CHAPTER
2
ANALYTIC FUNCTIONS

We now consider functions of a complex variable and develop a theory of differentiation for them. The main goal of the chapter is to introduce analytic functions, which play a central role in complex analysis.

11 FUNCTIONS OF A COMPLEX VARIABLE

**Definition:**
Let $S$ be a set of complex numbers. A function $f$ defined on $S$ is a rule that assigns to each $z$ in $S$ a complex number $w$.

The number $w$ is called the **value** of $f$ at $z$ and is denoted by $f(z)$; that is, $w = f(z)$.

The set $S$ is called the **domain of definition** of $f$.

**EXAMPLE 1.**
Real and imaginary part of the function:

Suppose that \( w = u + iv \) is the value of a function \( f \) at \( z = x + iy \), so that \( u + iv = f(x + iy) \)

Each of the real numbers \( u \) and \( v \) depends on the real variables \( x \) and \( y \), and it follows that \( f(z) \) can be expressed in terms of a pair of real-valued functions of the real variables \( x \) and \( y \):

\[
f(z) = u(x, y) + iv(x, y).
\]

If the polar coordinates \( r \) and \( \theta \), instead of \( x \) and \( y \), are used, then \( u + iv = f(re^{i\theta}) \),

where \( w = u + iv \) and \( z = re^{i\theta} \). In that case, we may write

\[
f(z) = u(r, \theta) + iv(r, \theta).
\]

Definitions:

1) If \( n \) is zero or a positive integer and if \( a_0, a_1, a_2, \ldots a_n \) are complex constants, where \( a_n \neq 0 \), the function

\[
P(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n.
\]

is a polynomial of degree \( n \).

Note: the sum here has a finite number of terms and that the domain of definition is the entire \( z \) plane.

2) Quotients \( \frac{P(z)}{Q(z)} \) of polynomials are called rational functions and are defined at each point \( z \) where \( Q(z) \neq 0 \).

Examples:
**Definition: Multiple-valued function:**

A multiple valued function is a rule that assigns more than one value to a point $z$ in the domain of definition.

**Notes:**

1) These *multiple-valued functions* occur in the theory of functions of a complex variable, just as they do in the case of real variables.

2) When multiple-valued functions are studied, usually just one of the possible values assigned to each point is taken, in a systematic manner, and a (single-valued) function is constructed from the multiple-valued function.

**Example:**
14. LIMITS

Let a function \( f \) be defined at all points \( z \) in some deleted neighborhood of \( z_0 \). The statement that the limit of \( f(z) \) as \( z \) approaches \( z_0 \) is a number \( w_0 \),

or means that the point \( w = f(z) \) can be made arbitrarily close to \( w_0 \) if we choose the point \( z \) close enough to \( z_0 \) but distinct from it. We now express the definition of limit in a precise and usable form.

**Definition:**

\[
\lim_{z \to z_0} f(z) = w_0,
\]

1) means that, for each positive number \( \varepsilon \), there is a positive number \( \delta \) such that

\[ |f(z) - w_0| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta. \]

2) When a limit of a function \( f(z) \) exists at a point, it is unique.

**Note:** Definition (1) requires that \( f \) be defined at all points in some deleted neighborhood of \( z_0 \). Such a deleted neighborhood, of course, always exists when \( z_0 \) is an interior point of a region on which \( f \) is defined.
We can extend the definition of limit to the case in which \( z_o \) is a boundary point of the region by agreeing that the first of inequalities in (1) need be satisfied by only those points \( z \) that lie in both the region and the deleted neighborhood.

2) If \( z_o \) is an interior point of the domain of definition of \( f \), and limit (1) is to exist, the first of inequalities (1) must hold for all points in the deleted neighborhood

\[ 0 < |z - z_o| < \delta. \]

Thus the symbol \( z \to z_o \) implies that \( z \) is allowed to approach \( z_o \) in an arbitrary manner, not just from some particular direction.

**EXAMPLE 1:** Show that if \( f(z) = iz/2 \) in the open disk \( |z| < 1 \), then

\[
\lim_{z \to 1} f(z) = \frac{i}{2},
\]

**EXAMPLE 2:** Show that

If

\[
f(z) = \frac{z}{\overline{z}},
\]

then

\[
\lim_{z \to 0} f(z) \quad \text{does not exist.}
\]
15. THEOREMS ON LIMITS

We can expedite our treatment of limits by establishing a connection between limits of functions of a complex variable and limits of real-valued functions of two real variables. Since limits of the latter type are studied in calculus, we use their definition and properties freely.

**Theorem 1.** Suppose that

\[ f(z) = u(x,y) + iv(x,y), \quad z_0 = x_0 + iy_0, \text{ and } w_0 = u_0 + iv_0. \]

Then

1. \[ \lim_{z \to z_0} f(z) = w_0 \]
   
   if and only if

2. \[ \lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0 \quad \text{and} \quad \lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0. \]

**Theorem 2.** Suppose that

\[ \lim_{z \to z_0} f(z) = w_0 \quad \text{and} \quad \lim_{z \to z_0} F(z) = W_0. \]

Then

1. \[ \lim_{z \to z_0} [f(z) + F(z)] = w_0 + W_0 \]
2. \[ \lim_{z \to z_0} [f(z) F(z)] = w_0 W_0 \quad \text{as } z \to z_0 \]
3. and, if \( W_0 \neq 0 \)
4) **Limits of polynomials:**

\[ P(z) = a_0 + a_1 z + a_2 z^2 + \ldots \ldots + a_n z^n. \]

as \( z \) approaches a point \( z_o \) is the value of the polynomial at that point

\[ \lim_{z \to z_0} P(z) = P(z_0). \]

**Examples:**
16. LIMITS INVOLVING THE POINT AT INFINITY

It is sometimes convenient to include with the complex plane the point at infinity, denoted by $\infty$, and to use limits involving it. The complex plane together with this point is called the extended complex plane.

**A stereographic projection:**

To visualize the point at infinity, one can think of the complex plane as passing through the equator of a unit sphere centered at the point $z = 0$. To each point $z$ in the plane there corresponds exactly one point $P$ on the surface of the sphere. The point $P$ is determined by the intersection of the line through the point $z$ and the north pole $N$ of the sphere with that surface.

By letting the point $N$ of the sphere correspond to the point at infinity, we obtain a one to one correspondence between the points of the sphere and the points of the extended complex plane. The sphere is known as the **Riemann sphere**, and the correspondence is called a **stereographic projection**.
**Definition:**
An \( \varepsilon \) neighborhood, or neighborhood, of \( \infty \):
for each small positive number \( \varepsilon \), those points in the
complex plane exterior to the circle \(|z| = 1/\varepsilon\)
correspond to points on the sphere close to \(N\). We thus
call the set \(|z| > 1/\varepsilon\) an \( \varepsilon \) neighborhood.

**Theorem.** If \(z_0\) and \(w_0\) are points in the \(z\) and \(w\) planes, respectively, then

\[
(1) \quad \lim_{z \to z_0} f(z) = \infty \quad \text{if and only if} \quad \lim_{z \to z_0} \frac{1}{f(z)} = 0
\]
and

\[
(2) \quad \lim_{z \to \infty} f(z) = w_0 \quad \text{if and only if} \quad \lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0.
\]

Moreover,

\[
(3) \quad \lim_{z \to \infty} f(z) = \infty \quad \text{if and only if} \quad \lim_{z \to 0} \frac{1}{f(1/z)} = 0.
\]
Proof:

Examples: Show that

\[ \lim_{z \to -1} \frac{iz + 3}{z + 1} = \infty \]

\[ \lim_{z \to \infty} \frac{2z + i}{z + 1} = 2 \quad \text{since} \]

\[ \lim_{z \to \infty} \frac{2z^3 - 1}{z^2 + 1} = \infty \]
17 Continuity

Definition:

A function \( f \) is \textit{continuous} at a point \( z_0 \) if all three of the following conditions are satisfied:

\begin{align*}
(1) & \quad \lim_{z \to z_0} f(z) \text{ exists,} \\
(2) & \quad f(z_0) \text{ exists,} \\
(3) & \quad \lim_{z \to z_0} f(z) = f(z_0).
\end{align*}

Definitions:

1) A function of a complex variable is said to be continuous in a region \( R \) if it is continuous at each point in \( R \).
2) If two functions are continuous at a point, their sum and product are also continuous at that point;
3) their quotient is continuous at any such point where the denominator is not zero.
4) A polynomial is continuous in the entire plane.

\textbf{Theorem 1}: \textit{A composition of continuous functions is itself continuous.}

Proof:

\textbf{Theorem 2}: \textit{If a function } \( f(z) \) \textit{is continuous and nonzero at a point } \( z_0 \), \textit{then } \( f(z) \neq 0 \) throughout some neighborhood of that point.}
Proof:

**Propositions:**

1) The function $f(z) = u(x,y) + iv(x,y)$ is continuous at a point $z_0 = (x_0, y_0)$ if and only if its component functions are continuous there.

2) If $f$ is continuous in a region $R$ that is both closed and bounded then $f$ is **bounded** on $R$ and $|f(z)|$ reaches a maximum value somewhere in $R$. That is, there exists a nonnegative real number $M$ such that

$$|f(z)| \leq M \quad \text{for all } z \in R,$$

where equality holds for at least one such $z$.

**Examples:**
18. DERIVATIVES

Definition:

Let $f$ be a function whose domain of definition contains a neighborhood of a point $z_0$.

The derivative of $f$ at $z_0$, written $f'(z_0)$, is defined by the equation

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

provided this limit exists. The function $f$ is said to be differentiable at $z_0$ when its derivative at $z_0$ exists.

By expressing the variable $z$ in definition (1) in terms of the new complex variable $\Delta z = z - z_0$

we can write that definition as

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$
If $\Delta w = f(z + \Delta z) - f(z)$, then

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}.$$  

Examples:
19. DIFFERENTIATION FORMULAS

Definition:

The derivative of a function $f$ at a point $z$ is denoted by

$$\frac{d}{dz} f(z) \quad \text{or} \quad f'(z),$$

either depending on which notation is more convenient.

Properties:

Let $c$ be a complex constant, and let $f$ be a function whose derivative exists at point $z$. It is easy to show that

$$(1) \quad \frac{d}{dz} c = 0, \quad \frac{d}{dz} z = 1, \quad \frac{d}{dz} [cf(z)] = cf'(z).$$

Also, if $n$ is a positive integer,

$$(2) \quad \frac{d}{dz} z^n = nz^{n-1}.$$  

This formula remains valid when $n$ is a negative integer, provided that $z \neq 0$.

If the derivatives of two functions $f$ and $F$ exist at a point $z$, then

$$(3) \quad \frac{d}{dz} [f(z) + F(z)] = f'(z) + F'(z),$$

$$(4) \quad \frac{d}{dz} [f(z)F(z)] = f(z)F'(z) + f'(z)F(z);$$

and, when $F(z) \neq 0$,

$$(5) \quad \frac{d}{dz} \left[ \frac{f(z)}{F(z)} \right] = \frac{F(z)f'(z) - f(z)F'(z)}{[F(z)]^2}.$$
There is also a chain rule for differentiating composite functions. Suppose that $f$ has a derivative at $z_0$ and that $g$ has a derivative at the point $f(z_0)$. Then the function $F(z) = g[f(z)]$ has a derivative at $z_0$, and

$$F'(z_0) = g'[f(z_0)]f'(z_0).$$

If we write $w = f(z)$ and $W = g(w)$, so that $W = F(z)$, the chain rule becomes

$$\frac{dW}{dz} = \frac{dW}{dw} \frac{dw}{dz}.$$

Proof:

Examples:
20. CAUCHY–RIEMANN EQUATIONS

In this section, we obtain a pair of equations that the first-order partial derivatives of the component functions $u$ and $v$ of a function

(1) \[ f(z) = u(x, y) + iv(x, y) \]

must satisfy at a point $z_0 = (x_0, y_0)$ when the derivative of $f$ exists there. We also show how to express $f'(z_0)$ in terms of those partial derivatives.

**Theorem.** Suppose that

\[ f(z) = u(x, y) + iv(x, y) \]

and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the first-order partial derivatives of $u$ and $v$ must exist at $(x_0, y_0)$, and they must satisfy the Cauchy–Riemann equations

(7) \[ u_x = v_y, \quad u_y = -v_x \]

there. Also, $f'(z_0)$ can be written

(8) \[ f'(z_0) = u_x + iv_x, \]

where these partial derivatives are to be evaluated at $(x_0, y_0)$.

Proof:

Examples:
21. SUFFICIENT CONDITIONS FOR DIFFERENTIABILITY

Satisfaction of the Cauchy-Riernann equations at a point \( z_0 = (x_0, y_0) \) is not sufficient to ensure the existence of the derivative of a function \( f(z) \) at that point. But, with certain continuity conditions, we have the following useful theorem.

**Theorem.** Let the function

\[
f(z) = u(x, y) + iv(x, y)
\]

be defined throughout some \( \varepsilon \) neighborhood of a point \( z_0 = x_0 + iy_0 \), and suppose that the first-order partial derivatives of the functions \( u \) and \( v \) with respect to \( x \) and \( y \) exist everywhere in that neighborhood. If those partial derivatives are continuous at \( (x_0, y_0) \) and satisfy the Cauchy–Riemann equations

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\]

at \( (x_0, y_0) \), then \( f'(z_0) \) exists.

Proof:

Examples:
22. POLAR COORDINATES

Assuming that $z_0 \neq 0$, we shall in this section use the coordinate transformation

(1) \[ x = r \cos \theta, \quad y = r \sin \theta \]

**Theorem.** Let the function

\[ f(z) = u(r, \theta) + iv(r, \theta) \]

be defined throughout some $\varepsilon$ neighborhood of a nonzero point $z_0 = r_0 \exp(i\theta_0)$, and suppose that the first-order partial derivatives of the functions $u$ and $v$ with respect to $r$ and $\theta$ exist everywhere in that neighborhood. If those partial derivatives are continuous at $(r_0, \theta_0)$ and satisfy the polar form

\[ ru_r = v_\theta, \quad u_\theta = -rv_r \]

of the Cauchy–Riemann equations at $(r_0, \theta_0)$, then $f'(z_0)$ exists.

The derivative $f'(z_0)$ here can be written (see Exercise 8)

(7) \[ f'(z_0) = e^{-i\theta}(u_r + iv_r), \]

where the right-hand side is to be evaluated at $(r_0, \theta_0)$.

**Proof:**

**Examples:**
23 ANALYTIC FUNCTIONS

Definitions:

1) A function $f$ of the complex variable $z$ is analytic in an open set if it has a derivative at each point in that set.
2) $f$ is analytic at a point $z_0$ if it is analytic throughout some neighborhood of $z_0$.
3) An entire function is a function that is analytic at each point in the entire finite plane.
4) Every polynomial is an entire function.
5) If a function $f$ fails to be analytic at a point $z_0$ but is analytic at some point in every neighborhood of $z_0$, then $z_0$ is called a singular point, or singularity, of $f$.
6) If two functions are analytic in a domain $D$, their sum and their product are both analytic in $D$.
7) Similarly, their quotient is analytic in $D$ provided the function in the denominator does not vanish at any point in $D$.
8) In particular, the quotient $P(z)/Q(z)$ of two polynomials is analytic in any domain throughout which $Q(z) \neq 0$.
9) A composition of two analytic functions is analytic.

That is, function $f(z)$ is analytic in a domain $D$ and that the image of $D$ under the transformation $w = f(z)$ is contained in the domain of definition of a function $g(w)$. 
Then the composition \( g[f(z)] \) is analytic in \( D \), with derivative

\[
\frac{d}{dz} g[f(z)] = g'[f(z)] f'(z).
\]

**Theorem.** If \( f'(z) = 0 \) everywhere in a domain \( D \), then \( f(z) \) must be constant throughout \( D \).
25. HARMONIC FUNCTIONS

Definition: A real-valued function \( H \) of two real variables \( x \) and \( y \) is said to be \textit{harmonic} in a given domain of the \( xy \) plane if, throughout that domain, it has \textit{continuous partial derivatives} of the first and second order \textit{and satisfies} the partial differential equation

\[
H_{xx}(x,y) + H_{yy}(x,y) = 0
\]

known as \textit{Laplace's equation}.

Example:

\textbf{Theorem 1.} If a function \( f(z) = u(x, y) + iv(x, y) \) is \textit{analytic} in a domain \( D \), \textit{then} its component functions \( u \) and \( v \) are \textit{harmonic} in \( D \).

Proof:

Examples:
Theorem 2. A function \( f(z) = u(x, y) + i v(x, y) \) is analytic in a domain \( D \) if and only if \( v \) is a harmonic conjugate of \( u \).

Proof:

Examples:
26. UNIQUELY DETERMINED ANALYTIC FUNCTIONS

Lemma. Suppose that

(i) a function \( f \) is analytic throughout a domain \( D \);
(ii) \( f(z) = 0 \) at each point \( z \) of a domain or line segment contained in \( D \).

Then \( f(z) = 0 \) in \( D \); that is, \( f(z) \) is identically equal to zero throughout \( D \).

Theorem. A function that is analytic in a domain \( D \) is uniquely determined over \( D \) by its values in a domain, or along a line segment, contained in \( D \).

27 REFLECTION PRINCIPLE

The theorem in this section concerns the fact that some analytic functions possess the property that \( \overline{f(z)} = f(\overline{z}) \) for all points \( z \) in certain domains, while others do not. We note, for example, that \( z + 1 \) and \( z^2 \) have that property when \( D \) is the entire finite plane; but the same is not true of \( z + i \) and \( iz^2 \). The theorem, which is known as the refection principle, provides a way of predicting when \( \overline{f(z)} = f(\overline{z}) \).
Theorem. Suppose that a function \( f \) is analytic in some domain \( D \) which contains a segment of the \( x \) axis and whose lower half is the reflection of the upper half with respect to that axis. Then

\[
f(z) = f(\bar{z})
\]

for each point \( z \) in the domain if and only if \( f(x) \) is real for each point \( x \) on the segment.