Integer programming models

Topic objectives:

- Present integer programming models for special cases in industrial engineering.
- See chapter 9.1 in the textbook.

- Integer programming models are either direct description of decision problems or as convenience for problems that are not integer by themselves.
- Integer programming models are either pure (all variables are integer) or mixed (some but not all are integer).
- Integer decision variables that are restricted to either 0 or 1 are called binary variables.

Capital budgeting

- The capital budgeting problem deals with the allocation of limited resources (fund, space, time, etc) to selected courses of action.
- An industrial engineer is planning the production of five products that will be produced during the following week.
- All products pass through the same three-stage process, but the stage costs are different for the different products:

	Stage cost (riyals)			
Product	1	2	3	Revenue
1	5	1	8	20
2	4	7	10	40
3	3	9	2	20
4	7	4	1	15
5	8	6	10	30
Allocated fund (riyals)	25	25	25	

- Given the limited funds allocated for each stage, the industrial engineer has to select the products that will maximize the revenue for the company.
- Define the binary variable

$$x_j = \begin{cases} 1 & \text{if product } j \text{ is selected} \\ 0 & \text{otherwise} \end{cases}$$

- The objective function will be

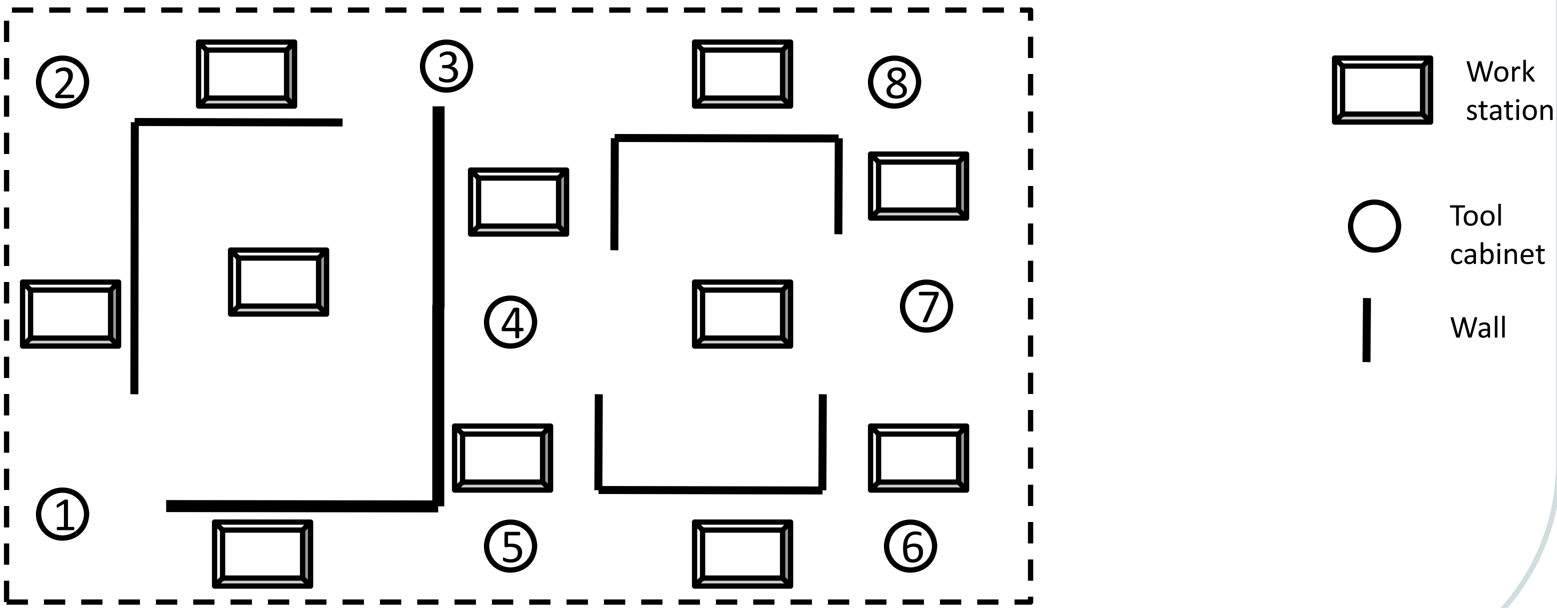
$$\max\ z = 20x_1 + 40x_2 + 20x_3 + 15x_4 + 30x_5$$

- The constraints impose the budget limitation:

$$5x_1 + 4x_2 + 3x_3 + 7x_4 + 8x_5 \leq 25$$
$$x_1 + 7x_2 + 9x_3 + 4x_4 + 6x_5 \leq 25$$
$$8x_1 + 10x_2 + 2x_3 + x_4 + x_5 \leq 25$$
$$x = (0,1)$$

Set covering

- The set covering problem can be described as allocating service stations that will cover all service demand locations.
- In a plant, there are 11 work stations and 8 possible locations of tool cabinets:



- Each tool cabinet can serve as number of work stations as desired.
- These tool cabinet locations are determined in a way that workers in work stations can find an adjacent cabinet.
- The industrial engineer has to determine the minimum number of tool cabinets that will serve the work stations.
- Define the binary decision variable

$$x_j = \begin{cases} 1 & \text{if tool cabinet is installed in position } j \\ 0 & \text{otherwise} \end{cases}$$

- Since the purpose is to install the smallest number of cabinets in the plant, the objective function then becomes

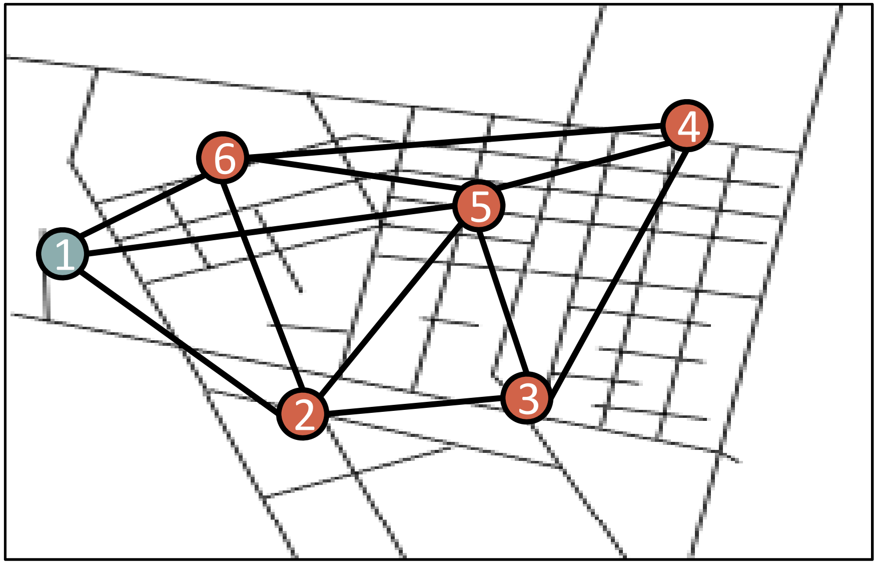
$$\min\ z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8$$

- Each work station has to be served by one a close-by cabinet, the constraints become

$$x_1 + x_2 \ge 1 \qquad x_4 + x_5 \ge 1 \qquad x_7 + x_8 \ge 1$$
$$x_1 + x_5 \ge 1 \qquad x_4 + x_7 \ge 1 \qquad x_j = (0,1)$$
$$x_2 + x_3 \ge 1 \qquad x_5 + x_6 \ge 1$$
$$x_3 + x_8 \ge 1 \qquad x_6 + x_7 \ge 1$$
$$x_3 + x_4 \ge 1 \qquad x_1 + x_3 \ge 1$$

Traveling salesman problem TSP

- TSP is a problem of finding the best route (path) or sequence to visit all locations and stop at the starting location with minimum cost, distance, or time.
- An industrial engineer has to plan for the distribution of products manufactured by the company.



- The all delivery points have to be visited and the truck returns back to the factory with the least distance.
- Define the binary variable

$$x_{ij} = \begin{cases} 1 & \text{if point } j \text{ is visited from point } i \\ 0 & \text{otherwise} \end{cases}$$

- Define the parameter

d_{ij} = distance between point i and j

- If point j cannot be visited directly from point i, set $d_{ij} = \infty$.
- By design, $d_{ii} = \infty$, point i cannot be visited directly from itself.

- The objective function for this problem becomes

$$\begin{aligned}\min\ z = {} & d_{12}x_{12} + d_{15}x_{15} + d_{16}x_{16} + d_{21}x_{21} + d_{23}x_{23} + d_{25}x_{25} \\ & + d_{26}x_{26} + d_{32}x_{32} + d_{34}x_{34} + d_{35}x_{35} + d_{43}x_{43} + d_{45}x_{45} + d_{46}x_{46} \\ & + d_{51}x_{51} + d_{52}x_{52} + d_{53}x_{53} + d_{54}x_{54} + d_{56}x_{56} + d_{61}x_{61} + d_{62}x_{62} \\ & + d_{64}x_{64} + d_{65}x_{65}\end{aligned}$$

- The model has two sets of constraints.
- The first set states that every point has to be connected to the following point:

$$x_{12} + x_{15} + x_{16} = 1$$

$$x_{21} + x_{23} + x_{25} + x_{26} = 1$$

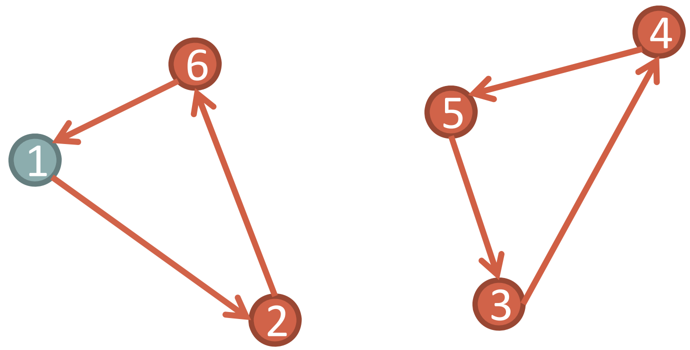
$$x_{32} + x_{34} + x_{35} = 1$$

$$x_{43} + x_{45} + x_{46} = 1$$
$$x_{51} + x_{52} + x_{53} + x_{54} + x_{56} = 1$$
$$x_{61} + x_{62} + x_{64} + x_{65} = 1$$

- The second set states that every point is connected to a preceding point:

$$x_{21} + x_{51} + x_{61} = 1$$
$$x_{12} + x_{32} + x_{52} + x_{62} = 1$$
$$x_{23} + x_{43} + x_{53} = 1$$
$$x_{34} + x_{54} + x_{64} = 1$$
$$x_{15} + x_{25} + x_{35} + x_{45} + x_{65} = 1$$
$$x_{16} + x_{26} + x_{46} + x_{56} = 1$$

- These constraints are not enough to produce a single path or a tour that will visit all points:



- A sub-tour elimination strategy has to be implemented to prevent sub-tours from happening.

- Notice that any sub-tour has links equal to the number of points.
- To prevent sub-tours from forming, we can impose a restriction that:
  - Any group of two points should not have more than one link;
  - Any group of three points should not have more than two links;
  - …..
  - Any group of five point should not have more than 4 links.
- Only the 6 points are allowed to have 6 links.

- So, we can write

$$\left.\begin{array}{l} x_{12} + x_{21} \le 1 \\ x_{15} + x_{51} \le 1 \\ x_{16} + x_{61} \le 1 \\ x_{23} + x_{32} \le 1 \\ \vdots \end{array}\right\} \text{Two points or nodes}$$

$$\left.\begin{array}{l} x_{12} + x_{26} + x_{16} \le 2 \\ x_{16} + x_{62} + x_{21} \le 2 \\ x_{15} + x_{56} + x_{61} \le 2 \\ x_{16} + x_{65} + x_{51} \le 2 \\ \vdots \end{array}\right\} \text{Three nodes}$$

$$\left.\begin{array}{l} x_{12} + x_{25} + x_{56} + x_{61} \le 3 \\ x_{16} + x_{65} + x_{52} + x_{21} \le 3 \\ \vdots \end{array}\right\} \text{Four nodes}$$

$$\left.\begin{array}{l} x_{12} + x_{23} + x_{35} + x_{56} + x_{61} \le 4 \\ \vdots \end{array}\right\} \text{Five nodes}$$

$$x_{ij} = (0,1)$$

- The traveling salesman problem can be written compactly as

$$\min \ z = \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} x_{ij}$$

$$\text{s.t.} \qquad \sum_{j=1}^{n} x_{ij} = 1, \qquad i = 1, \ldots, n$$

$$\sum_{i=1}^{n} x_{ij} = 1, \qquad j = 1, \ldots, n$$

$$\sum_{i \in S} \sum_{j \in S} x_{ij} \le |S| - 1, \qquad S \subset N, 2 \le |S| \le n - 1$$

$$x_{ij} = (0,1)$$

Fixed charge

- Manufacturing costs usually have two components, fixed cost and variable cost.
- The fixed cost does not change with the volume of production.
- The variable cost increases with the production volume.
- When there is no production, the manufacturing cost is equal to zero.
- Therefore, if x = the production volume, then

$$C(x) = \begin{cases} F + Vx & \text{for } x > 0 \\ 0 & \text{for } x = 0 \end{cases}$$

- Dealing with this form of cost function is difficult since it is discontinuous at $x = 0$.
- Integer programming easily and conveniently is going to put the problem into a mathematical model.
- The industrial engineer is planning the purchase of raw materials used for production.
- The raw materials can be purchased from three local manufacturing plants:

Company	Fixed cost (riyals)	Unit variable cost (riyals)
Saudi Fabrication	1600	250
National Manufacture	2500	210
Expert Staff	1800	220

- It has been estimated that 200 units are needed for production.
- Define the integer variables:

x_1 = number of units ordered from the first company

x_2 = number of units ordered from the second company

x_3 = number of units ordered from the third company

- Define the binary variable:

$$y_j = \begin{cases} 1 & \text{if raw materials ordered from company } j \\ 0 & \text{otherwise} \end{cases}$$

- To ensure that y_j =1 for x_j > 0, we can write

$$x_j < M\, y_j$$

- M is any large number greater than 200 (needed raw materials).
- The integer model is

$$\min\ z = 250x_1 + 210x_2 + 220x_3 + 1600y_1 + 2500y_2 + 1800y_3$$

$$\begin{aligned}
\text{s.t.}\quad & x_1 + x_2 + x_3 = 200 \\
& x_1 \le 200y_1 \\
& x_2 \le 200y_2 \\
& x_3 \le 200y_3 \\
& x_j \ge 0 \\
& y_j = (0,1)
\end{aligned}$$

Either-or and if-then constraints

- In general LP models, all the constraints are enforced.
- Integer programming is used to model cases where logical conditions are encountered, such as either one restriction is enforced or another; or conditions in which a restriction is enforced only when a specific situation happens.
- An industrial engineer is planning the sequence or order of 3 jobs on a machine:

Job	Processing time p_j (days)	Due date d_j	Late penalty (riyals per day)
1	5	25	19
2	20	22	12
3	15	35	34

- The penalty value signifies the importance of finishing the job on time, and it does not mean that any money will be paid.
- Hence, the best sequence is the one that has the least penalty cost.
- Define the decision variable:

$$x_j = \text{start day of job } j$$

- In this problem, there are two groups of constraints
  - Processing of a single job on the machine in any given time
  - Job finish time calculation

- Two jobs i and j will not be processed at the same time if either job i starts after job j completes or job j starts after job i completes:

$$x_i - x_j \geq p_j \quad \text{or} \quad x_j - x_i \geq p_i$$

- Introduce the binary variable

$$y_{ij} = \begin{cases} 1 & \text{if job } i \text{ is before job } j \\ 0 & \text{if job } j \text{ is before job } i \end{cases}$$

- The either-or constraint above can be enforced as

$$My_{ij} + (x_i - x_j) \geq p_j$$

$$M(1 - y_{ij}) + (x_j - x_i) \geq p_i$$

for M very large.

- When $y_{ij} = 0$, then
  $x_i - x_j \geq p_j$ (binding)
  and
  $M + x_j - x_i \geq p_i \rightarrow x_j - x_i \geq p_i - M$ (non binding)
- Hence, job i comes after job j.
- When $y_{ij} = 1$, then
  $M + x_i - x_j \geq p_j \rightarrow x_i - x_j \geq p_j - M$ (non binding)
  and
  $x_j - x_i \geq p_i$ (binding)
- Hence, job j comes after job i.

- The deviation of job j from its due date is

$$s_j = d_j - (x_j + p_j)$$

- Job j will not be penalized if $s_j \geq 0$.
- Replace s_j by two positive deviational variables:

$$s_j = s^-_j - s^+_j$$

- Hence, job j is delayed if $s^+_j > 0$.
- The objective function of the job sequence problem is

$$\text{Min } z = 19s^+_1 + 12s^+_2 + 34s^+_3$$

- Assume that there is an additional restriction that if job i comes before job j, then job k should come before job m.
- This restriction can be written as

$$\text{If } x_i + p_i \leq x_j \quad \text{then } x_k + p_k \leq x_m$$

- It can be also written as

$$\text{If } x_j - (x_i + p_i) > 0 \text{ then } x_k + p_k - x_m < 0$$

- Introduce a binary variable y = (0,1).
- The if-then restriction can be explained by two constraints:

$$x_j - (x_i + p_i) \leq M(1 - y) - \varepsilon$$

$$x_k + p_k - x_m \leq My$$

- If the optimal solution requires $x_j - (x_i + p_i) > 0$, that is job i is after job j, then y has to be 0, and hence $x_k + p_k - x_m \leq 0$, that is job m has to come after job k.
- If the optimal solution is for $x_j - (x_i + p_i) < 0$, then y can be 0 or 1 and it has no effect on the second constraint.