CHAPTER 3

MECHANICS OF MATERIALS

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Torsion

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Torsional Loads on Circular Shafts

- Interested in stresses and strains of circular shafts subjected to twisting couples or *torques*.
- Turbine exerts torque $T$ on the shaft.
- Shaft transmits the torque to the generator.
- Generator creates an equal and opposite torque $T'$. 
Net Torque Due to Internal Stresses

- Net of the internal shearing stresses is an internal torque, equal and opposite to the applied torque,

\[ T = \int \rho \, dF = \int \rho (\tau \, dA) \]

- Although the net torque due to the shearing stresses is known, the distribution of the stresses is not

- Distribution of shearing stresses is statically indeterminate – must consider shaft deformations

- Unlike the normal stress due to axial loads, the distribution of shearing stresses due to torsional loads can not be assumed uniform.
Axial Shear Components

- Torque applied to shaft produces shearing stresses on the faces perpendicular to the axis.

- Conditions of equilibrium require the existence of equal stresses on the faces of the two planes containing the axis of the shaft.

- The existence of the axial shear components is demonstrated by considering a shaft made up of axial slats.

The slats slide with respect to each other when equal and opposite torques are applied to the ends of the shaft.
• From observation, the angle of twist of the shaft is proportional to the applied torque and to the shaft length.

\[ \phi \propto T \]

\[ \phi \propto L \]

• When subjected to torsion, every cross-section of a circular shaft remains plane and undistorted.

• Cross-sections for hollow and solid circular shafts remain plain and undistorted because a circular shaft is axisymmetric.

• Cross-sections of noncircular (non-axisymmetric) shafts are distorted when subjected to torsion.
Shearing Strain

- Consider an interior section of the shaft. As a torsional load is applied, an element on the interior cylinder deforms into a rhombus.

- Since the ends of the element remain planar, the shear strain is equal to angle of twist.

- It follows that

\[ L\gamma = \rho\phi \quad \text{or} \quad \gamma = \frac{\rho\phi}{L} \]

- Shear strain is proportional to twist and radius

\[ \gamma_{\text{max}} = \frac{c\phi}{L} \quad \text{and} \quad \gamma = \frac{\rho}{c}\gamma_{\text{max}} \]
Stresses in Elastic Range

- Multiplying the previous equation by the shear modulus,
  \[ G \gamma = \frac{\rho}{c} G \gamma_{\text{max}} \]
  From Hooke’s Law, \( \tau = G \gamma \), so
  \[ \tau = \frac{\rho}{c} \tau_{\text{max}} \]
  The shearing stress varies linearly with the radial position in the section.

- Recall that the sum of the moments from the internal stress distribution is equal to the torque on the shaft at the section,
  \[ T = \int \rho \tau \, dA = \frac{\tau_{\text{max}}}{c} \int \rho^2 \, dA = \frac{\tau_{\text{max}}}{c} J \]
  The results are known as the elastic torsion formulas,
  \[ \tau_{\text{max}} = \frac{Tc}{J} \quad \text{and} \quad \tau = \frac{T\rho}{J} \]
Normal Stresses

- Elements with faces parallel and perpendicular to the shaft axis are subjected to shear stresses only. Normal stresses, shearing stresses or a combination of both may be found for other orientations.

- Consider an element at $45^\circ$ to the shaft axis,

  $$F = 2(\tau_{\text{max}} A_0) \cos 45^\circ = \tau_{\text{max}} A_0 \sqrt{2}$$

  $$\sigma_{45^\circ} = \frac{F}{A} = \frac{\tau_{\text{max}} A_0 \sqrt{2}}{A_0 \sqrt{2}} = \tau_{\text{max}}$$

- Element $a$ is in pure shear.

- Element $c$ is subjected to a tensile stress on two faces and compressive stress on the other two.

- Note that all stresses for elements $a$ and $c$ have the same magnitude.
Torsional Failure Modes

- Ductile materials generally fail in shear. Brittle materials are weaker in tension than shear.

- When subjected to torsion, a ductile specimen breaks along a plane of maximum shear, i.e., a plane perpendicular to the shaft axis.

- When subjected to torsion, a brittle specimen breaks along planes perpendicular to the direction in which tension is a maximum, i.e., along surfaces at 45° to the shaft axis.
Shaft $BC$ is hollow with inner and outer diameters of 90 mm and 120 mm, respectively. Shafts $AB$ and $CD$ are solid of diameter $d$. For the loading shown, determine $(a)$ the minimum and maximum shearing stress in shaft $BC$, $(b)$ the required diameter $d$ of shafts $AB$ and $CD$ if the allowable shearing stress in these shafts is 65 MPa.

**SOLUTION:**

- Cut sections through shafts $AB$ and $BC$ and perform static equilibrium analysis to find torque loadings.
- Apply elastic torsion formulas to find minimum and maximum stress on shaft $BC$.
- Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter.
Sample Problem 3.1

- Cut sections through shafts $AB$ and $BC$ and perform static equilibrium analysis to find torque loadings.

\[ \sum M_x = 0 = (6 \text{kN} \cdot \text{m}) - T_{AB} \]

\[ T_{AB} = 6 \text{kN} \cdot \text{m} = T_{CD} \]

\[ \sum M_x = 0 = (6 \text{kN} \cdot \text{m}) + (14 \text{kN} \cdot \text{m}) - T_{BC} \]

\[ T_{BC} = 20 \text{kN} \cdot \text{m} \]
Sample Problem 3.1

- Apply elastic torsion formulas to find minimum and maximum stress on shaft $BC$

$$J = \frac{\pi}{2} \left( c_2^4 - c_1^4 \right) = \frac{\pi}{2} \left[ (0.060)^4 - (0.045)^4 \right]$$

$$= 13.92 \times 10^{-6} \text{ m}^4$$

$$\tau_{\text{max}} = \tau_2 = \frac{T_{BC} c_2}{J} = \frac{(20 \text{ kN} \cdot \text{m})(0.060 \text{ m})}{13.92 \times 10^{-6} \text{ m}^4}$$

$$= 86.2 \text{ MPa}$$

$$\frac{\tau_{\text{min}}}{\tau_{\text{max}}} = \frac{c_1}{c_2} \quad \frac{\tau_{\text{min}}}{86.2 \text{ MPa}} = \frac{45 \text{ mm}}{60 \text{ mm}}$$

$$\tau_{\text{min}} = 64.7 \text{ MPa}$$

- Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter

$$\tau_{\text{max}} = \frac{T_c}{J} = \frac{T_c}{\frac{\pi}{2} c^4}$$

$$65 \text{ MPa} = \frac{6 \text{ kN} \cdot \text{m}}{\frac{\pi}{2} c^3}$$

$$c = 38.9 \times 10^{-3} \text{ m}$$

$$d = 2c = 77.8 \text{ mm}$$
Angle of Twist in Elastic Range

- Recall that the angle of twist and maximum shearing strain are related,
  \[ \gamma_{\text{max}} = \frac{c \phi}{L} \]
- In the elastic range, the shearing strain and shear are related by Hooke’s Law,
  \[ \gamma_{\text{max}} = \frac{\tau_{\text{max}}}{G} = \frac{Tc}{JG} \]
- Equating the expressions for shearing strain and solving for the angle of twist,
  \[ \phi = \frac{TL}{JG} \]
- If the torsional loading or shaft cross-section changes along the length, the angle of rotation is found as the sum of segment rotations
  \[ \phi = \sum_{i} \frac{T_i L_i}{J_i G_i} \]
Given the shaft dimensions and the applied torque, we would like to find the torque reactions at A and B.

From a free-body analysis of the shaft,
\[ T_A + T_B = 90 \text{lb} \cdot \text{ft} \]
which is not sufficient to find the end torques. The problem is statically indeterminate.

Divide the shaft into two components which must have compatible deformations,
\[ \phi = \phi_1 + \phi_2 = \frac{T_A L_1}{J_1 G} - \frac{T_B L_2}{J_2 G} = 0 \quad T_B = \frac{L_1 J_2}{L_2 J_1} T_A \]

Substitute into the original equilibrium equation,
\[ T_A + \frac{L_1 J_2}{L_2 J_1} T_A = 90 \text{lb} \cdot \text{ft} \]