.

**Continued.**

Critical Values for $X^2$

**Example continued:**

Appendix B: Table 6: $X^2$-Distribution

<table>
<thead>
<tr>
<th>Degrees of freedom</th>
<th>0.995</th>
<th>0.99</th>
<th>0.975</th>
<th>0.95</th>
<th>0.90</th>
<th>0.10</th>
<th>0.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.010</td>
<td>0.020</td>
<td>0.051</td>
<td>0.103</td>
<td>0.211</td>
<td>4.605</td>
<td>5.891</td>
</tr>
<tr>
<td>3</td>
<td>0.072</td>
<td>0.115</td>
<td>0.216</td>
<td>0.352</td>
<td>0.584</td>
<td>6.251</td>
<td>7.815</td>
</tr>
</tbody>
</table>

$X^2_R = 24.769$

$X^2_L = 10.085$
### Confidence Intervals for \( \sigma^2 \) and \( \sigma \)

A **confidence interval** for a population variance and standard deviation is as follows.

**Confidence Interval for \( \sigma^2 \):**

\[
\frac{(n-1)s^2}{X^2_R} < \sigma^2 < \frac{(n-1)s^2}{X^2_L}
\]

**Confidence Interval for \( \sigma \):**

\[
\sqrt{\frac{(n-1)s^2}{X^2_R}} < \sigma < \sqrt{\frac{(n-1)s^2}{X^2_L}}
\]

The probability that the confidence intervals contain \( \sigma^2 \) or \( \sigma \) is \( c \).
Confidence Intervals for $\sigma^2$ and $\sigma$

Constructing a Confidence Interval for a Variance and a Standard Deviation

In Words

5. Find the left and right endpoints and form the confidence interval.

6. Find the confidence interval for the population standard deviation by taking the square root of each endpoint.

In Symbols

$$\frac{(n-1)s^2}{\chi^2_{L}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{R}}$$

$$\sqrt{\frac{(n-1)s^2}{\chi^2_{L}}} < \sigma < \sqrt{\frac{(n-1)s^2}{\chi^2_{R}}}$$

Example:

You randomly select and weigh 41 samples of 16-ounce bags of potato chips. The sample standard deviation is 0.05 ounce. Assuming the weights are normally distributed, construct a 90% confidence interval for the population standard deviation.

d.f. = $n - 1 = 41 - 1 = 40$ degrees of freedom,

Area to the right of $\chi^2_R = \frac{1 - c}{2} = \frac{1 - 0.9}{2} = 0.05$

Area to the right of $\chi^2_L = \frac{1 + c}{2} = \frac{1 + 0.9}{2} = 0.95$

The critical values are $\chi^2_R = 55.758$ and $\chi^2_L = 26.509$. 

Continued.
Constructing a Confidence Interval

Example continued:
\[ \chi^2_L = 26.509 \]
\[ \chi^2_R = 55.758 \]

Left endpoint = 0.04
Right endpoint = 0.06

With 90% confidence we can say that the population standard deviation is between 0.04 and 0.06 ounces.

Hypothesis Testing for Variance and Standard Deviation
Critical Values for the $\chi^2$-Test

**Finding Critical Values for the $\chi^2$ Distribution**

1. Specify the level of significance $\alpha$.
2. Determine the degrees of freedom $d.f. = n - 1$.
3. The critical values for the $\chi^2$-distribution are found in Table 6 of Appendix B. To find the critical value(s) for $\alpha$
   a. right-tailed test, use the value that corresponds to $d.f.$ and $\alpha$.
   b. left-tailed test, use the value that corresponds to $d.f.$ and $1 - \alpha$.
   c. two-tailed test, use the values that corresponds to $d.f.$ and $\frac{1}{2}\alpha$ and $d.f.$ and $1 - \frac{1}{2}\alpha$.

---

**Finding Critical Values for the $\chi^2$**

**Example:**
Find the critical value for a left-tailed test when $n = 19$ and $\alpha = 0.05$.

There are 18 d.f. The area to the right of the critical value is $1 - \alpha = 1 - 0.05 = 0.95$.

From Table 6, the critical value is $\chi^2_0 = 9.390$.

**Example:**
Find the critical value for a two-tailed test when $n = 26$ and $\alpha = 0.01$.

There are 25 d.f. The areas to the right of the critical values are $\frac{1}{2}\alpha = 0.005$ and $1 - \frac{1}{2} \alpha = 0.995$.

From Table 6, the critical values are $\chi^2_L = 10.520$ and $\chi^2_R = 46.928$. 
The Chi-Square Test

The $\chi^2$-test for a variance or standard deviation is a statistical test for a population variance or standard deviation. The $\chi^2$-test can be used when the population is normal.

The **test statistic** is $s^2$ and the **standardized test statistic**

$$
\chi^2 = \frac{(n-1)s^2}{\sigma^2}
$$

follows a chi-square distribution with degrees of freedom d.f. = $n - 1$.

---

Using the $\chi^2$-Test for a Variance or Standard Deviation

<table>
<thead>
<tr>
<th>In Words</th>
<th>In Symbols</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. State the claim mathematically and verbally. Identify the null and alternative hypotheses.</td>
<td>State $H_0$ and $H_a$.</td>
</tr>
<tr>
<td>2. Specify the level of significance.</td>
<td>Identify $\alpha$.</td>
</tr>
<tr>
<td>3. Determine the degrees of freedom and sketch the sampling distribution.</td>
<td>d.f. = $n - 1$</td>
</tr>
<tr>
<td>4. Determine any critical values.</td>
<td>Use Table 6 in Appendix B.</td>
</tr>
</tbody>
</table>

Continued.
The Chi-Square Test

Using the $\chi^2$-Test for a Variance or Standard Deviation

<table>
<thead>
<tr>
<th>In Words</th>
<th>In Symbols</th>
</tr>
</thead>
<tbody>
<tr>
<td>5. Determine any rejection regions.</td>
<td>$\chi^2 = \frac{\sum (O - E)^2}{E}$</td>
</tr>
<tr>
<td>6. Find the standardized test statistic.</td>
<td>$\chi^2 = \frac{(n - 1)s^2}{\sigma^2}$</td>
</tr>
<tr>
<td>7. Make a decision to reject or fail to reject the null hypothesis.</td>
<td>If $\chi^2$ is in the rejection region, reject $H_0$. Otherwise, fail to reject $H_0$.</td>
</tr>
<tr>
<td>8. Interpret the decision in the context of the original claim.</td>
<td></td>
</tr>
</tbody>
</table>

Hypothesis Test for Standard Deviation

**Example:**
A college professor claims that the standard deviation for students taking a statistics test is less than 30. 10 tests are randomly selected and the standard deviation is found to be 28.8. Test this professor's claim at the $\alpha = 0.01$ level.

$H_0$: $\sigma \geq 30$  
$H_a$: $\sigma < 30$ (Claim)

This is a left-tailed test with d.f. = 9 and $\alpha = 0.01$.  

![Graph showing the rejection region for a left-tailed test.]

Continued.
Hypothesis Test for Standard Deviation

Example continued:
A college professor claims that the standard deviation for students taking a statistics test is less than 30. 10 tests are randomly selected and the standard deviation is found to be 28.8. Test this professor’s claim at the $\alpha = 0.01$ level.

$H_0: \sigma \geq 30$

$H_a: \sigma < 30$ (Claim)

$\chi^2 \approx 8.29$

At the 1% level of significance, there is not enough evidence to support the professor’s claim.

Hypothesis Test for Variance

Example:
A local balloon company claims that the variance for the time its helium balloons will stay afloat is 5 hours. A disgruntled customer wants to test this claim. She randomly selects 23 customers and finds that the variance of the sample is 4.5 seconds. At $\alpha = 0.05$, does she have enough evidence to reject the company’s claim?

$H_0: \sigma^2 = 5$ (Claim)  

$H_a: \sigma^2 \neq 5$

This is a two-tailed test with d.f. = 22 and $\alpha = 0.05$. 

Continued.
**Example continued:**
A local balloon company claims that the variance for the time its helium balloons will stay afloat is 5 hours. A disgruntled customer wants to test this claim. She randomly selects 23 customers and finds that the variance of the sample is 4.5 seconds. At $\alpha = 0.05$, does she have enough evidence to reject the company’s claim?

$H_0$: $\sigma^2 = 5$ (Claim)  
$H_a$: $\sigma^2 \neq 5$

The critical values are $\chi^2_L = 10.982$ and $\chi^2_R = 36.781$.

At $\alpha = 0.05$, there is not enough evidence to reject the claim that the variance of the float time is 5 hours.