We consider here various elementary functions studied in calculus and define corresponding functions of a complex variable. To be specific, we define analytic functions of a complex variable \( z \) that reduce to the elementary, functions in calculus when \( z = x + i\theta \). We start by defining the complex exponential function and then use it to develop the others.

### 28. THE EXPONENTIAL FUNCTION

**Definition:** The exponential function:

The exponential function \( e^z \) by writing

\[
e^z = e^x e^{iy} \quad (z = x + iy),
\]

where Euler's formula

\[
e^{iy} = \cos y + i \sin y
\]
is used and \( y \) is to be taken in radians.

**Notes:**

1) \( e^z \) reduces to the usual exponential function in calculus when \( y = 0 \).
2) when \( z = 1/n \) (\( n = 2, 3, \ldots \)), then \( e^{z} = e^{i/n} \) is the set of nth roots of \( e \).

**Properties of \( e^z \):**

1) \( e^{z_1 + z_2} = e^{z_1} e^{z_2} \)
2) \( \frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2} \)
3) \( e^z \) is entire and \( \frac{d}{dz}e^z = e^z \).
4) \( e^z \neq 0 \) for any complex number \( z \).
5) \( e^z \) is periodic, with a pure imaginary period \( 2ni \):

\[
e^{z+2n\pi} = e^z.
\]
6) while \( e^x \) is never negative, there are values of \( e^z \) that are.

**Examples:**
29. THE LOGARITHMIC FUNCTION

Our motivation for the definition of the logarithmic function is based on solving the equation

$$e^w = z$$

for $w$, where $z$ is any nonzero complex number.

To do this, we note that when $z$ and $w$ are written $z = re^{i\Theta}$ ($-\pi < \Theta \leq \pi$) and $w = u + iv$, the above equation becomes

$$e^u e^{iv} = r e^{i\Theta}$$

then:

$$e^u = r \quad \text{and} \quad v = \Theta + 2n\pi \quad \text{where} \quad n \text{ is any integer.}$$

That is; $u = \ln r$, $w$ has one of the values

$$w = \ln r + i(\Theta + 2n\pi) \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Thus, if we write

$$\log z = \ln r + i(\Theta + 2n\pi) \quad (2)$$

we have the simple relation

$$e^{\log z} = z \quad (z \neq 0). \quad (3)$$

Note:

It is not true that the left-hand side of equation (3) with the order of the exponential and logarithmic functions reversed reduces to just $z$.

More precisely, since expression (2) can be written

$$\log z = \ln |z| + i \arg z$$

and since $|e^z| = e^x$ and $\arg(e^z) = y + 2n\pi \quad (n = 0, \pm 1, \pm 2, \ldots)$

when $z = x + iy$, we know that

$$\log(e^z) = \ln |e^z| + i \arg(e^z) = \ln(e^x) + i(y + 2n\pi) = (x + iy) + 2n\pi i$$

($n = 0, \pm 1, \pm 2, \ldots$).

That is,

$$\log (e^z) = z + 2n\pi i, \quad (n = 0, \pm 1, \pm 2, \ldots).$$
The principal value of $\log z$ is the value obtained from equation (2) when $n=0$ there and is denoted by $\log z$. Thus

$$\log z = \ln r + i\Theta$$

and $\log z = \log z + 2n\pi i$, \hspace{1cm} (n=0,\pm 1,\pm 2,\ldots)

It reduces to the usual logarithm in calculus when $z$ is a positive real number $z = r$.

Examples:
30. BRANCHES AND DERIVATIVES OF LOGARITHMS

If $z = r e^{i\theta}$ is a nonzero complex number, the argument $\theta$ has any one of the values $\theta = \Theta + 2n\pi \ (n = 0, \pm 1, \pm 2, \ldots)$, where $\Theta = \text{Arg} \ z$. Hence the definition

$$\log z = \ln r + i(\Theta + 2n\pi i)$$

of the multiple-valued logarithmic function in Sec. 29 can be written

$$\log z = \ln r + i \theta, \quad r > 0, \quad \alpha < \theta < \alpha + 2\pi.$$ 

where $\alpha$ denote any real number.

with components $u(r, \theta) = \ln r, \quad v(r, \theta) = \theta.$

is single-valued and continuous in the stated domain.

Notes:

1) Note that if the function (2) were to be defined on the ray $\theta = \alpha$, it would not be continuous there.

2) The function (2) is not only continuous but also analytic in the domain $r > 0, \quad \alpha < \theta < \alpha + 2\pi$, since the first-order partial derivatives of $u$ and $v$ are continuous there and satisfy the polar form $r u_r = v_\theta, \quad u_\theta = -r v_r.$

of the Cauchy-Riemann equations.

3) Since

$$\frac{d}{dz} \log z = e^{-i\theta} (u_r + i v_r) = e^{-i\theta} \left( \frac{1}{r} + i 0 \right) = \frac{1}{re^{i\theta}};$$

$$\frac{d}{dz} \log z = \frac{1}{z} \quad (|z| > 0, \quad \alpha < \text{arg} \ z < \alpha + 2\pi).$$

$$\frac{d}{dz} \log z = \frac{1}{z} \quad (|z| > 0, \quad -\pi < \text{Arg} \ z < \pi).$$
**Definition:** A branch of a multiple-valued function $f$ is any single-valued function $F$ that is analytic in some domain at each point $Z$ of which the value $F(z)$ is one of the values $f(z)$.

Observe that, for each fixed $\alpha$, the single-valued function (2) is a branch of the multiple-valued function (1). The function $\log z = \ln r + i\Theta (r>0, -\pi<\Theta<\pi)$ is called the principal branch.

**Note that:**

1) A branch cut is a portion of a line or curve that is introduced in order to define a branch $F$ of a multiple-valued function.

2) Points on the branch cut for $F$ are singular points of $F$, and any point that is common to all branch cuts of $f$ is called a branch point.

3) The origin and the ray $\theta = \alpha$ make up the branch cut for the branch (2) of the logarithmic function.

4) The branch cut for the principal branch consists of the origin and the ray $\Theta = \pi$.

5) The origin is evidently a branch point for branches of the multiple-valued logarithmic function.

Examples:
Let \( z_1 \) and \( z_2 \) denote any two nonzero complex numbers,

1) \( \log(z_1z_2) = \log z_1 + \log z_2 \).
   This statement is not, in general, valid when \( \log \) is replaced everywhere by \( \text{Log} \).

2) \( \log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2 \).

3) If \( z \) is a nonzero complex number, then \( z^n = e^{n \log z}, \ (n=0, \pm 1, \pm 2, \ldots) \).

4) It is also true that when \( z \neq 0 \),
\[
z^{1/n} = \exp\left(\frac{1}{n} \log z\right) \quad (n = 1, 2, \ldots).
\]
   This property is also true when \( n \) is negative integer.

Examples:
32. COMPLEX EXPONENTS

**Definition:**
When $z \neq 0$ and the exponent $c$ is any complex number, the function $z^c$ is defined by means of the equation

$$z^c = e^{c \log z}$$

where $\log z$ denotes the multiple-valued logarithmic function.

**EXAMPLE:**

**Properties:**

1) \[ \frac{1}{z^c} = z^{-c} \]

2) \[ \frac{d}{dz} z^c = cz^{c-1} \]

3) The principal value of $z^c$ occurs when $\log z$ is replaced by $\text{Log } z$. P.V. \[ z^c = e^{c \log z} \]

Examples:
Definition: The exponential function with base $c$, where $c$ is any nonzero complex constant, is written

$$c^z = e^{z \log c}$$

Note that although $e^z$ is, in general, multiple-valued the usual interpretation of $e^z$ occurs when the principal value of the logarithm is taken.

When a value of $\log c$ is specified, $c^z$ is an entire function of $z$. In fact,

$$\frac{d}{dz} c^z = c^z \log c.$$  

Examples:
33. TRIGONOMETRIC FUNCTIONS

**Definition:**

1) The sine and cosine functions of a complex variable \( z \) is defined as follows:

\[
\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}.
\]

2) These functions are **entire** since they are linear combinations of the entire functions \( e^{iz} \) and \( e^{-iz} \).

3) Knowing the derivatives of those exponential functions, we find that

\[
\frac{d}{dz} \sin z = \cos z, \quad \frac{d}{dz} \cos z = -\sin z
\]

4) It is easy to see from definitions (1) that

\[
\sin(-z) = -\sin z \quad \text{and} \quad \cos(-z) = \cos z;
\]

A variety of other identities from trigonometry are valid with complex variables.

5) \( \sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2, \)

6) \( \cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2; \)

7) \( \sin^2 z + \cos^2 z = 1, \)

8) \( \sin 2z = 2 \sin z \cos z, \quad \cos 2z = \cos^2 z - \sin^2 z, \)

9) \( \sin(z + \pi/2) = \cos z, \quad \sin (z - \pi/2) = -\cos z. \)

10) \( \sin z = \sin x \cosh y + i \cos x \sinh y, \)

11) \( \cos z = \cos x \cosh y - i \sin x \sinh y, \)

12) \( \sin(z + 2\pi) = \sin z, \quad \sin(z + \pi) = -\sin z; \)
13) \( \cos(z+2\pi) = \cos z \), \( \cos(z+\pi) = -\cos z \);

14) \( |\sin z|^2 = \sin^2 x + \sinh^2 y \);

15) \( |\cos z|^2 = \cos^2 x + \sinh^2 y \);

16) from the last two equations that \( \sin z \) and \( \cos z \) are not bounded on the complex plane, where as the absolute values of \( \sin x \) and \( \cos x \) are less than or equal to unity for all values of \( x \).

17) A zero of a given function \( f(z) \) is a number \( z_0 \) such that \( f(z_0) = 0 \). Thus:

\[
\sin z = 0 \quad \text{if and only if } \quad z = n\pi \quad (n = 0, \pm 1, \pm 2, \ldots).
\]

\[
\cos z = 0 \quad \text{if and only if } \quad z = \frac{\pi}{2} + n\pi \quad (n = 0, \pm 1, \pm 2, \ldots).
\]

Definition of the other trigonometric functions:

\[
\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z},
\]

\[
\sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}.
\]

Note that \( \tan z \) and \( \sec z \) are analytic everywhere except at the singularities

\[
z = \frac{\pi}{2} + n\pi \quad (n = 0, \pm 1, \pm 2, \ldots),
\]

While \( \cot z \) and \( \csc z \) have singularities at the zeros of \( \sin z \) namely;

\[
z = n\pi \quad (n = 0, \pm 1, \pm 2, \ldots).
\]

Their derivatives are:

\[
\frac{d}{dz} \tan z = \sec^2 z, \quad \frac{d}{dz} \cot z = -\csc^2 z,
\]

\[
\frac{d}{dz} \sec z = \sec z \tan z, \quad \frac{d}{dz} \csc z = -\csc z \cot z.
\]
The **period** of each of these trigonometric functions follows from their definitions, that is:

\[ \tan(z + \pi) = \tan z, \]
\[ \cot(z + \pi) = \cot z, \text{ while;} \]
\[ \sec(z + 2\pi) = \sec z, \]
\[ \csc(z + 2\pi) = \csc z. \]

Examples:
34. HYPERBOLIC FUNCTIONS

The *hyperbolic sine* and the *hyperbolic cosine* of a complex variable are defined as they are with a real variable; that is,

\[
\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}.
\]

Since \(e^z\) and \(e^{-z}\) are entire, it follows from definitions (1) that \(\sinh z\) and \(\cosh z\) are entire. Furthermore

\[
\frac{d}{dz} \sinh z = \cosh z, \quad \frac{d}{dz} \cosh z = \sinh z.
\]

The *hyperbolic sine* and *cosine* functions are closely related to those trigonometric functions:

\[
-i \sinh(iz) = \sin z, \quad \cosh(iz) = \cos z,
\]

\[
-i \sin(iz) = \sinh z, \quad \cos(iz) = \cosh z.
\]

Some of the most frequently used identities involving hyperbolic sine and cosine functions are

\[
\sinh(-z) = -\sinh z, \quad \cosh(-z) = \cosh z,
\]

\[
\cosh^2 z - \sinh^2 z = 1,
\]

\[
\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2,
\]

\[
\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2
\]

and

\[
\sinh z = \sinh x \cos y + i \cosh x \sin y,
\]

\[
\cosh z = \cosh x \cos y + i \sinh x \sin y,
\]

\[
|\sinh z|^2 = \sinh^2 x + \sin^2 y,
\]

\[
|\cosh z|^2 = \sinh^2 x + \cos^2 y,
\]

Example:
The period of \( \sinh z \) and \( \cosh z \) is \( 2\pi i \) and

\[
\sinh z = 0 \quad \text{if and only if} \quad z = n\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)
\]

and

\[
\cosh z = 0 \quad \text{if and only if} \quad z = \left( \frac{\pi}{2} + n\pi \right) i \quad (n = 0, \pm 1, \pm 2, \ldots).
\]

The hyperbolic tangent of \( z \) is defined by the equation

\[
\tanh z = \frac{\sinh z}{\cosh z}
\]

and is analytic in every domain in which \( \cosh z \neq 0 \)

**Differentiation Formulas:**

\[
\frac{d}{dz} \tanh z = \text{sech}^2 z, \quad \frac{d}{dz} \coth z = - \text{csch}^2 z,
\]

\[
\frac{d}{dz} \text{sech} z = - \text{sech} z \tanh z, \quad \frac{d}{dz} \text{csch} z = - \text{csch} z \coth z.
\]

**Examples:**
35. INVERSE TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

Inverses of the trigonometric and hyperbolic functions can be described in terms of logarithms. In order to define the inverse sine function $\sin^{-1} z$, we write $w = \sin^{-1} z$ when $z = \sin w$. That is, $w = \sin^{-1} z$ when

$$z = \frac{e^{iw} - e^{-iw}}{2i}.$$

If we put this equation in the form

$$(e^{iw})^2 - 2iz(e^{iw}) - 1 = 0,$$

which is quadratic in $e^{iw}$, and solve for $e^{iw}$ [see Exercise 8(a), Sec. 9], we find that

(1) $$e^{iw} = iz + (1 - z^2)^{1/2},$$

where $(1 - z^2)^{1/2}$ is, of course, a double-valued function of $z$. Taking logarithms of each side of equation (1) and recalling that $w = \sin^{-1} z$, we arrive at the expression

(2) $$\sin^{-1} z = -i \log[iz + (1 - z^2)^{1/2}].$$

Example:

One can apply the technique used to derive expression (2) for $\sin^{-1} z$ to show that

(3) $$\cos^{-1} z = -i \log[z + i(1 - z^2)^{1/2}]$$

and that

(4) $$\tan^{-1} z = \frac{i}{2} \log \frac{i + z}{i - z}.$$
The derivatives of these three functions are readily obtained from the above expressions. The derivatives of the first two depend on the values chosen for the square roots:

\[
\frac{d}{dz} \sin^{-1} z = \frac{1}{(1 - z^2)^{1/2}},
\]

\[
\frac{d}{dz} \cos^{-1} z = \frac{-1}{(1 - z^2)^{1/2}}.
\]

The derivative of the last one,

\[
\frac{d}{dz} \tan^{-1} z = \frac{1}{1 + z^2},
\]

Inverse hyperbolic functions can be treated in a corresponding manner. It turns out that

\[
\sinh^{-1} z = \log[z + (z^2 + 1)^{1/2}],
\]

\[
\cosh^{-1} z = \log[z + (z^2 - 1)^{1/2}],
\]

and

\[
\tanh^{-1} z = \frac{1}{2} \log \frac{1 + z}{1 - z}.
\]

Examples: