Branch and bound method

Objectives of the topic:

- Present the general structure of all integer programming solution techniques.
- Apply the branch and bound method.
- See chapter 9.2 in the textbook.

General structure

- Integer programming solution methods implement a structure that sequentially impose restrictions on continuous solutions, until an integer solution is reached.
- The general structure of the solution methods is

Step 1: remove the integer restriction on all integer variables. This process is called relaxation:

$$x_j \geq 0 \text{ and integer} \quad \xrightarrow{\text{convert to}} \quad x_j \geq 0$$

$$x_k = (0,1) \quad \xrightarrow{\text{convert to}} \quad 0 \leq x_k \leq 1$$

Step 2: solve the relaxed LP model.

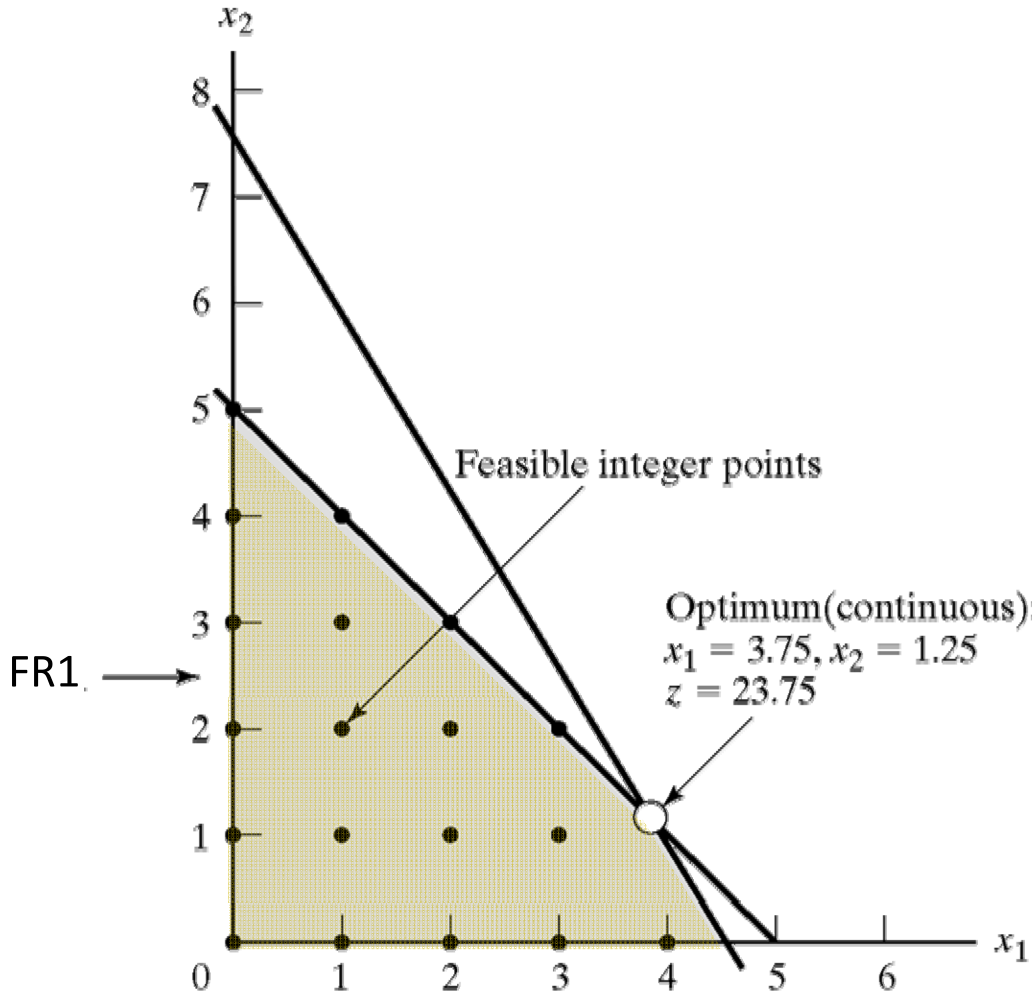
Step 3: start from the relaxed solution and add certain constraints that will lead to an integer solution.

Branch and bound basics

- B&B method is a systematic refinement of the feasible space that will lead to a feasible optimal integer solution.
- Consider the MIP model

$$\begin{aligned}
\max \quad & z = 5x_1 + 4x_2 \\
\text{s.t.} \quad & x_1 + x_2 \le 5 \\
& 10x_1 + 6x_2 \le 45 \\
& x_1, x_2 \ge 0 \text{ and integer}
\end{aligned}$$

- The feasible space is illustrated by



- The optimal solution of the relaxed MIP model is

  $x_1 = 3.75$

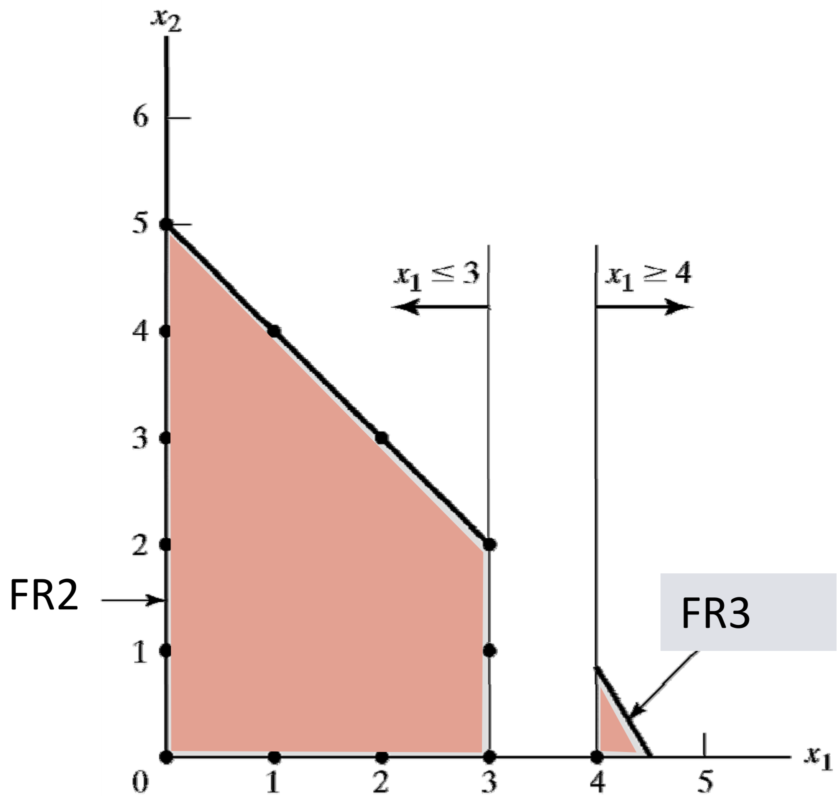
  $x_2 = 1.25$

  $z = 23.75$
- This solution is not integer.
- Consider first x_1. The optimal integer solution is either for $x_1 \geq 4$ or $x_1 \leq 3$.
- Hence, the area of the feasible region FR1 that has $3 < x_1 < 4$ can be eliminated.
- This breakup in FR1 leads to two sub feasible regions.

- The two sub regions can be labeled as:

$$FR2 = FR1 \cap \{x_1 \leq 3\}$$

$$FR3 = FR1 \cap \{x_1 \geq 4\}$$



- Notice that the feasible region defined by FR2 $\cup$ FR3 contains the same integer solutions as FR1.
- Continuing removing the regions that do not include integer solutions will lead to an LP model whose optimum solution is integer.
- Hence, the solution of MIP is reached by solving a sequence of LP problems.
- Choosing $x_1 \geq 4$ and $x_1 \leq 3$ is called branching, and x_1 is called the branching variable.
- The optimal integer solution lies either in FR2 or FR3.
- Hence, z is optimized with respect to FR2 and FR3, and when an integer solution is found, the procedure is terminated. Otherwise, the process of branching continues.

- Solve the sub problem LP2:

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 5 \\ & 10x_1 + 6x_2 \le 45 \\ & x_1 \le 3 \\ & x_1, x_2 \ge 0 \end{aligned}$$

- The solution is

$x_1 = 3$

$x_2 = 2$

$z = 23$

- The solution of LP2 is integer, hence, no further branching is required.
- This solution cannot be said to be also the optimal solution to the original problem, because the sub problem for $x_1 \geq 4$ might give a better solution.
- It can only be said that = 23 is a lower bound on the optimum solution.
- This means that any sub problem that cannot yield a better solution than the lower bound must be discarded.

- Solve the second sub problem LP3:

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 5 \\ & 10x_1 + 6x_2 \le 45 \\ & x_1 \ge 4 \\ & x_1, x_2 \ge 0 \end{aligned}$$

- The solution is

  $x_1 = 4$

  $x_2 = 0.83$

  $z = 23.33$

- Since z of LP3 is greater than 23 (lower bound), FR3 has to be examined further.
- The branching variable now is x_2, hence, $x_2 \leq 0$ or ≥ 1.
- Solve LP4:

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 5 \\ & 10x_1 + 6x_2 \leq 45 \\ & x_1 \geq 4 \\ & x_2 \leq 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- The optimal solution is: $x_1 = 4.5$, $x_2 = 0$, $z = 22.5$.

- Since z of LP4 is lower than the current lower bound, branching from LP4 is stopped.
- Solve LP5:

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 5 \\ & 10x_1 + 6x_2 \leq 45 \\ & x_1 \geq 4 \\ & x_2 \geq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- LP5 has no feasible solution.
- The branch and bound method can now be terminated.

Practice

- Use branch and bound method to solve

$$\begin{aligned} \min \quad & z = 5x_1 + 4x_2 \\ \text{s.t.} \quad & 3x_1 + 2x_2 \geq 5 \\ & 2x_1 + 3x_2 \geq 7 \\ & x_1, x_2 \geq 0 \text{ and integer} \end{aligned}$$

- First, solve the relaxed problem LP1:

$$\begin{aligned} \min \quad & z = 5x_1 + 4x_2 \\ \text{s.t.} \quad & 3x_1 + 2x_2 \geq 5 \\ & 2x_1 + 3x_2 \geq 7 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- Write LP in equation form:

$$\begin{aligned} \min \quad & z = 5x_1 + 4x_2 + 100R_1 + 100R_2 \\ \text{s.t.} \quad & 3x_1 + 2x_2 - s_1 + R_1 = 5 \\ & 2x_1 + 3x_2 - s_2 + R_2 = 7 \\ & x, R \geq 0 \end{aligned}$$

- The simplex steps to solve LP1 are:

Basic	x_1	x_2	s_1	s_2	R_1	R_2	Solution
z	-5	-4	0	0	-100	-100	0
R_1	3	2	-1	0	1	0	5
R_2	2	3	0	-1	0	1	7

Basic	x_1	x_2	s_1	s_2	R_1	R_2	Solution
z	495	496	-100	-100	0	0	1200
R_1	3	2	-1	0	1	0	5
R_2	2	3	0	-1	0	1	7

Basic	x_1	x_2	s_1	s_2	R_1	R_2	Solution
z	164 1/3	0	-100	65 1/3	0	-165 1/3	42 2/3
R_1	1 2/3	0	-1	2/3	1	- 2/3	1/3
x_2	2/3	1	0	- 1/3	0	1/3	2 1/3

Basic	x_1	x_2	s_1	s_2	R_1	R_2	Solution
z	0	0	-1 ⅖	- 2/5	-98 ⅗	-99 ⅗	9 ⅘
x_1	1	0	- 3/5	2/5	3/5	- 2/5	⅕
x_2	0	1	2/5	- 3/5	- 2/5	3/5	2 ⅕

- The optimal relaxed solution is

$$x_1 = 0.2 \quad x_2 = 2.2 \quad z = 9.8$$

- The optimal integer solution > 9.8.
- Either x_1 or x_2 can be chosen as the branching variable at this point. Choose x_1.
- The optimal integer solution is either for $x_1 \leq 0$ or $x_1 \geq 1$.

- Solve problem LP2:

$$\begin{aligned} \min \quad & z = 5x_1 + 4x_2 \\ \text{s.t.} \quad & 3x_1 + 2x_2 \geq 5 \\ & 2x_1 + 3x_2 \geq 7 \\ & x_1 \leq 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- Adding a constraint to the primal LP is equivalent to adding a new row to the simplex tableau:

Basic	x_1	x_2	s_1	s_2	s_3	Solution
z	0	0	-1 2/5	- 2/5	0	9 4/5
x_1	1	0	- 3/5	2/5	0	1/5
x_2	0	1	2/5	- 3/5	0	2 1/5
s_3	1	0	0	0	1	0

Basic	x_1	x_2	s_1	s_2	s_3	Solution
z	0	0	-1 2/5	- 2/5	0	9 ⅘
x_1	1	0	- 3/5	2/5	0	1/5
x_2	0	1	2/5	- 3/5	0	2 ⅕
s_3	0	0	3/5	- 2/5	1	-1/5

Basic	x_1	x_2	s_1	s_2	s_3	Solution
z	0	0	-2	0	-1	10
x_1	1	0	0	0	1	0
x_2	0	1	- 1/2	0	-1 ½	2 ½
s_2	0	0	-1 ½	1	-2 ½	1/2

- The optimal solution is

$$x_1 = 0 \quad x_2 = 2.5 \quad z = 10$$

- The solution of LP2 is not integer, hence, further branching is required.
- Branching on x_2, two sub problems will be formed.
- Solve LP3:

$$\begin{aligned} \min \quad & z = 5x_1 + 4x_2 \\ \text{s.t.} \quad & 3x_1 + 2x_2 \geq 5 \\ & 2x_1 + 3x_2 \geq 7 \\ & x_1 \leq 0 \\ & x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Basic	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	0	0	-2	0	0	0	10
x_1	1	0	0	0	1	0	0
x_2	0	1	- 1/2	0	-1 1/2	0	2 1/2
s_2	0	0	-1 1/2	1	-2 1/2	0	1/2
s_4	0	1	0	0	0	1	2

Basic	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	0	0	-2	0	0	0	10
x_1	1	0	0	0	1	0	0
x_2	0	1	- 1/2	0	-1 ½	0	2 ½
s_2	0	0	-1 ½	1	-2 ½	0	1/2
s_4	0	0	1/2	0	1 ½	1	- 1/2

- LP3 has no feasible solution.
- For $x_2 \geq 3$, solve LP4:

$$\begin{aligned}
\min \quad & z = 5x_1 + 4x_2 \\
\text{s.t.} \quad & 3x_1 + 2x_2 \geq 5 \\
& 2x_1 + 3x_2 \geq 7 \\
& x_1 \leq 0 \\
& x_2 \geq 3 \\
& x_1, x_2 \geq 0
\end{aligned}$$

Basic	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	0	0	-2	0	0	0	10
x_1	1	0	0	0	1	0	0
x_2	0	1	- 1/2	0	-1 1/2	0	2 1/2
s_2	0	0	-1 1/2	1	-2 1/2	0	1/2
s_4	0	1	0	0	0	-1	3

Basic	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	0	0	-2	0	0	0	10
x_1	1	0	0	0	1	0	0
x_2	0	1	- 1/2	0	-1 1/2	0	2 1/2
s_2	0	0	-1 1/2	1	-2 1/2	0	1/2
s_4	0	0	1/2	0	1 1/2	-1	1/2

Basic	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	0	0	-2	0	0	0	10
x_1	1	0	0	0	1	0	0
x_2	0	1	- 1/2	0	-1 ½	0	2 ½
s_2	0	0	-1 ½	1	-2 ½	0	1/2
s_4	0	0	- 1/2	0	-1 ½	1	- 1/2

Basic	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	0	0	-2	-0	0	0	10
x_1	1	0	- 1/3	0	0	2/3	- 1/3
x_2	0	1	0	0	0	-1	3
s_2	0	0	- 2/3	1	0	-1 ⅔	1 ⅓
s_3	0	0	1/3	0	1	- 2/3	1/3

Basic	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	-6	0	0	0	0	-4	12
s_1	-3	0	1	0	0	-2	1
x_2	0	1	0	0	0	-1	3
s_2	-2	0	0	1	0	-3	2
s_3	1	0	0	0	1	0	-0

- The optimal solution is

$$x_1 = 0 \quad x_2 = 3 \quad z = 12$$

- Hence, the upper bound on the optimal solution is 12.

- Back to x_1, solve LP5:

$$\begin{aligned} \min \quad & z = 5x_1 + 4x_2 \\ \text{s.t.} \quad & 3x_1 + 2x_2 \geq 5 \\ & 2x_1 + 3x_2 \geq 7 \\ & x_1 \geq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Basic	x_1	x_2	s_1	s_2	s_5	Solution
z	0	0	-1 2/5	- 2/5	0	9 4/5
x_1	1	0	- 3/5	2/5	0	1/5
x_2	0	1	2/5	- 3/5	0	2 1/5
s_5	1	0	0	0	-1	1

Basic	x_1	x_2	s_1	s_2	s_5	Solution
z	0	0	-1 2/5	- 2/5	0	9 4/5
x_1	1	0	- 3/5	2/5	0	1/5
x_2	0	1	2/5	- 3/5	0	2 1/5
s_5	0	0	3/5	- 2/5	-1	4/5

Basic	x_1	x_2	s_1	s_2	s_5	Solution
z	0	0	-1 ⅖	- 2/5	0	9 ⅘
x_1	1	0	- 3/5	2/5	0	1/5
x_2	0	1	2/5	- 3/5	0	2 ⅕
s_5	0	0	- 3/5	2/5	1	- 4/5

Basic	x_1	x_2	s_1	s_2	s_5	Solution
z	0	0	0	-1 ⅓	-2 ⅓	11 ⅔
x_1	1	0	0	0	-1	1
x_2	0	1	0	- 1/3	⅔	1 ⅔
s_1	0	0	1	- 2/3	-1 ⅔	1 ⅓

- The optimal solution is

$$x_1 = 1 \quad x_2 = 1.667 \quad z = 11.667$$

- Since z of LP5 is lower than the current upper bound (12), the branch and bound method continues.

- Branching on x_2, solve LP6:

$$\begin{aligned} \min \quad & z = 5x_1 + 4x_2 \\ \text{s.t.} \quad & 3x_1 + 2x_2 \geq 5 \\ & 2x_1 + 3x_2 \geq 7 \\ & x_1 \geq 1 \\ & x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Basic	x_1	x_2	s_1	s_2	s_5	s_6	Solution
z	0	0	0	-1 1/3	-2 1/3	0	11 2/3
x_1	1	0	0	0	-1	0	1
x_2	0	1	0	- 1/3	2/3	0	1 2/3
s_1	0	0	1	- 2/3	-1 2/3	0	1 1/3
s_6	0	1	0	0	0	1	1

Basic	x_1	x_2	s_1	s_2	s_5	s_6	Solution
z	0	0	0	-1 1/3	-2 1/3	0	11 2/3
x_1	1	0	0	0	-1	0	1
x_2	0	1	0	- 1/3	2/3	0	1 2/3
s_1	0	0	1	- 2/3	-1 2/3	0	1 1/3
s_6	0	0	0	1/3	- 2/3	1	- 2/3

Basic	x_1	x_2	s_1	s_2	s_5	s_6	Solution
z	0	0	0	-2 ½	0	-3 ½	14
x_1	1	0	0	- 1/2	0	-1 ½	2
x_2	0	1	0	0	0	1	1
s_1	0	0	1	-1 ½	0	-2 ½	3
s_5	0	0	0	- 1/2	1	-1 ½	1

- The optimal solution is

$$x_1 = 2 \quad x_2 = 1 \quad z = 14$$

- This solution is greater than the upper bound (12), hence it should be discarded.
- Solve LP7:

$$\begin{aligned} \min \quad & z = 5x_1 + 4x_2 \\ \text{s.t.} \quad & 3x_1 + 2x_2 \geq 5 \\ & 2x_1 + 3x_2 \geq 7 \\ & x_1 \geq 1 \\ & x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Basic	x_1	x_2	s_1	s_2	s_5	s_7	Solution
z	0	0	0	-1 1/3	-2 1/3	0	11 2/3
x_1	1	0	0	0	-1	0	1
x_2	0	1	0	- 1/3	2/3	0	1 2/3
s_1	0	0	1	- 2/3	-1 2/3	0	1 1/3
s_7	0	1	0	0	0	-1	2

Basic	x_1	x_2	s_1	s_2	s_5	s_7	Solution
z	0	0	0	-1 1/3	-2 1/3	0	11 2/3
x_1	1	0	0	0	-1	0	1
x_2	0	1	0	- 1/3	2/3	0	1 2/3
s_1	0	0	1	- 2/3	-1 2/3	0	1 1/3
s_7	0	0	0	1/3	- 2/3	-1	1/3

Basic	x_1	x_2	s_1	s_2	s_5	s_7	Solution
z	0	0	0	-1 1/3	-2 1/3	0	11 2/3
x_1	1	0	0	0	-1	0	1
x_2	0	1	0	- 1/3	2/3	0	1 2/3
s_1	0	0	1	- 2/3	-1 2/3	0	1 1/3
s_7	0	0	0	- 1/3	2/3	1	- 1/3

Basic	x_1	x_2	s_1	s_2	s_5	s_7	Solution
z	0	0	0	0	-5	-4	13
x_1	1	0	0	0	-1	0	1
x_2	0	1	0	0	0	-1	2
s_1	0	0	1	0	-3	-2	2
s_2	0	0	0	1	-2	-3	1

- The optimal solution is

$$x_1 = 1 \quad x_2 = 2 \quad z = 13$$

- This solution is greater than the upper bound, so it must be discarded.
- Since no further branching can be made, the branch and bound algorithm is terminated.
- The optimal integer solution is

$$x_1 = 0 \quad x_2 = 3 \quad z = 12$$