Chapter 2

First-Order Partial Differential Equations

2.1 Linear Partial Differential Equations of First Order

The most general first-order linear PDE in two independent variables \( x \) and \( t \) has the form

\[
au_x + bu_t = cu + d \tag{2.1.1}
\]

where \( a, b, c, d \) are functions of \( x \) and \( t \) only. We single out the variable \( t \) (often “time” in physical problems) and write the first-order general PDE in the “normal” form

\[
u_t + F(x, t, u, u_x) = 0.
\]

The general solution of a first-order PDE involves an arbitrary function. In applications one is usually interested not in obtaining the general solution of a PDE, but a solution subject to some additional condition such as an initial condition (IC) or a boundary condition (BC) or both.

A basic problem for first-order PDEs is to solve

\[
u_t + F(x, t, u, u_x) = 0, \quad x \in R, \quad t > 0 \tag{2.1.2}
\]

subject to the IC

\[
u(x, 0) = u_0(x), \quad x \in R \tag{2.1.3}
\]

where \( u_0(x) \) is a given function. (The interval of interest for \( x \) may be finite.) This is called a Cauchy problem; it is a pure initial value problem. It may be viewed as a signal or wave at time \( t = 0 \). The initial signal or wave is a space distribution of \( u \), and a “picture” of the wave may be obtained by drawing the graph of \( u = u_0(x) \) in the \( xu \)-space. Then the PDE
(2.1.2) may be interpreted as the equation that describes the propagation of the wave as time increases.

We first consider the wave equation

$$u_t + cu_x = 0 \quad (2.1.4)$$

with the IC

$$u(x, 0) = u_0(x), \quad (2.1.5)$$

where $c$ is a constant.

If $x = x(t)$ defines a smooth curve $C$ in the $(x, t)$ plane, the total derivative of $u = u(x, t)$ along a curve is found by using the chain rule:

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \frac{dx}{dt}.$$ 

The left-hand side of (2.1.4) is a total derivative of $u$ along the curves defined by the equation $\frac{dx}{dt} = c$. Therefore, equation (2.1.4) is equivalent to the statement

$$\frac{du}{dt} = 0 \text{ along the curves } \frac{dx}{dt} = c. \quad (2.1.6)$$

From (2.1.6) we find that

$$u = \text{ constant along the curves } x - ct = \xi \quad (2.1.7)$$

where $\xi$ is constant of integration. For different values of $\xi$ we get a family of curves in the $(x, t)$ plane. A curve of the family through an arbitrary point $(x, t)$ intersects the $x$-axis at $(\xi, 0)$. Since $u$ is constant on this curve, its value $u(x, t)$ is equal to its value $u(\xi, 0)$ at the initial time:

$$u = u(x, t) = u(\xi, 0) = u_0(\xi) = u_0(x - ct) \quad (2.1.8)$$

$u_0(x - ct)$ is the solution to the IVP (2.1.4) - (2.1.5).

The curves defined by (2.1.6) are called “characteristic curves” or simply characteristics of the PDE (2.1.4). A characteristic in the $xt$-space represents a moving wavelet in the $x$-space, $\frac{dx}{dt}$ being its speed. The greater the inclination of the line with the $t$-axis, the greater will be the speed of the corresponding wavelet. Signals or wavelets are propagated along the characteristics. Also, along the characteristics the PDE reduces to a system of ODEs (see (2.1.6)). At the initial time $t = 0$ the wave has the form $u_0(x)$. At a later time $t$ the wave profile is $u_0(x - ct)$. This shows that in time $t$ the initial profile is translated to the right a distance $ct$. Thus, $c$ represents the speed of the wave.
Example 1
\[ \frac{\partial u}{\partial t} + t^2 \frac{\partial u}{\partial x} = 0, \quad u(x, 0) = f(x). \]

It is clear that \( \frac{du}{dt} = 0 \) along the characteristic curves \( \frac{dx}{dt} = t^2 \). On integration we get \( x = \frac{t^3}{3} + \xi \) so that
\[ u = \text{constant on } x = \xi + \frac{t^3}{3}. \]
Therefore,
\[ u(x, t) = u(\xi, 0) = f(\xi) = f \left( x - \frac{t^3}{3} \right). \]
The solution \( u(x, t) = f \left( x - \frac{t^3}{3} \right) \) has a travelling wave form \( u(x, t) = f(\eta), \eta = x - \frac{t^3}{3} \). The travelling wave moves with a nonconstant speed \( t^2 \) and a nonconstant acceleration \( 2t \).

The method of characteristics can also be applied to solve IVP for a nonhomogeneous PDE of the form \( u_t + c(x, t) u_x = f(x, t), x \in \mathbb{R}, t > 0, \ u(x, 0) = u_0(x). \)

Example 2
\[ \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = e^{-3x}, \ u(x, 0) = f(x). \]
We note that
\[ \frac{du}{dt} = e^{-3x} \text{ along } \frac{dx}{dt} = c. \]
This pair of ODEs can be solved subject to the IC \( x = \xi, \ u = f(\xi) \) at \( t = 0. \)
We get
\[ x = ct + \xi \]
and
\[ \frac{du}{dt} = e^{-3(ct+\xi)}. \]
On integration we have
\[ u(x, t) = \frac{e^{-3ct}}{-3c} e^{-3\xi} + g(\xi) \]
where \( g \) is the function of integration. Applying the IC we get
\[ g(\xi) = \frac{e^{-3\xi}}{3c} + f(\xi). \]

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Thus,

\[ u(x,t) = e^{-\xi} (1 - e^{-3ct}) + f(\xi) = e^{-\frac{3(x-ct)}{3c}} (1 - e^{-3ct}) + f(x-ct). \]

The solution here is of the similarity form \( u(x,t) = \alpha(x,t) + \beta(\eta) \), where \( \eta = x - ct \) is the similarity variable, a linear combination of the independent variables \( x \) and \( t \).

**Example 3**

\[ \frac{\partial u}{\partial t} + x \frac{\partial u}{\partial x} = t, \quad u(x,0) = f(x). \]

Here \( \frac{du}{dt} = t \) along \( \frac{dx}{dt} = x \),

which on integration yields

\[ x = \xi e^t \]

and

\[ u(x,t) = \frac{t^2}{2} + g(\xi). \]

At \( t = 0 \), \( x = \xi \), \( u = f(\xi) \); therefore, \( g(\xi) = f(\xi) \). Thus

\[ u = \frac{t^2}{2} + f(\xi) = \frac{t^2}{2} + f(xe^{-t}). \]

The solution here has the similarity form

\[ u = \alpha(x,t) + \beta(\eta) \]

where \( \eta = xe^{-t} \) is the similarity variable.

**Example 4**

\[ xu_x + (x^2 + y)u_y + \left( \frac{y}{x} - x \right) u = 1. \]

The characteristics are given by

\[ \frac{dx}{dt} = x, \quad \frac{dy}{dt} = x^2 + y, \quad \frac{du}{dt} + \left( \frac{y}{x} - x \right) u = 1, \]

the first two of which give the locus in the \((x,y)\) plane, the so-called traces,

\[ \frac{dy}{dx} = x + \frac{y}{x} \]
which on integration become
\[ \frac{y}{x} - x = \text{constant}. \]

It is often easier to find the general solution of the PDE by introducing the variable describing the trace curves as a new independent variable: \( \phi = \frac{y}{x} - x \). The given PDE then becomes
\[ x \left( \frac{\partial u}{\partial x} \right) \phi + \phi u = 1 \]
which on integration with respect to \( \phi \) gives
\[ u = \phi^{-1} + x^{-\phi} f(\phi) \]
where \( f \) is an arbitrary function of \( \phi \).

2.2 Quasilinear Partial Differential Equations of First Order

The general first-order quasilinear equation has the form
\[ au_x + bu_t = c, \quad (2.2.1) \]
where \( a, b, \) and \( c \) are functions of \( x, t, \) and \( u \). Quasilinear PDEs are simpler to treat than fully nonlinear ones for which \( u_x \) and \( u_t \) may not occur linearly.

The solution \( u = u(x,t) \) of (2.2.1) may be interpreted geometrically as a surface in \((x,t,u)\)-space, called an “integral surface.”

The Cauchy problem for (2.2.1) requires that \( u \) assume prescribed values on some plane curve \( C \). If \( s \) is a parametric on \( C \), its representation is \( x = x(s), \ t = t(s) \). We may prescribe \( u = u(s) \) on \( C \). The ordered triple \((x(s),t(s),u(s))\) defines a curve \( \Gamma \) in the \((x,t,u)\)-space; \( C \) is the projection of \( \Gamma \) onto the \((x,t)\) plane. Thus, generally, the problem is to find the solution or an integral surface \( u = u(x,t) \) containing the three-dimensional curve \( \Gamma \). The direction cosines of the normal \( \vec{n} \) to the surface \( u(x,t) - u = 0 \) are proportional to the components of \( \text{grad} \ (u(x,t) - u) = (u_x, u_t, -1) \).

If we define the vector \( \vec{e} = (a,b,c) \), then the PDE (2.2.1) can be written as \( \vec{e} \cdot \vec{n} = 0 \). In other words, the vector direction \((a,b,c)\) is tangential to the integral surface at each point. The direction \((a,b,c)\) at any point on the surface is called the “characteristic direction.” A space curve whose tangent at every point coincides with the characteristic direction is called a “characteristic curve” and is given by the equations
\[ \frac{dx}{a} = \frac{dt}{b} = \frac{du}{c}. \quad (2.2.2) \]
The characteristics are curves in the \((x,t,u)\)-space and lie on the integral surface. The projections of the characteristic curves onto the \((x,t)\) plane are called “base characteristics” or “ground characteristics.” Integration of (2.2.2) is not easy as \(a, b, c\) now depend upon \(u\) as well. Prescribing \(u\) at one point of the characteristic enables one to determine \(u\) all along it. We assume that all the smoothness conditions on the functions \(a, b, \) and \(c\) are satisfied so that the system of ODEs (2.2.2) has a unique solution starting from a point on the initial curve. Lagrange proved that solution of Equation (2.2.1) is given by

\[ F(\phi, \psi) = 0 \quad \text{or} \quad \phi = f(\psi), \]

where \(\phi(x,t,u)\) and \(\psi(x,t,u)\) are independent functions (that is, normals to the surfaces \(\phi = \text{constant}\) and \(\psi = \text{constant}\) are not parallel at any point of intersection) such that

\[ a\phi_x + b\phi_t + c\phi_u = 0, \quad a\psi_x + b\psi_t + c\psi_u = 0 \quad (2.2.3) \]

(The functions \(F\) and \(f\) are themselves arbitrary). \(F(\phi, \psi) = 0\), called the “general integral,” is an implicit relation between \(x, t,\) and \(u\). Oftentimes it is possible to solve for \(u\) in terms of \(x\) and \(t\). If \(\phi = \text{constant}\) is a first integral of (2.2.2), it satisfies (2.2.3). A second integral of (2.2.2), \(\psi = \text{constant}\), also satisfies (2.2.3). Equation (2.2.2) represents the curves of intersection of the surfaces \(\phi = c_1\) and \(\phi = c_2\), where \(c_1\) and \(c_2\) are arbitrary constants. We thus have a two-parameter family of curves. If we impose the condition \(F_1(c_1, c_2) = 0\) we get a one-parameter family of characteristics. An integral surface can be constructed by drawing characteristics from each point of the initial curve. Note that (2.2.2) may be written in the parametric form

\[ \frac{dx}{d\tau} = a, \quad \frac{dt}{d\tau} = b, \quad \frac{du}{d\tau} = c \quad (2.2.4) \]

where \(\tau\) is a parameter measured along the characteristic.

One may also obtain a solution of (2.2.4) in the form \(x = x(s,\tau), t = t(s,\tau),\) and \(u = u(s,\tau)\), where \(s\) is a parameter measured along the initial curve. Solving for \(s\) and \(\tau\) in terms of \(x\) and \(t\) from the first two equations and substituting in \(u = u(s,\tau)\), one gets \(u\) as a function of \(x\) and \(t\).

**Example 1**

Find the general solution of \((t + u)u_x + tu_t = x - t\). Also find the integral surface containing the curve \(t = 1, u = 1 + x, -\infty < x < \infty\).

The characteristics of the given PDE are given by

\[ \frac{dx}{t + u} = \frac{dt}{t} = \frac{du}{x - t}. \]
It is easy to see that

\[
\frac{d(x + u)}{x + u} = \frac{dt}{t}.
\]

On integration we have

\[
\frac{x + u}{t} = c_1
\]

where \(c_1\) is a constant. Again

\[
\frac{d(x - t)}{u} = \frac{du}{x - t},
\]

implying

\[
(x - t)^2 - u^2 = c_2,
\]

where \(c_2\) is another constant.

The general solution, therefore, is

\[
(x - t)^2 - u^2 = f\left(\frac{x + u}{t}\right).
\]

If the integral surface contains the given curve \(t = 1, u = 1 + x\), we have

\[
(x - 1)^2 - (1 + x)^2 = f(1 + 2x),
\]

or

\[
f(1 + 2x) = -4x
\]

implying that

\[
f(z) = -2(z - 1)
\]

and so

\[
f\left(\frac{x + u}{t}\right) = -2\left(\frac{x + u}{t} - 1\right).
\]

The solution therefore is

\[
(x - t)^2 - u^2 = \frac{2}{t}(x + u - t).
\]

Solving for \(u\), we have

\[
u = \frac{1}{t} \pm \left(x - t + \frac{1}{t}\right).
\]

The condition \(u = 1 + x\) when \(t = 1\) is satisfied only if we take the positive sign. Thus, the solution of the IVP is

\[
u = \frac{2}{t} + x - t.
\]

Clearly, the solution is defined only for \(t > 0\).

While the general solution is quite implicit, the solution of IVP has the form \(u = f(t) + g(\eta), \eta = x - t\), and may be found by similarity methods.

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Example 2

Find the general solution of

\[(t^2 - u^2)u_x - xt u_t = xu.\]

Also find the integral surface containing the curve \(x = t = u, \quad x > 0.\)

The characteristics of the given PDE are

\[\frac{dx}{t^2 - u^2} = \frac{dt}{-xt} = \frac{du}{xu}.\]

A first integral obtained from the second pair is \(\phi(x, t, u) \equiv ut = c_1,\) say.

Each of the above ratios is equal to

\[\frac{xdx + tdt + udu}{x(t^2 - u^2) + t(-xt) + u(xu)} = \frac{xdx + tdt + udu}{0}.\]

Therefore, a second integral is \(\psi(x, t, u) \equiv x^2 + t^2 + u^2 = c_2,\) say. The general solution, therefore, is \(\phi = f(\psi),\) that is,

\[ut = f(x^2 + t^2 + u^2).\]

Applying the initial condition \(x = t = u,\) we get

\[x^2 = f(3x^2),\]

giving

\[f(z) = \frac{z}{3}.\]

Therefore we get the special solution satisfying the IC as

\[ut = \frac{x^2 + t^2 + u^2}{3}.\]

Solving the quadratic in \(u\) we find that

\[u = \frac{3t - (5t^2 - 4x^2)^{1/2}}{2},\]

the root with the negative sign satisfying the given conditions.

Here, again, the general solution is rather implicit. The special solution satisfying given IC may be obtained by the similarity approach.

Conservation Laws

Considerable interest attaches to the quasilinear equations of the form

\[u_t + (f(u))_x = 0;\]
it is a divergence form or a conservation law. A simple model of traffic on a highway yields a conservation law of this type.

Consider a single-lane highway occupied by moving cars. We can define a density function $u(x, t)$ as the number of cars per unit length at the point $x$ measured from some fixed point on the road at time $t$. The flux of vehicles $\phi(x, t)$ is the number of cars per unit time (say, hour) passing a fixed place $x$ at time $t$. Here we regard $u$ and $\phi$ as continuous functions of the distance $x$. If we consider an arbitrary section of the highway between $x = a$ and $x = b$, then the number of cars between $x = a$ and $x = b$ at time $t$ is equal to $\int_a^b u(x, t) \, dx$. Assuming that there are neither entries nor exits on this section of the road, the time rate of change of the number of cars in the section $[a, b]$ equals the number of cars per unit time entering at $x = a$ minus the number of cars per unit time leaving at $x = b$. That is

$$\frac{d}{dt} \int_a^b u(x, t) \, dx = \phi(a, t) - \phi(b, t)$$

or

$$\int_a^b \frac{du}{dt} \, dx = - \int_a^b \frac{d\phi}{dx} (x, t) \, dx.$$

This yields the conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} = 0 \quad (2.2.5)$$

since the interval $[a, b]$ is arbitrary. If we assume that the flux $\phi$ depends on the traffic density $u$, then the conservation equation becomes

$$\frac{\partial u}{\partial t} + \phi'(u) \frac{\partial u}{\partial x} = 0$$

or

$$\frac{\partial u}{\partial t} + c(u) \frac{\partial u}{\partial x} = 0$$

where $c(u) = \phi'(u)$.

Considering this, we see that $\frac{du}{dt} = 0$ along the characteristic $\frac{dx}{dt} = c(u)$. Unlike the linear case, the characteristic curves cannot in general be determined in advance since $u$ is yet unknown. But, in the special case considered here, since $u$ and $c(u)$ remain constant on a characteristic, the latter must be a straight line in the $(x, t)$ plane. If, through an arbitrary point $(x, t)$, we draw a characteristic back in time, it will cut the $x$-axis at the point $(\xi, 0)$. If $u = u_0(x)$ at $t = 0$, the equation of this characteristic is

$$x = \xi + c(u_0(\xi))t. \quad (2.2.6)$$

Since $u$ remains constant along this characteristic,

$$u(x, t) = u(\xi, 0) = u_0(\xi). \quad (2.2.7)$$
As $\xi$ varies, we get different characteristics. Equations (2.2.6) and (2.2.7) give the implicit solution $u(x,t) = u_0[x - c(u_0(\xi))t]$.

**Shock waves**

In the case of quasilinear equations, two characteristics may intersect. Consider the characteristics $C_1$ and $C_2$, starting from the points $x = \xi_1$ and $x = \xi_2$, respectively. Along $C_1$, $u(x,t) = u_0(\xi_1) = u_1$, say. Along $C_2$, $u(x,t) = u_0(\xi_2) = u_2$. The speeds of the characteristics are $c(u_1)$ and $c(u_2)$. If $c(u_1) > c(u_2)$, the angle characteristic $\xi_1$ makes with the $t$-axis is greater than that which the characteristic $\xi_2$ makes with it, and so they intersect. This means that, at the point of intersection $P$, $u$ has simultaneously two values, $u_1$ and $u_2$. This is unphysical since $u$ (usually a density in physical problems) cannot have two values at the same time. To overcome this difficulty we assume that the solution $u$ has a jump discontinuity. It is found that the discontinuity in $u$ propagates along special loci in space time. The trajectory $x = x_s(t)$ in the $(x,t)$ plane along which the discontinuity, called a shock, propagates is referred to as the “shock path” or “shock trajectory;” $\frac{dx_s(t)}{dt}$ is the shock speed. The shock path is not a characteristic curve.

Let $u(x,0)$ be the initial distribution of $u$ (some density). The dependence of $c$ on $u$ produces nonlinear distortion of the wave as it propagates. When $c'(u) > 0$ ($c$ is an increasing function of $u$), higher values of $u$ propagate faster than the lower ones. As a result, the initial wave profile distorts. The density distribution becomes steeper as time increases and the slope becomes infinite at some finite time, called the “breaking time.”

We now determine how the discontinuity is formed and propagates. At the discontinuity the PDE itself does not apply (We assume that all the derivatives exist in the flow region). Equation $u_t + c(u)u_x = 0$ holds on either side. It may be written in the conservation form

$$u_t + \phi_x = 0$$

where $\phi'(u) = c(u)$. If $v(x,t)$ is the velocity at $(x,t)$, then the flux $\phi(x,t) = u(x,t)v(x,t)$. Conservation of density at the discontinuity requires (relative inflow equals relative outflow)

$$u(x_s-, t) \left[ v(x_s-, t) - \frac{dx_s}{dt} \right] = u(x_s+, t) \left[ v(x_s+, t) - \frac{dx_s}{dt} \right].$$

Solving for $\frac{dx_s}{dt}$, we get the shock velocity as
\[
\frac{dx_s}{dt} = \frac{\phi(x_s+,t) - \phi(x_s-,t)}{u(x_s+,t) - u(x_s-,t)} = \frac{[\phi]}{[u]} \quad (2.2.8)
\]

where \([\phi]\) and \([u]\) denote jumps in \(\phi\) and \(u\) across the shock, respectively.

Consider the IVP

\[
u_t + uu_x = 0 \quad (2.2.9)
\]

\[
u(x,0) = \begin{cases} 
1 & x < 0 \\
0 & x > 0
\end{cases}
\]

Equation (2.2.9) can be written in the conservation form as

\[
u_t + \phi_x = 0
\]

where the flux \(\phi = \frac{u^2}{2}\). The jump condition (2.2.8) becomes

\[
\frac{dx_s}{dt} = \frac{[\phi]}{[u]} = \frac{\phi_+ - \phi_-}{u_+ - u_-} = \frac{u_+^2 - u_-^2}{u_+ - u_-} = \frac{u_+ + u_-}{2}
\]

where the subscripts \(+\) and \(-\) indicate that the quantity is evaluated at \(x_s+\) and \(x_s-\), respectively. Thus, the shock speed is the average of the values of \(u\) ahead of and behind the shock.

Again, (2.2.9) implies that \(\frac{du}{dt} = 0\) along the characteristic \(\frac{dx}{dt} = u\); in other words, \(u\) = constant along the straight line characteristics having speed \(u\). Characteristics starting from the \(x\)-axis have speed unity if \(x < 0\) and zero if \(x > 0\). So at \(t = 0^+\), the characteristics intersect and a shock is produced. The shock speed \(\frac{dx_s}{dt} = 0 + \frac{1}{2} = \frac{1}{2}\), and hence the shock path is \(x = \frac{t}{2}\). The initial discontinuity at \(x = 0\) propagates along this path with speed \(\frac{1}{2}\). A solution to the IVP is

\[
u(x, t) = \begin{cases} 
1 & \text{if } x < \frac{1}{2}t \\
0 & \text{if } x > \frac{1}{2}t
\end{cases}
\]

In the present example there is a discontinuity in the initial data and a shock is formed immediately. Even when the initial condition \(\nu(x,0) = \nu_0(x)\) is continuous, a discontinuity may be formed in a finite time.

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Consider the characteristics coming out of point \( x = \xi \) on the initial line

\[ x = \xi + F(\xi)t, \]

where \( F(\xi) = c(u_0(\xi)) \). Differentiating this equation with respect to \( t \) we get

\[ 0 = \xi_t + F(\xi) + F'(\xi)\xi_t \]

or

\[ \xi_t = \frac{-F(\xi)}{1 + F'(\xi)t}. \]

Since

\[ u = u_0(\xi), \]

we have

\[ u_t = u'_0(\xi)\xi_t \]

\[ = \frac{-u'_0(\xi)F(\xi)}{1 + F'(\xi)t}. \]

It is clear that for \( u_t \) (and hence \( u_x \)) to become infinite we must have \( F'(\xi) < 0 \). The breaking of the wave first occurs on the characteristic \( \xi = \xi_B \) for which \( F'(\xi) < 0 \) and \( |F'(\xi)| \) is a maximum. The time of first breaking of the wave is

\[ t_B = -\frac{1}{F'(\xi_B)}. \]

**Example 1**

\[ u_t + 2uu_x = 0 \]

\[ u(x, 0) = \begin{cases} 3 & x < 0 \\ 2 & x > 0 \end{cases} \]

The given PDE in conservation form is

\[ u_t + \phi_x = 0 \]

where \( \phi = u^2 \). Here, \( \frac{du}{dt} = 0 \) along \( \frac{dx}{dt} = 2u \), that is, \( u \) is constant along the straight line characteristics having speed \( 2u \). For \( x < 0 \) the speed of the characteristic is \( \frac{dx}{dt} = 6 \), an integration yields the equation of the characteristic as

\[ x = 6t + \xi \]

where \( \xi \) is constant of integration. For \( x > 0 \) the characteristic speed is 4 and the corresponding characteristics are \( x = 4t + \xi \). For \( t > 0 \) the
characteristics collide immediately and a shock wave is formed. The slope of the shock is given by
\[ \frac{dx_s}{dt} = \frac{\phi}{u} = \frac{\phi(3) - \phi(2)}{3 - 2} = 5. \]

The shock path is clearly \( x = 5t \). The solution of the problem is \( u(x, t) = 3 \) for \( x < 5t \) and \( u(x, t) = 2 \) for \( x > 5t \).

We now consider examples of the form
\[ u_t + c(x, t, u)u_x = f(x, t, u), \quad x \in \mathbb{R}, \, t > 0 \]
\[ u(x, 0) = u_0(x), \quad x \in \mathbb{R}. \]

**Example 2**

\[ u_t - u^2u_x = 3u, \quad x \in \mathbb{R}, \, t > 0 \]
\[ u(x, 0) = u_0(x), \quad x \in \mathbb{R} \]

Here, \( \frac{du}{dt} = 3u \) along the characteristics \( \frac{dx}{dt} = -u^2 \). This system of ODEs must be solved subject to the IC \( u = u_0(\xi), \quad x = \xi \) at \( t = 0 \). We have, on integration of the first, the result \( u = ke^{3t} \) where \( k \) is constant of integration. Since \( u = u_0(\xi) \) at \( t = 0 \), we have
\[ u = u_0(\xi)e^{3t}. \quad (2.2.10) \]

Now \( \frac{dx}{dt} = -u_0^2(\xi)e^{6t} \). Therefore, using the initial condition \( x = \xi \) at \( t = 0 \), we get
\[ x = \xi + \frac{u_0^2(\xi)}{6}(1 - e^{6t}). \quad (2.2.11) \]

Equations (2.2.10) and (2.2.11) constitute (an implicit) solution of the given initial value problem.

**Example 3**

\[ u_t + uu_x = -u, \quad x \in \mathbb{R}, \, t > 0 \]
\[ u(x, 0) = -\frac{x}{2}, \quad x \in \mathbb{R} \]

Here,
\[ \frac{du}{dt} = -u \quad \text{along} \quad \frac{dx}{dt} = u. \]
Solving the first equation with 1C \( x = \xi, \ u = -\frac{\xi}{2} \) at \( t = 0 \) we have

\[ u = -\frac{\xi}{2} e^{-t}. \tag{2.2.12} \]

Integrating \( \frac{dx}{dt} = -\frac{\xi}{2} e^{-t} \) and using the initial conditions, we get

\[ x = \frac{\xi}{2}(1 + e^{-t}). \tag{2.2.13} \]

Substituting \( \xi \) from (2.2.13) into (2.2.12) we get the solution

\[ u(x,t) = -\frac{xe^{-t}}{1 + e^{-t}}. \]

**Example 4**

Consider the IVP

\[ u_t + uu_x = 0, \quad x \in \mathbb{R}, t > 0 \]

\[ u(x,0) = 0, \text{ if } x < 0; \quad u(x,0) = 1 \text{ if } x > 0. \]

Here, \( \frac{du}{dt} = 0 \) along characteristics \( \frac{dx}{dt} = u \). Characteristics issuing from the \( x \)-axis have speed zero if \( x < 0 \) and 1 if \( x > 0 \). There is a void between \( x = 0 \) and \( x = t \) for \( t > 0 \). We can imagine that all values of \( u \) between 0 and 1 are present initially at \( x = 0 \). In this void, continuous solution can be constructed which connects the solution \( u = 1 \) ahead to the solution \( u = 0 \) behind. We insert a fan of characteristics (which are straight lines here) passing through the origin. Each member of the fan has a different (constant) slope. The value of \( u \) these characteristics carry varies continuously from 0 to 1. That is, \( u = c \) (constant), \( 0 < c < 1 \), on the characteristic \( x = ct \). Thus, the solution is

\[ u(x,t) = \begin{cases} 
0 & \text{for } x < 0 \\
\frac{x}{t} & \text{for } 0 < \frac{x}{t} < 1 \\
1 & \text{for } x > t.
\end{cases} \tag{2.2.14} \]

A solution of this form is called a “centred expansion wave”; it is clearly a similarity solution.

**Example 5**

\[ x(y^2 + u)u_x - y(x^2 + u)u_y = (x^2 - y^2)u \]

\[ u = 1 \quad \text{on } x + y = 0 \]
The characteristic equations are
\[ \frac{dx}{x(y^2 + u)} = \frac{dy}{-y(x^2 + u)} = \frac{du}{(x^2 - y^2)u} \]
which, on some manipulation, give
\[ \frac{dx}{x} + \frac{dy}{y} + \frac{du}{u} = 0 \]  \hspace{1cm} (2.2.15)
and
\[ xdx + ydy - du = 0. \]  \hspace{1cm} (2.2.16)
Equations (2.2.15) and (2.2.16) integrate to give
\[ xyu = C_1 \]
and
\[ x^2 + y^2 - 2u = C_2 \]
where \( C_1 \) and \( C_2 \) are arbitrary constants. The general solution therefore is
\[ x^2 + y^2 - 2u = f(xy). \]  \hspace{1cm} (2.2.17)
The initial data \( u = 1 \) on \( x + y = 0 \) gives \( f(-x^2) = 2x^2 - 2 \) or \( f(x^2) = -2x^2 - 2 \). Thus, the general solution (2.2.17) in this case reduces to
\[ 2xyu + x^2 + y^2 - 2u + 2 = 0. \]

**Example 6**

\[ xu_x + yu_y = x \exp(-u) \]
\[ u = 0 \text{ on } y = x^2 \]
The characteristic equations are
\[ \frac{dx}{x} = \frac{dy}{y} = \frac{du}{x \exp(-u)} \]  \hspace{1cm} (2.2.18)
and have the first integrals
\[ \frac{y}{x} = C_1 \]
and
\[ e^u = x + C_2 \]
from the first and second and first and third of (2.2.18), respectively. \( C_1 \) and \( C_2 \) are arbitrary constants. The general solution of the given PDE therefore is
\[ e^u = x + g(y/x), \]  \hspace{1cm} (2.2.19)
which is a similarity form for the dependent variable \( U = e^u \). If we use the given 1C, we get \( g(x) = 1 - x \), and so (2.2.19) in this case becomes
\[
e^u = x + 1 - \frac{y}{x}
\]
or
\[
u = \ln \left( x + 1 - \frac{y}{x} \right).
\]

**Direct Similarity Approach for First-Order PDEs**

Although we discuss self-similar solutions in detail in Chapter 3, here we give two examples to illustrate the simple approach of Clarkson and Kruskal (1989) which is direct and does not require group theoretic ideas.

**Example 1**

\[
u_t + uu_x = 0. \tag{2.2.20}
\]

We assume that (2.2.20) has solution of the form
\[
u(x, t) = \alpha(x, t) + \beta(x, t)H(\eta), \eta = \eta(x, t), \beta(x, t) \neq 0. \tag{2.2.21}
\]

Differentiating (2.2.21) to get \( \nu_t \) and \( \nu_x \) and, hence, substituting in (2.2.20), we have
\[
\beta^2 \eta_xHH' + \beta \beta_x H^2 + \beta(\eta_t + \alpha \eta_x)H' + (\beta_t + \alpha \beta_x + \beta \alpha_x)H + (\alpha_t + \alpha \alpha_x) = 0. \tag{2.2.22}
\]

Equation (2.2.22) becomes an ODE for the determination of the similarity function \( H(\eta) \) if
\[
\beta \beta_x = \beta^2 \eta_x \Gamma_1(\eta) \tag{2.2.23}
\]
\[
\beta(\eta_t + \alpha \eta_x) = \beta^2 \eta_x \Gamma_2(\eta) \tag{2.2.24}
\]
\[
\beta_t + \alpha \beta_x + \beta \alpha_x = \beta^2 \eta_x \Gamma_3(\eta) \tag{2.2.25}
\]
\[
\alpha_t + \alpha \alpha_x = \beta^2 \eta_x \Gamma_4(\eta). \tag{2.2.26}
\]

Equation (2.2.22) then becomes
\[
HH' + \Gamma_1(\eta)H^2 + \Gamma_2(\eta)H' + \Gamma_3(\eta)H + \Gamma_4(\eta) = 0. \tag{2.2.27}
\]

We solve (2.2.23) - (2.2.26) to obtain the unknown functions \( \alpha, \beta, \eta, \Gamma_1, \Gamma_2, \Gamma_3, \) and \( \Gamma_4 \). In the process of solution the following remarks are found useful.

**Remark 1**

If \( \alpha(x, t) \) has the form \( \alpha(x, t) = \dot{\alpha}(x, t) + \beta(x, t)\Omega(\eta) \), then we may set \( \Omega \equiv 0 \).
**Remark 2**

If \( \beta(x, t) \) is to be determined from an equation of the form \( \beta(x, t) = \hat{\beta}(x, t)\Omega(\eta) \), then we may put \( \Omega \equiv 1 \).

Setting \( \Gamma_1(\eta) = \frac{\Omega_1'(\eta)}{\Omega_1(\eta)} \) in (2.2.23) and integrating with respect to \( x \), we obtain

\[
\beta = B(t)\Omega_1(\eta). \tag{2.2.28}
\]

Using **Remark 2** in (2.2.28), we set \( \Omega_1(\eta) \equiv 1 \) so that

\[
\Gamma_1(\eta) = 0 \tag{2.2.29}
\]

and from (2.2.23)

\[
\beta = B(t). \tag{2.2.30}
\]

Substituting (2.2.30) in (2.2.25) we get

\[
\alpha_x = B(t)\eta_x\Gamma_3(\eta) - \frac{B'(t)}{B(t)} \tag{2.2.31}
\]

Setting \( \Gamma_3(\eta) = \Omega_3'(\eta) \) in (2.2.31) and integrating with respect to \( x \), we obtain

\[
\alpha = B(t)\Omega_3(\eta) - \frac{B'(t)}{B(t)}x + A(t). \tag{2.2.32}
\]

Making use of **Remark 1** in (2.2.32) we set \( \Omega_3 \equiv 0 \), and so

\[
\Gamma_3 \equiv 0 \tag{2.2.33}
\]

from (2.2.31) and (2.2.32). Equation (2.2.32) now reduces to

\[
\alpha = A(t) - \frac{B'(t)}{B(t)}x. \tag{2.2.34}
\]

Using (2.2.30), (2.2.24) is written as a first-order PDE for \( \eta \),

\[
\eta_t + \left[ \alpha - B(t)\Gamma_2(\eta) \right] \eta_x = 0,
\]

with the characteristic equations

\[
\frac{dt}{1} = \frac{dx}{\alpha - B\Gamma_2(\eta)} = \frac{d\eta}{0}. \tag{2.2.35}
\]

The second equation in (2.2.35) gives

\[
\eta = \text{constant} = S, \text{ say}, \tag{2.2.36}
\]

as the similarity variable. Setting \( \Gamma_2(\eta) = l \), where \( l \) is a constant, the first equation in (2.2.35) becomes

\[
\frac{dx}{dt} + \frac{B'}{B}x = A(t) - lB(t). \tag{2.2.37}
\]
Here we have used (2.2.34) for $\alpha$. The general solution of (2.2.37) is

$$\eta = xB(t) - \int A(t)B(t)dt + l \int B^2(t)dt \quad (2.2.38)$$

where $\eta$, the constant of integration, serves as the similarity variable. Using (2.2.30) and (2.2.34), the solution form (2.2.21) becomes

$$u(x, t) = A(t) - x \frac{B'(t)}{B(t)} + B(t)H(\eta). \quad (2.2.39)$$

Using (2.2.30), (2.2.34), and (2.2.38) in (2.2.26), we get

$$\left( A' - \frac{B'}{B} A \right) + x \left( 2 \frac{B'^2}{B^2} - \frac{B''}{B} \right) = B^3 \Gamma_4(\eta). \quad (2.2.40)$$

Equations (2.2.38) and (2.2.40) imply that

$$\Gamma_4(\eta) = m\eta + k \quad (2.2.41)$$

where $m$ and $k$ are arbitrary constants. Substituting (2.2.38) and (2.2.41) in (2.2.40), we get

$$\left( A' - \frac{B'}{B} A \right) + x \left( 2 \frac{B'^2}{B^2} - \frac{B''}{B} \right) = B^3 \left[ mxB - m \int ABdt + ml \int B^2 dt + k \right].$$

Equating coefficients of $x$ and terms free of $x$ on both sides of this equation, we get

$$BB'' - 2B'^2 + mB^6 = 0 \quad (2.2.42)$$

$$A' - \frac{B'}{B} A = B^3 \left[ k - m \int ABdt + ml \int B^2 dt \right]. \quad (2.2.43)$$

Equations (2.2.42) and (2.2.43) are solved for two special cases.

(i) $m = 0$

In this case, equation (2.2.42) becomes

$$BB'' - 2B'^2 = 0, \quad (2.2.44)$$

giving

$$B(t) = bt^{-1} \quad (2.2.45)$$

where $b$ is an arbitrary constant.

Using (2.2.45) and $m = 0$ in (2.2.43), we have

$$\frac{dA}{dt} + \frac{1}{t} A = \frac{kb^3}{t^3}$$

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yielding a special solution

$$A(t) = -kb^3 t^{-2}. \quad (2.2.46)$$

On using (2.2.45) and (2.2.46) in (2.2.38) and (2.2.39), we have the similarity variable and the solution as

$$\eta(x,t) = bx t^{-1} - \frac{kb^4}{2} t^{-2} - lb^2 t^{-1} \quad (2.2.47)$$

and

$$u(x,t) = -kb^3 t^{-2} + xt^{-1} + bt^{-1} H(\eta), \quad (2.2.48)$$

respectively. Using (2.2.29), (2.2.33), (2.2.41), and \(\Gamma_2(\eta) = l\) in (2.2.27), we find that \(H(\eta)\) satisfies the first order ODE

$$(H + l)H' + k = 0 \quad (2.2.49)$$

with the solution

$$\frac{H^2}{2} + lH + k\eta = p$$

where \(p\) is the constant of integration; solving for \(H\) we have

$$H(\eta) = -l \pm \sqrt{l^2 + 2p - 2k\eta}. \quad (2.2.50)$$

Substituting (2.2.50) and (2.2.47) into (2.2.48), we get an explicit solution of (2.2.20) as

$$u(x,t) = xt^{-1} - kb^3 t^{-2} + bt^{-1} \left[ -l \pm \sqrt{l^2 + 2p - 2kxt^{-1} + 2kbl^2 t^{-1} + k^2 b^4 t^{-2}} \right].$$

(ii) \(l = k = 0\)

Another special solution of (2.2.42) is

$$B(t) = qt^{-1/2} \quad (2.2.51)$$

provided

$$4q^4 + 1 = 0 \quad (2.2.52)$$

where \(q\) is a constant. Correspondingly, a solution of (2.2.43) may be found to be

$$A(t) = c - at^{-1} \quad (2.2.53)$$

where \(a\) and \(c\) are arbitrary constants. Making use of (2.2.51)–(2.2.53) in (2.2.38) and (2.2.39), we get the following similarity reduction of (2.2.20):

$$\eta(x,t) = q(x - 2at^{-1/2} - 2qct^{1/2} \quad (2.2.54)$$

$$u(x,t) = c - at^{-1} + \frac{x}{2} t^{-1} + qt^{-1/2} H(\eta). \quad (2.2.55)$$

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The ODE for this special case \( l = k = 0 \) governing \( H(\eta) \) is obtained by using the results \( \Gamma_i(\eta) = 0, \ i = 1, 2, 3, \Gamma_4(\eta) = m\eta, \ 4q^4m + 1 = 0, \) and (2.2.54) in (2.2.27):

\[
HH' - \frac{1}{4q^4\eta} = 0
\]

which immediately integrates to give

\[
H(\eta) = \pm \sqrt{\frac{\eta^2}{4q^4}} + r
\]

where \( r \) is constant of integration. Using (2.2.56) and (2.2.54) in (2.2.55), we get another explicit solution of (2.2.20):

\[
u(x, t) = c - at^{-1} + \frac{x}{2}t^{-1} \pm \sqrt{\frac{1}{4q^2}[(x-2a)t^{-1/2} - 2ct^{1/2}]^2}.
\]

(2.2.57)

Example 2

\[(t + u)u_x + tu_t = x - t \] (2.2.58)

Assume a solution of the form

\[u(x, t) = \alpha(x, t) + \beta(x, t)H(\eta), \eta = \eta(x, t), \beta(x, t) \neq 0. \] (2.2.59)

Differentiating (2.2.59) to get \( u_x \) and \( u_t \) and substituting in (2.2.58) and rearranging terms, we get

\[
\beta^2 \eta_x HH' + \beta (t\eta_x + \alpha \eta_x + t\eta_t)H' + \beta \beta_x H^2
\]

\[
+ (t\beta_x + \alpha \beta_x + \beta t + \alpha_x^{\beta})H
\]

\[
+ (\alpha \alpha_x + \alpha \alpha_x + \alpha \alpha_t - x + t) = 0.
\]

(2.2.60)

Equation (2.2.60) will be an ODE for \( H(\eta) \) only if

\[
\beta[(t + \alpha)\eta_x + t\eta_t] = \beta^2 \eta_x \Gamma_1(\eta) \] (2.2.61)

\[
\beta \beta_x = \beta^2 \eta_x \Gamma_2(\eta) \] (2.2.62)

\[
(t + \alpha)\beta_x + t\beta_t + \alpha_x \beta = \beta^2 \eta_x \Gamma_3(\eta) \] (2.2.63)

\[
(t + \alpha)\alpha_x + ta_t - x + t = \beta^2 \eta_x \Gamma_4(\eta). \] (2.2.64)

It then takes the form

\[
HH' + \Gamma_1(\eta)H' + \Gamma_2(\eta)H^2 + \Gamma_3(\eta)H + \Gamma_4(\eta) = 0. \] (2.2.65)
In (2.2.62), let \( \Gamma_2(\eta) = \frac{\Omega'_2(\eta)}{\Omega_2(\eta)} \) so that it integrates and gives
\[
\beta(x,t) = B(t)\Omega_2(\eta). \tag{2.2.66}
\]
Using Remark 2 in Example 1, we may put \( \Omega_2 \equiv 1 \) in (2.2.66) and obtain
\[
\beta(x,t) = B(t) \tag{2.2.67}
\]
and
\[
\Gamma_2(\eta) \equiv 0. \tag{2.2.68}
\]
In (2.2.63) we put \( \Gamma_3(\eta) = \Omega'_3(\eta) \), use (2.2.67), and integrate with respect to \( x \) to get
\[
\alpha(x,t) = \left[ A(t) - xt\frac{B'(t)}{B(t)} \right] + B(t)\Omega_3(\eta). \tag{2.2.69}
\]
Using Remark 1 of Example 1, we may put \( \Omega_3 \equiv 0 \) in (2.2.69) and have
\[
\alpha(x,t) = \left[ A(t) - xt\frac{B'(t)}{B(t)} \right]. \tag{2.2.70}
\]
Since \( \Gamma_3(\eta) = \Omega'_3(\eta) \), we also have
\[
\Gamma_3(\eta) \equiv 0. \tag{2.2.71}
\]
Substituting (2.2.70) in (2.2.61), we get
\[
\left[ t + A(t) - xt\frac{B'(t)}{B(t)} - B(t)\Gamma_1(\eta) \right] \eta_x + t\eta_t = 0. \tag{2.2.72}
\]
The characteristics of (2.2.72) are
\[
\frac{dx}{t + A(t) - xt\frac{B'(t)}{B(t)} - B(t)\Gamma_1(\eta)} = \frac{dt}{t} = \frac{d\eta}{0}. \tag{2.2.73}
\]
A first integral from (2.2.73) is clearly \( \eta = \text{constant} \); this is the similarity variable.
Setting
\[
\Gamma_1(\eta) = l, \text{ a constant} \tag{2.2.74}
\]
in the first equation of (2.2.73), we have
\[
\frac{dx}{dt} + \frac{B'(t)}{B(t)} x = 1 + \frac{A(t) - lB(t)}{t}. \tag{2.2.75}
\]
The solution of (2.2.75) gives the similarity variable
\[
\eta = xB(t) - \int \left[ 1 + \frac{A(t) - lB(t)}{t} \right] B(t)dt. \tag{2.2.76}
\]
Substituting (2.2.76) into (2.2.64) and using (2.2.70), we get

\[- \left[ t + A(t) - xt \frac{B'(t)}{B(t)} \right] \frac{tB'(t)}{B(t)} + t \left[ A'(t) - x \frac{B'(t)}{B(t)} - xt \frac{B''(t)}{B(t)} + xt \frac{B^2(t)}{B^2(t)} \right]
\]

\[-x + t = B^3(t) \Gamma_4(\eta). \tag{2.2.77}\]

Equations (2.2.76) and (2.2.77) imply that

\[\Gamma_4(\eta) = m\eta + k \tag{2.2.78}\]

where \(m\) and \(k\) are constants. Equations (2.2.76) and (2.2.78), when used in (2.2.77), give

\[-t^2 \frac{B'}{B} = t \frac{AB'}{B} + xt^2 \frac{B^2}{B^2} + tA' - xt \frac{B'}{B} - xt^2 \frac{B''}{B} + xt^2 \frac{B^2}{B^2} - x + t
\]

\[= B^3 \left[ m x B - m \int \left\{ 1 + \frac{A - lB}{t} \right\} Bdt + k \right]. \tag{2.2.79}\]

Equating coefficients of \(x\) and terms free of \(x\) on both sides of (2.2.79), we get

\[2t^2 B^2 - tBB' - t^2 BB'' - B^2 = mB^6, \tag{2.2.80}\]

\[-t^2 B' - tAB' + tA' B + tB = kB^4 - mB^4 \int \left[ 1 + \frac{A - lB}{t} \right] Bdt. \tag{2.2.81}\]

For the special case \(m = 0\), (2.2.80) gives

\[B(t) = bt. \tag{2.2.82}\]

Substituting (2.2.82) in (2.2.81), we get

\[\frac{dA}{dt} - \frac{1}{t} A = kb^3 t^2. \]

A special solution of this linear equation is

\[A(t) = \frac{k}{2} b^3 t^3. \tag{2.2.83}\]

On using (2.2.82), (2.2.83), (2.2.70), and (2.2.67) in (2.2.76) and (2.2.59), we get

\[\eta = bxt - \frac{bt^2}{2} - \frac{k}{8} b^4 t^4 + \frac{lb^2}{2} t^2 \tag{2.2.84}\]

\[u = \frac{k}{2} b^3 t^3 - x + btH(\eta). \tag{2.2.85}\]
On using (2.2.67), (2.2.70), and (2.2.82), Equation (2.2.60) becomes

\[ HH' + lH' + k = 0 \]  

(2.2.86)

and integrates to give

\[ H = -l \pm \sqrt{l^2 + 2p - 2k\eta} \]  

(2.2.87)

where \( p \) is the constant of integration. Using (2.2.87) and (2.2.84) in (2.2.85), we get a similarity solution of the given PDE. It may be explicitly written as

\[ u(x,t) = \frac{k}{2} b^3 t^3 - x - bt \]

\[ \pm bt \sqrt{l^2 + 2p - 2kbxt + kbt^2 - klb^2t^2 + \frac{k^2}{4} b^4 t^4}. \]

We have obtained some special exact solutions of (2.2.58) via the direct similarity approach. A richer class of solutions may be obtained if the intermediate equations can be solved more generally.

### 2.3 Reduction of \( u_t + u^n u_x + H(x, t, u) = 0 \) to the form \( U_t + U^n U_x = 0 \)

A large number of physical models are described by special cases of the generalised Burgers equation (GBE) (see Chapter 6)

\[ u_t + u^n u_x + H(x, t, u) = \delta u_{xx}, \]  

(2.3.1)

where \( \delta \) is the coefficient of viscosity. The inviscid limit of (2.3.1) as \( \delta \to 0 \) is

\[ u_t + u^n u_x + H(x, t, u) = 0. \]  

(2.3.2)

The term \( H(x, t, u) \) in (2.3.2) may represent the effects of damping, geometrical spreading, or sources of some sort. Equation (2.3.2) plays an important role in the analytical theory of GBEs.

We seek the most general transformation of the type

\[ \tau = \tau(x, t) \]  

(2.3.3)

\[ y = y(x, t) \]  

(2.3.4)

\[ U(y, \tau) = f(x, t)u(x, t) \]  

(2.3.5)

which reduces (2.3.2) to the form

\[ U_\tau + U^n U_y = 0. \]  

(2.3.6)
A more general form \( U = F(x, t, u) \) is not considered in order that Rankine-Hugoniot conditions for (2.3.2) and (2.3.6) remain the same.

We assume that \( f(x, t) > 0 \) and

\[
J = \begin{vmatrix} y_t & y_x \\ \tau_t & \tau_x \end{vmatrix} \neq 0. \tag{2.3.7}
\]

Differentiating (2.3.5) with respect to \( x \) and \( t \), we get

\[
U_y y_x + U_{\tau_x} = f u_x + f_x u \\
U_y y_t + U_{\tau_t} = f u_t + f_t u. \tag{2.3.8}
\]

Solving for \( U_{\tau_x} \) and \( U_y \) from (2.3.8), we have

\[
U_{\tau} = -\frac{1}{J} [(y_x f_t - y_t f_x) u + y_x f u_t - y_t f u_x] \]

\[
U_y = -\frac{1}{J} [(\tau_t f_x - \tau_x f_t) u + \tau_t f u_x - \tau_x f u_t]. \tag{2.3.9}
\]

Substituting (2.3.9) in (2.3.6), we get

\[
-\frac{1}{J} [(y_x f_t - y_t f_x) u + y_x f u_t - y_t f u_x] - \frac{f^n u^n}{J} [(\tau_t f_x - \tau_x f_t) u + \tau_t f u_x - \tau_x f u_t] = 0
\]

or

\[
u_t + \left( \frac{f}{f} - \frac{y t f}{y x f} \right) u - \frac{y t}{y x} u_x \\
+ f^n u^n \left[ \left( \frac{\tau_x f_x}{y_x f} - \frac{\tau_t f_t}{y_x f} \right) u + \frac{\tau_t}{y_x} u_x - \frac{\tau_x}{y_x} u_t \right] = 0. \tag{2.3.10}
\]

For (2.3.10) to be of the form (2.3.2), we must have

\[
y_t = 0, \quad \tau_x = 0, \quad \text{and} \quad \frac{f^n \tau_t}{y_x} = 1. \tag{2.3.11}
\]

Equation (2.3.10) then takes the form

\[
u_t + u^n u_x + \frac{f_t}{f} u + \frac{f_x}{f} u^{n+1} = 0. \tag{2.3.12}
\]

From (2.3.11) we see that \( y \) is a function of \( x \) alone and \( \tau \) is a function of \( t \) alone. Equation (2.3.11) then becomes

\[
f = \left[ \frac{y'(x)}{\tau'(t)} \right]^{1/n}. \tag{2.3.13}
\]
Let
\[ G(t) = \frac{f_t}{f} = -\frac{1}{n} \frac{d^2 \tau}{d\tau} = -\frac{d}{dt} \ln \left[ \left( \frac{dx}{dt} \right)^n \right] \]  \ \ \ \ (2.3.14)\]
and
\[ F(x) = \frac{f_x}{f} = \frac{1}{n} \frac{d^2 y}{dy} = \frac{d}{dx} \ln \left[ \left( \frac{dy}{dx} \right)^n \right]. \ \ \ \ (2.3.15)\]
Equation (2.3.12) can now be written as
\[ u_t + u^n u_x + G(t)u + F(x)u^{n+1} = 0 \]  \ \ \ \ (2.3.16)\]
where \( G(t) \) and \( F(x) \) are given by (2.3.14) and (2.3.15). Thus, \( H(x,t,u) \) in (2.3.2) must be of the form \( G(t)u + F(x)u^{n+1} \). Conversely, for given \( G(t) \) and \( F(x) \), the relations (2.3.14) and (2.3.15) determine the transformation functions \( \tau \) and \( y \) in (2.3.3) and (2.3.4).
Equation (2.3.14) may be written as
\[ \tau(t) = \int_t^t \left( \exp \left( \int_s^s G(s)ds_1 \right) \right)^{-n} ds. \]  \ \ \ \ (2.3.17)\]
Similarly, from (2.3.15)
\[ y(x) = \int_x^x \left( \exp \left( \int_s^s F(s)ds_1 \right) \right)^n ds. \]  \ \ \ \ (2.3.18)\]
Therefore,
\[ f(x,t) = \exp \left( \int_t^t G(s)ds \right) \exp \left( \int_x^x F(s)ds_1 \right). \]  \ \ \ \ (2.3.19)\]
Thus, we have the following result: the most general equation of the form (2.3.2) that can be reduced to (2.3.6) by the transformation (2.3.3)-(2.3.5) is (2.3.16); the transformation itself is given by (2.3.17) - (2.3.19).
Equations of the form (2.3.16) appear in many physical applications. Nimmo and Crighton (1986) considered the case \( n = 1 \) with \( F(x) \equiv 0 \) and \( G(t) = \left( \frac{j}{2t} + \alpha \right) \), \( j = 0, 1, 2 \). In this case, (2.3.16) takes the form
\[ u_t + uu_x + \left( \frac{j}{2t} + \alpha \right) u = 0. \]  \ \ \ \ (2.3.20)\]
From (2.3.17), (2.3.18), and (2.3.19) we get the transformation
\[ \tau = \int_t^t \left\{ \exp \left( \int_s^s \left( \frac{j}{2s_1} + \alpha \right) ds_1 \right) \right\}^{-1} ds \]
\[ y = \int_x^t (e^0)^1 ds = x; U(y, \tau) = f(x, t)u \] (2.3.21)

where
\[ f(x, t) = \exp \int_t^t \left( \frac{j}{2s} + \alpha \right) ds \cdot \exp \int_x^x 0ds_1 = tj/2e^{ot}. \]

This changes (2.3.20) to the form
\[ U_t + UU_y = 0. \]

Lefloch (1988) considered the special case of (2.3.16) for \( n = 1, G(t) \equiv 0, \) and \( F(x) = \frac{\beta}{x}; \)
\[ u_t + uu_x + \frac{\beta}{x}u^2 = 0. \] (2.3.22)

The transformation which reduces (2.3.22) to \( U_\tau + UU_y = 0 \) is
\[ y = \int_x^x \left( \exp \left( \int_s^x \frac{\beta}{s_1} ds_1 \right) \right) ds \]
\[ \frac{\beta + 1}{\beta + 1} \]
\[ \tau = \int_t^t \left( \exp \int_0^s ds_1 \right)^{-1} ds \]
\[ = t \]
\[ U = f(x, t)u = x^{\beta}u \]
since
\[ f(x, t) = \exp \left( \int_t^t 0 ds \right) \cdot \exp \left( \int_x^x \frac{\beta}{s_1} ds_1 \right) = x^{\beta}. \]

The inviscid limit of Burgers-Fisher equation
\[ u_t + uu_x + u(u - 1) = \frac{\delta}{2}u_{xx} \] (2.3.23)
is
\[ u_t + uu_x + u^2 - u = 0. \] (2.3.24)
This is a special case of (2.3.16) with \( n = 1, F(x) = 1, \) and \( G(t) = -1. \)
The transformation which changes (2.3.24) to \( U_\tau + UU_y = 0 \) is
\[ y = \int_x^x \left( \exp \left( \int_s^x ds_1 \right) \right) ds = \int_x^x e^s ds = e^x \]
\[ \tau = \int^t \left( \exp \left( \int^s (-1) ds_1 \right) \right)^{-1} ds = \int^t (e^{-s})^{-1} ds = e^t \]

\[ U(y, \tau) = f(x, t)u = e^{x-t}u(x, t) \]

since

\[ f(x, t) = \exp \left( \int^t (-1) ds \right) \exp \left( \int^x 1. ds_1 \right) = e^{-t} \cdot e^x. \]

Murray (1970) considered the equation  \[ u_t + g(u)u_x + \lambda u^\alpha = 0 \] where \( g'(u) > 0 \) for \( u > 0 \) and \( \lambda > 0 \) is a constant (see Section 2.4). We consider a special case \( g(u) = u \) and \( \alpha = 2 \), namely \[ u_t + uu_x + \lambda u^2 = 0. \] This is (2.3.16) with \( F(x) = \lambda, G(t) = 0 \) and \( n = 1 \):

\[ u_t + uu_x + \lambda u^2 = 0. \tag{2.3.25} \]

The transformation which reduces (2.3.25) to \( U_\tau + UU_y = 0 \) is

\[ y = \int^x \left( \exp \left( \int^s \lambda ds_1 \right) \right) ds = \int^x e^{\lambda s} ds = \frac{e^{\lambda x}}{\lambda} \]

\[ \tau = \int^t \left( \exp \left( \int^s 0. ds_1 \right) \right)^{-1} ds = \int^t ds = t \]

\[ U = f(x, t)u = e^{\lambda x}u \]

since

\[ f(x, t) = \exp \left( \int^t 0. ds \right) \cdot \exp \left( \int^x \lambda ds_1 \right) = e^{\lambda x}. \]

In the problem of propagation of waves in tubes we get the following equation for right-running waves (Shih (1974)):

\[ u_t + \left( a_0 + \frac{\gamma + 1}{2} u \right) u_x + \frac{F}{4D} u^2 = 0 \tag{2.3.26} \]

where \( F, D \) are a constants (see also Crighton (1979)).

With

\[ t' = \frac{F}{4D} t \quad \text{and} \quad x' = \frac{2}{\gamma + 1} \frac{F}{4D} x, \]

Equation (2.3.26) reduces (after dropping primes) to

\[ u_t + (a + u)u_x + u^2 = 0, a = \frac{2a_0}{\gamma + 1}. \tag{2.3.27} \]

With \( \tau = x - at \), (2.3.27) changes to

\[ u_t + uu_\tau + u^2 = 0. \tag{2.3.28} \]
Equation (2.3.28) is a special case of (2.3.25) with $\lambda = 1$. Therefore, the transformation $\bar{t} = t$, $y = e^\bar{x}$, $U = e^{\bar{x}}u(x,t) = yu(ln y, t)$ reduces (2.3.28) to the form $U_t + UU_y = 0$; here we assume that $y > 0, t > 0$.

We carried out a detailed analysis for the reduction of $u_t + uu_x = 0$ to an ODE by the direct approach of Clarkson and Kruskal (1989) in Section 2.2. A similar analysis may be done for (2.3.6) for $n \geq 2$ to find its symmetries and, hence, the solution.

2.4 Initial Value Problem for

$u_t + g(u)u_x + \lambda h(u) = 0$

An obvious generalization of the equation $u_t + u^n u_x = 0$ discussed in detail in Section 2.3 is

$u_t + g(u)u_x + \lambda h(u) = 0 \quad (2.4.1)$

where $\lambda \geq 0$ is a parameter and $g(u)$ and $h(u)$ are nonnegative functions of $u$ such that $g_u(u) > 0, h_u(u) > 0$ for $u > 0$.

Many model equations in applications are special cases of (2.4.1). In particular, when $h(u)$ can be negative for some $u$, interesting phenomena appear; they occur in a model for the Gunn effect (Murray (1970)) (see also Section 2.3). While it is not possible to give an explicit general discussion of (2.4.1), much progress can be made when $h(u) = O(u^\alpha)$, $\alpha > 0$, $0 < \alpha << 1$. Indeed, Murray (1970) has shown that in this case, a finite initial disturbance zero outside a finite range in $x$ decays (i) within a finite time and finite distance for $0 < \alpha < 1$ and is unique under certain conditions, (ii) within an infinite time like $O(\exp -\lambda t)$ and in a finite distance for $\alpha = 1$, and (iii) within an infinite time and distance like $O(t^{-1/(\alpha-1)})$ for $1 < \alpha \leq 3$ and $O(t^{-1/2})$ for $\alpha \geq 3$. The asymptotic speed of propagation of the discontinuity was given in each case together with its role in the decay process. We follow Murray (1970) closely in this section. After giving some results regarding the general Equation (2.4.1), we give a detailed analysis for the simpler case $u_t + (u + a)u_x + \lambda u = 0$, which displays many interesting features and is itself a descriptor of some physical phenomenon. It is a limiting case of the Burgers equation with damping, $u_t + (u + a)u_x + \lambda u = \delta u_{xx}$, as $\delta \to 0$, and plays an important role in its analysis. In the following section we shall discuss more recent work of Bukiet, Pelesko, Li, and Sachdev (1996), where special cases of (2.4.1) admitting similarity form of solutions would be studied. In this work, a numerical scheme for (2.4.1) was developed and the asymptotic nature of the exact solutions confirmed.

An initial-boundary value problem for (2.4.1) is posed as follows:

$u(0, t) = 0, \quad t > 0$
\[ u(x, 0) = u_0(x) = \begin{cases} 
0 & x < 0 \\
f(x) & 0 < x < X \\
0 & X < x 
\end{cases} \quad (2.4.2) \]

where \( f(x) \) is such that
\[ 0 \leq f(x) \leq 1. \quad (2.4.3) \]

With \( g(u) \) a monotonic increasing function, weak or discontinuous solutions of (2.4.1) occur when \( \lambda = 0 \) for some value of \( t > 0 \), even for smooth functions \( u_0(x) \) (see Section 2.2). If a discontinuity exists at \( t = 0 \), its propagation and decay are considered from the beginning.

Let the path of the shock discontinuity in the \((x,t)\)-plane be given by
\[ x = x_s(t). \quad (2.4.4) \]

The Rankine-Hugoniot condition which holds across the shock is
\[ \frac{dx_s}{dt} = \frac{1}{u_1 - u_2} \int_{u_2}^{u_1} g(u)\,du \quad (2.4.5) \]

where \( u_1(t) \) and \( u_2(t) \) are the values of \( u(x,t) \) at \( x_s^- \) and \( x_s^+ \), respectively. This can be obtained by applying the Gauss theorem to (2.4.1) across the shock. For simplicity we require \( u = 0 \) to be a solution of (2.4.1), implying that \( h(0) = 0 \). Equation (2.4.1) shows that, along the characteristics, we have
\[ \frac{dx}{dt} = g(u) \]
\[ \frac{du}{dt} + \lambda h(u) = 0. \quad (2.4.6) \]

In parametric form we have
\[ \frac{dx}{d\sigma} = g(u), \quad \frac{dt}{d\sigma} = 1 \]
\[ \frac{du}{d\sigma} = -\lambda h(u) \quad (2.4.7) \]

where \( \sigma \) is a parameter measured along the characteristics.

The solution of (2.4.7) may be obtained as
\[ x(\sigma) = \xi + \int_0^\sigma g[u(x(\tau), \tau)]\,d\tau \]
\[ t(\sigma) = \sigma \]
\[ \int_{f(\xi)}^{\xi} \frac{ds}{h(s)} = -\lambda \sigma. \quad (2.4.8) \]
Here, $t = 0$ when $\sigma = 0$, and $\xi$ is the value of $x$ at $t = 0$. Let $t_c$ be the critical time beyond which the solution (2.4.8) ceases to be single-valued and a shock is formed.

Let
\[
\int_0^u \frac{ds}{h(s)} = H(u) - H(f(\xi))
\] (2.4.9)
so that
\[
H'(u) = \frac{1}{h(u)}.
\] (2.4.10)

The integration of (2.4.8) yields
\[
H(u) = H(f(\xi)) - \lambda \sigma
\] (2.4.11)

\[
u(\sigma) = H^{-1}[H(f(\xi)) - \lambda \sigma]
\] (2.4.12)

where the inverse function $G = H^{-1}$ exists since $H$ is monotonic. On using (2.4.12), we get from (2.4.8)

\[
x = \xi + \int_0^t g\{G[H(f(\xi)) - \lambda \tau]\} d\tau.
\] (2.4.13)

To find when the solution ceases to be single-valued, we differentiate (2.4.13) with respect to $\xi$ and equate the result to zero. We find that the earliest time $t_c$ at which the shock is formed satisfies

\[
0 = 1 + \int_0^{t_c} g'\{G[H(f(\xi)) - \lambda \tau]\} G'[H(f(\xi)) - \lambda \tau] H'(f(\xi)) f'(\xi) d\tau,
\]

that is,

\[
1 = \int_0^{t_c} \frac{d}{d\tau} g\{G[H(f(\xi)) - \lambda \tau]\} \frac{1}{h(f(\xi))} f'(\xi) d\tau
\]
\[
= \frac{1}{\lambda h(f(\xi))} \left[ g(G[H(f(\xi)) - \lambda \tau]) - g(f(\xi)) \right].
\] (2.4.14)

Here we have made use of the fact that $GH(f(\xi)) = f(\xi)$. When $\lambda = 0$, (2.4.12) gives $u(\sigma) = GH(f(\xi)) = f(\xi)$. Therefore, from (2.4.12) and (2.4.14) we get

\[
t_c = \left\{ \left[ -g'(f(\xi)) f'(\xi) \right]^{-1} \right\}_{\min}
\] (2.4.15)

Now we consider in some detail the special case

\[
u_t + (u + a) u_x + \lambda u = 0, a \geq 0, \lambda > 0; \]

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here \( g(u) = u + a \) and \( h(u) = u \), and so

\[
H(u) = \int \frac{du}{u} = \ln u
\]

\[
G(u) = H^{-1}(u) = e^u
\]

\[
H(f(\xi)) - \lambda \sigma = \ln(f(\xi)) - \lambda \sigma
\]

\[
u(\sigma) = G\{ H(f(\xi)) - \lambda \sigma \} = e^{\ln(f(\xi)) - \lambda \sigma}
\]

\[
= f(\xi)e^{-\lambda \sigma}.
\]  

(2.4.17a)

Using (2.4.8) we have

\[
x(\sigma) = \xi + \int_0^\sigma [u(x(\tau), \tau) + a]d\tau
\]

\[
t = \sigma.
\]  

(2.4.17b)

With smooth initial data, a shock will form at the time \( t = t_c \) obtained from (2.4.14):

\[
1 = \frac{1}{\lambda} f'(\xi) [f(\xi)e^{-\lambda t_c} + a - (f(\xi) + a)]
\]

or

\[
\lambda = f'(\xi)(e^{-\lambda t_c} - 1).
\]

The earliest time for shock formation, therefore, is

\[
t_c = \frac{1}{\lambda} \left[ \ln \frac{f'(\xi)}{f'(\xi) + \lambda} \right]_{\min}.
\]  

(2.4.18)

For \( t_c \) to be positive we must have \(-f'(\xi) > \lambda\) for some \( 0 \leq \xi < X \).

If a \( t_c \) does not exist, then the solution of IVP is given by (2.4.17a) for all \( t \geq 0 \). It decays exponentially as \( t \to \infty \). When \( a = 0 \), (2.4.17b) gives

\[
x = \xi + \int_0^\sigma u(x(\tau), \tau)d\tau
\]

\[
= \xi + \int_0^\sigma f(\xi)e^{-\lambda \tau} d\tau
\]

\[
= \xi + \frac{f(\xi)}{\lambda} (1 - e^{-\lambda \sigma}).
\]

Recalling that \( \sigma = t \), we have

\[
x \to \xi + \frac{f(\xi)}{\lambda} \text{ as } t \to \infty.
\]

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From this it follows that in the limit $t \to \infty$,

$$x \leq x_m = \left[ \xi + \frac{1}{\lambda} f(\xi) \right]_{\max}, \quad 0 \leq \xi \leq X. \quad (2.4.19)$$

Thus, the solution $u(x, t)$ decays to zero in a finite distance but exponentially in time. The solution does not decay in a finite distance if $a > 0$.

We consider a form of initial condition $u_0(x)$ with a shock present at $x = X$ having $u_2 = 0$ for all $t \geq 0$. The characteristic solution (2.4.17a)-(2.4.17b) holds for all $x$ and $t$, including $x = x_s \pm$, that is, $u(x_s-, t) = u_1(t)$, $u(x_s+, t) = u_2(t) = 0$. The shock speed is given by (2.4.5):

$$\frac{dx_s}{dt} = \frac{1}{u_1 - 0} \int_0^{u_1} (u + a) du = \frac{1}{2} u_1(t) + a. \quad (2.4.20)$$

Put $x = x_s(\equiv x_s-)$ in (2.4.17b) to get

$$x_s = \xi + \int_0^t [u(x(\tau), \tau) + a] d\tau, \sigma = t \quad (2.4.21)$$

just behind the shock.

On differentiation $x_s$ with respect to $t$, we have

$$\frac{dx_s}{dt} = \frac{d}{dt} \xi(x_s, t) + u(x_s, t) + a + \int_0^t \frac{\partial}{\partial \tau} \left[ f(\xi(x_s, \tau)) e^{-\lambda \tau} + a \right] d\tau
= \frac{d\xi}{dt} + u_1(t) + a - \frac{1}{\lambda} (e^{-\lambda t} - 1) \frac{df(\xi)}{dt}. \quad (2.4.22)$$

From (2.4.17a) we get

$$f(\xi(x_s, t)) = e^{\lambda t} u_1(t) \quad (2.4.23)$$

and

$$\frac{d}{dt} f(\xi(x_s, t)) = e^{\lambda t} \left[ \frac{du_1}{dt} + \lambda u_1 \right].$$

Since $f(x)$ is a monotonic increasing function for $0 \leq x < X$, its inverse exists. Let $f^{-1} = F$. Then (2.4.23) becomes

$$\xi(x_s, t) = F(e^{\lambda t} u_1(t)). \quad (2.4.24)$$

Differentiating (2.4.24), we get

$$\frac{d}{dt} \xi(x_s, t) = F'(e^{\lambda t} u_1) \left[ \frac{du_1}{dt} + \lambda u_1 \right] e^{\lambda t}. \quad (2.4.25)$$

Equating (2.4.20) and (2.4.22) and using (2.4.25) and (2.4.23) therein, we get
\[ \frac{1}{2} u_1(t) + a = F'(e^{\lambda t} u_1) \left[ \frac{du_1}{dt} + \lambda u_1 \right] e^{\lambda t} + u_1(t) + a \]

\[ = \frac{1}{\lambda} (e^{-\lambda t} - 1) e^{\lambda t} \left[ \frac{du_1}{dt} + \lambda u_1 \right] \]

or

\[ \frac{1}{\lambda} \frac{du_1}{dt} = (1 - e^{\lambda t})^{-1} \left\{ u_1 \left( e^{\lambda t} - \frac{1}{2} \right) + e^{\lambda t} F'(e^{\lambda t} u_1) \left[ \frac{du_1}{dt} + \lambda u_1 \right] \right\}. \]

(2.4.26)

The solutions \( u_1(t) \) of equation (2.4.26) will now be studied. Considering \( f(x) \) as in Figure 2.1 and letting \( \delta \to 0 \), we get the top-hat situation as shown in Figure 2.2. For \( \delta = 0 \) we have initially \( u = 0 \) for \( x < 0 \) and \( u = 1 \) for \( 0 < x < X \). Thus, we have a centered simple wave at \( x = 0 \). Therefore, for \( t \geq 0 \), (2.4.17a) holds with \( u = u_1 \) and \( f(\xi) = 1 \) and, since \( \sigma = t \), we have

\[ u_1(t) = e^{-\lambda t}. \]

(2.4.27)

Equation (2.4.20) now becomes

\[ \frac{dx_s}{dt} = \frac{1}{2} e^{-\lambda t} + a \]

which, on integration from 0 to \( t \), gives

\[ x_s(t) = X + at + \frac{1}{2\lambda} (1 - e^{-\lambda t}) \]

(2.4.28)

where \( x_s(0) = X \).

Figure 2.1. Typical initial profile for \( u(x, 0) \).
Figure 2.2. Top hat initial condition $u(x, 0)$.

Figure 2.3. $u(x, t)$ when the point A never overtakes B, and for all $x$ such that $u(x, t) > 0$, $x - X - at < 1/(2\lambda)$ for all $t \geq 0$.

Equation (2.4.27) can be obtained also from (2.4.26) by letting $F' \to \infty$.

The solution at this stage is shown in Figure 2.3.

This solution is valid for $t \leq t_0$ where $t_0$ is the time at which the first characteristic of the centred wave at $x = 0$ catches up with the shock, that is, when the point A in the Figure 2.3 catches up with the shock at B. From (2.4.12) we get the distance travelled by $A$ by putting $\xi = 0$ and $f(\xi) = 1$. At $t = t_0$ (2.4.17b) gives

$$x(t_0) = \int_0^{t_0} (e^{-\lambda \tau} + a)d\tau$$

$$= at_0 + \frac{1}{\lambda}(1 - e^{-\lambda t_0}).$$

(2.4.29)
From (2.4.28) we get
\[ x_s(t_0) = X + at_0 + \frac{1}{2\lambda}(1 - e^{-\lambda t_0}) \]
\[ = at_0 + \frac{1}{\lambda}(1 - e^{-\lambda t_0}) \]
on using (2.4.29). Therefore,
\[ e^{-\lambda t_0} = 1 - 2\lambda X. \] (2.4.30)
The equation satisfied by \( u_1(t) \) for \( t \geq t_0 \) is given by (2.4.26) (assuming \( t_0 \) exists) with \( F' = 0 \):
\[ \frac{1}{\lambda} \frac{du_1}{dt} = (1 - e^{-\lambda t})^{-1} u_1 \left( e^{\lambda t} - \frac{1}{2} \right). \] (2.4.31)
This solution must match the solution obtained from (2.4.27) at \( t = t_0 \). Therefore, on using (2.4.30), we have
\[ u_1(t_0) = e^{-\lambda t_0} = 1 - 2\lambda X. \] (2.4.32)
We consider two cases arising from (2.4.30).
i) \( 2\lambda X > 1 \).

In this case, \( t_0 \) in (2.4.30) does not exist, so \( u_1(t) \) given by (2.4.27) is valid for all \( t \geq 0 \), \( x_s(t) \) is given by (2.4.28), and \( u \) is found parametrically from (2.4.17) as
\[ u(\xi, t) = f(\xi)e^{-\lambda t} \]
\[ x = \int_0^t [f(\xi)e^{-\lambda \tau} + a]d\tau \]
\[ = at + \frac{f(\xi)}{\lambda}(1 - e^{-\lambda t}). \] (2.4.33)

ii) \( 2\lambda X < 1 \).

A finite \( t_0 \) exists, and \( u_1(t) \) is given by (2.4.27) for \( t < t_0 \). For \( t > t_0 \), we solve (2.4.31) subject to (2.4.32) and obtain
\[ x_s(t) = X + at + \frac{1}{2\lambda}(1 - e^{-\lambda t_0})^{1/2}[2(1 - e^{-\lambda t_1/2} - (1 - e^{-\lambda t_0}))^{1/2}] \] (2.4.34)
where we have used the condition \( x_s(t_0) \) from (2.4.28). It follows from (2.4.34) that
where we have used (2.4.32). For $t \geq t_0$, the solution for $u(x,t)$ for $at \leq x < x_s(t)$ is given parametrically by (2.4.33).

As mentioned in the introduction to this section, Murray (1970) also considered the general Equation (2.4.1) with $h(u) = u^\alpha, \alpha > 0, u << 1$, including the limiting case $\lambda \to 0$.

Bukiet, Pelesko, Li, and Sachdev (1996) devised a characteristic-based numerical scheme for first-order PDEs and verified the asymptotic results of Murray with reference to the following initial conditions for the special case of (2.4.1), namely

\[ u_t + (\gamma u^\beta)_x + \lambda u^\alpha = 0. \quad (2.4.36) \]

(i) Smooth IC

\[ u(x,0) = \begin{cases} 
0 & x < 0 \\
\sin^2(\pi x) & 0 \leq x \leq 1 \\
0 & 1 < x 
\end{cases} \quad (2.4.37) \]

The parameters in (2.4.36) were chosen to be $\gamma = 1/2, \beta = 2, \lambda = \pi/2$, and $\alpha = 1$.

(ii) Top hat IC

\[ u(x,0) = \begin{cases} 
0 & x < 0 \\
h & 0 < x < X \\
0 & X < x 
\end{cases} \quad (2.4.38) \]

The parameters in (2.4.36) for this IC were

\[ \gamma = \frac{1}{2}, \beta = 2, \lambda = 1, \text{ and } \alpha = 1.5, 2.5, 4. \]

For the continuous IC (2.4.37), the formation of the shock and its subsequent propagation were studied numerically. Asymptotic decay law agreed with the analytic formulae of Murray (1970).

For the top hat IC (2.4.38), different cases were considered: when the rarefaction wave catches up to the shock and when it does not. Again, the analytic results of Murray (1970) were confirmed numerically.
2.5 Initial Value Problem for
\[ u_t + u^\alpha u_x + \lambda u^\beta = 0 \]

In this section we continue the analysis of Section 2.4, but restrict ourselves to Equation (2.5.1) below, which is a special case of (2.4.1) with \( g(u) = u^\alpha, h(u) = u^\beta \). With this choice it becomes possible to find explicit solutions for many cases either by the method of characteristics or by reduction to an ODE via similarity analysis. Apart from finding explicit solutions, the concern here is to demonstrate the limiting nature of the similarity solution. We follow the work of Bukiet, Pelesko, Li, and Sachdev (1996). An important contribution of this paper is the development of a characteristic-based numerical scheme for nonlinear scalar hyperbolic equations, which involves the solving of ODEs. The solution thus computed displays sharp, well-defined shocks when they exist. The analytic solutions found here demonstrate the efficacy of the numerical scheme developed by Bukiet et al. (1996).

Consider the equation
\[ u_t + u^\alpha u_x + \lambda u^\beta = 0 \] (2.5.1)

with the top hat initial data
\[ u(x, 0) = \begin{cases} 
0 & x < 0 \\
h & 0 < x < X \\
0 & X < x 
\end{cases} \] (2.5.2)

where \( h, \alpha \) and \( \beta \) are positive constants; \( \lambda > 0 \) is the dissipative constant.

If \( \alpha = 0 \), the solution is a decaying travelling wave moving to the right with speed 1. For \( \lambda = 0 \), (2.5.1) reduces to
\[ u_t + u^\alpha u_x = 0, \] (2.5.3)

and so
\[ \frac{du}{dt} = 0 \text{ along the characteristic curves } \frac{dx}{dt} = u^\alpha. \] (2.5.4)

That is, \( u \) is constant along the characteristics \( \frac{dx}{dt} = u^\alpha \) which are straight lines in the \((x, t)\) plane. The initial condition \( u = 0 \) for \( x < 0 \) and \( u = h \) for \( x > 0 \) give rise to a rarefaction wave centred at \( x = 0 \). All values from 0 to \( h \) propagate along the characteristics of the rarefaction wave. Since \( \alpha > 0 \), the characteristic with value \( h \) for \( u \) has the highest speed: \( \frac{dx}{dt} = h^\alpha \). The equation of this characteristic is obtained by integrating (2.5.4):
\[ x_F = h^\alpha t \] (2.5.5)
where we have used the condition \( x_F = 0 \) at \( t = 0 \), \( x_F \) denoting the front of the rarefaction wave. A shock originates at \( x = X \) (since \( u = h \) for \( x < X \) and \( u = 0 \) for \( x > X \)). The motion of this shock is given by

\[
\frac{dx_s}{dt} = \frac{1}{0 - u_b} \int_{u_a}^{0} u^\alpha du
= \frac{u_b^\alpha}{\alpha + 1}.
\] (2.5.6)

Here, \( u_a \) and \( u_b \) are values of \( u \) immediately ahead of and behind the shock, respectively. Before the rarefaction catches up with the shock, the value of \( u_b \) is \( h \). Therefore,

\[
\frac{dx_s}{dt} = \frac{h^\alpha}{\alpha + 1}
\]

or

\[
x_s = \frac{h^\alpha}{\alpha + 1} t + X, \quad 0 \leq t \leq t_0,
\] (2.5.7)

since \( x_s = X \) at \( t = 0 \); \( t_0 \) is the time at which the rarefaction catches up with the shock. Thus the solution for \( x_F < x < x_s \) is \( u = h \). In order to solve for \( u \) in the rarefaction (\( 0 \leq x \leq x_F \)), we use (2.5.4). Thus,

\[
\frac{dx}{dt} = u^\alpha, \quad u = C \text{ (constant),}
\]

therefore,

\[
x = C^\alpha t,
\]

since \( x = 0, t = 0 \) in the rarefaction wave. We readily have

\[
u = \left(\frac{x}{t}\right)^{\frac{1}{\alpha}}.
\] (2.5.8)

From (2.5.5) and (2.5.7) we find that the rarefaction wave catches up with the shock at the time \( t_0 \) when \( x_s = x_F \), that is,

\[
\frac{h^\alpha}{\alpha + 1} t_0 + X = h^\alpha t_0
\]

or

\[
t_0 = \frac{(\alpha + 1)X}{\alpha h^\alpha}.
\] (2.5.9)

At this time the position of the shock is

\[
x_s = \frac{h^\alpha}{\alpha + 1} \frac{(\alpha + 1)X}{\alpha h^\alpha} + X
= \frac{\alpha + 1}{\alpha} X
\] (2.5.10)

where we have used (2.5.7) and (2.5.9).
For \( t > t_0 \), the motion of the shock can be found from Equations (2.5.6) and (2.5.8). The value of \( u_b \) is now less than \( h \). On using (2.5.8), (2.5.6) becomes
\[
\frac{dx_s}{dt} = \frac{1}{\alpha + 1} \left( \frac{x_s}{t} \right)
\]
which, on integration, yields
\[
x_s = K t^{\frac{1}{\alpha + 1}}
\]
(2.5.11)
where \( K \) is a constant. From (2.5.10) we have \( x_s = \frac{\alpha + 1}{\alpha} X \) at \( t = t_0 \); therefore,
\[
\frac{\alpha + 1}{\alpha} X = K t_0^{\frac{1}{\alpha + 1}}.
\]
(2.5.12)
Equation (2.5.12) gives the value of \( K \) as
\[
K = \frac{\alpha + 1}{\alpha} \frac{X}{t_0^{\frac{1}{\alpha + 1}}}
= (\alpha + 1) \left[ \frac{1}{\alpha + 1} \left( \frac{hX}{\alpha} \right)^{\alpha} \right]^{\frac{1}{\alpha + 1}}
\]
where we have made use of (2.5.9). Using this value of \( K \) in (2.5.11), we have
\[
x_s = (\alpha + 1) \left[ \frac{t}{\alpha + 1} \left( \frac{hX}{\alpha} \right)^{\alpha} \right]^{\frac{1}{\alpha + 1}}.
\]
(2.5.13)
The results can now be summarised as follows:
\[
u(x,t) = \begin{cases} 
\left( \frac{x}{t} \right)^{1/\alpha} & 0 \leq x \leq x_F \\
\frac{h}{x_F} & x_F < x < x_s, \; t \leq t_0 \\
0 & x > x_s \end{cases}
\]
(2.5.14)
and
\[
u(x,t) = \begin{cases} 
\left( \frac{x}{t} \right)^{1/\alpha} & 0 \leq x \leq x_s \\
0 & x > x_s, \; t \geq t_0 
\end{cases}
\]
(2.5.15)
\[\text{ii)} \lambda \neq 0, \beta = 1. \text{ Here we have the equation} \]
\[
u_t + \nu^\alpha \nu_x + \nu = 0
\]
(2.5.16)
so that
\[
\frac{du}{dt} + \nu = 0 \text{ along the characteristic curves } \frac{dx}{dt} = \nu^\alpha.
\]
(2.5.17)
Integration of (2.5.17) gives
\[ u = u_0 e^{-\lambda t} \]
where \( u = u_0 \) at \( t = 0 \). Since \( u \) varies from 0 to \( h \) in the rarefaction wave, decay of the height of the top hat is given by \( \max u = h e^{-\lambda t} \). Using this result in the characteristic direction in (2.5.17), the position of the wavefront is found to be
\[ x_F = \frac{h^\alpha}{\lambda \alpha} \left( 1 - e^{-\lambda \alpha t} \right), \quad (2.5.18) \]
where we have inserted the 1C \( x_F = 0 \) at \( t = 0 \). The equation for the motion of the shock wave is
\[ \frac{dx_s}{dt} = \int_{u_b}^{\frac{h}{u_b}} \frac{0 - u}{u} \, du = \frac{(h e^{-\lambda t})^\alpha}{\alpha + 1} \quad (2.5.19) \]
which, on integration and use of 1C \( x_s = X \) at \( t = 0 \), gives
\[ x_s = X + \frac{h^\alpha}{\lambda \alpha (\alpha + 1)} \left( 1 - e^{-\lambda \alpha t} \right). \quad (2.5.20) \]
Equation (2.5.20) gives the shock trajectory. The time \( t_0 \) at which the front of the rarefaction catches up with the shock is obtained by equating (2.5.18) and (2.5.20),
\[ \frac{h^\alpha}{\lambda \alpha} \left( 1 - e^{-\lambda \alpha t_0} \right) = X + \frac{h^\alpha}{\lambda \alpha (\alpha + 1)} \left( 1 - e^{-\lambda \alpha t_0} \right), \quad (2.5.21) \]
yielding
\[ t_0 = -\frac{1}{\lambda \alpha} \ln \left[ 1 - \frac{(\alpha + 1)X}{h^\alpha} \right]. \quad (2.5.22) \]
Thus, the rarefaction wave catches up with the shock only if
\[ 1 - \frac{(\alpha + 1)\lambda \alpha X}{h^\alpha} > 0, \]
that is,
\[ h \geq \left[ (\alpha + 1)\lambda \alpha X \right]^\frac{1}{\alpha}. \quad (2.5.23) \]
If \( t_0 \) exists, the location of the shock at this time is given by
\[ x_s = X + \frac{h^\alpha}{\lambda \alpha (\alpha + 1)} \cdot \frac{\lambda (\alpha + 1)X}{h^\alpha} = \frac{(\alpha + 1)^\alpha}{\alpha} X, \]
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where we have used (2.5.20) and (2.5.22). Suppose a characteristic in the rarefaction has a value \( u = c \) at time \( t \). Then, as in (2.5.17), we have

\[
\frac{dx}{dt} = (ce^{-\lambda t})^\alpha = c^\alpha e^{-\lambda \alpha t}
\]

which, on integration and use of the IC \( x = 0, \ t = 0 \), gives

\[
x = \frac{c^\alpha}{\lambda \alpha} (1 - e^{-\lambda \alpha t}).
\]

The solution in the rarefaction wave is

\[
u(x, t) = ce^{-\lambda t} = \left( \frac{\lambda \alpha x}{e^{\lambda \alpha t} - 1} \right)^{\frac{1}{\alpha}}.
\] (2.5.24)

After \( t = t_0 \), the motion of the shock is given by

\[
\frac{dx_s}{dt} = \frac{u^\beta_{0 s}}{\alpha + 1} = \frac{\lambda \alpha x_s}{(\alpha + 1)(e^{\lambda \alpha t} - 1)} = \frac{\lambda \alpha e^{-\lambda \alpha t} x_s}{(\alpha + 1)(1 - e^{-\lambda \alpha t})}.
\] (2.5.25)

On integrating (2.5.25), using the condition \( x_s = \frac{\alpha + 1}{\alpha} X \) at \( t = t_0 \) and recalling (2.5.22), we get

\[
x_s = \frac{\alpha + 1}{\alpha} \left( hX \right)^{\frac{\alpha}{\alpha + 1}} \frac{1}{[\lambda(\alpha + 1)]^{1/(\alpha + 1)}} (1 - e^{-\lambda \alpha t})^{1/(\alpha + 1)}, \ \text{for} \ \ t > t_0.
\] (2.5.26)

It is clear that, whether the rarefaction catches up with the shock or not, the shock decays in infinite time but in finite distance.

If the rarefaction does not catch up with the shock, it follows from (2.5.20) that the location of the shock, as \( t \to \infty \), is

\[
x_{s, \infty} = X + \frac{h^\alpha}{\lambda \alpha (\alpha + 1)}.
\] (2.5.27)

If the rarefaction does catch up with the shock, that is, if \( t_0 \) exists, then the location of the shock from (2.5.26) in the limit \( t \) tending to infinity is

\[
x_{s, \infty} = \frac{\alpha + 1}{\alpha} \frac{1}{[\lambda(\alpha + 1)]^{1/(\alpha + 1)}} (hX)^{\alpha/(\alpha + 1)}.
\] (2.5.28)

To summarise, if \( \beta = 1 \),

\[
u(x, t) = \begin{cases} 
\left( \frac{\lambda \alpha x}{e^{\lambda \alpha t} - 1} \right)^{1/\alpha} & 0 \leq x \leq x_F \\
he^{-\lambda t} & x_F < x < x_s, \ t \leq t_0 \\
0 & x > x_s
\end{cases}
\] (2.5.29)
where \( t < t_0 \) and

\[
    u(x,t) = \begin{cases} \frac{\lambda \alpha xe^{-\lambda \alpha t}}{1 - e^{-\lambda \alpha t}} & 0 \leq x \leq x_s \\ 0 & x > x_s, \ t \geq t_0 \end{cases}
\]  

(2.5.30)

where \( t \geq t_0 \).

(iii) \( \beta \neq 1, \beta \neq \alpha \neq 1 \)

Here the solution with top hat initial data will be found. Along the characteristic curves we have

\[
    \frac{dx}{dt} = u, \quad \frac{du}{dt} + \lambda u^\beta = 0.
\]  

(2.5.31)

Integrating (2.5.31) subject to conditions \( u = h \) at \( t = 0 \), we get

\[
    u = [h^{1-\beta} - \lambda(1 - \beta)t]^{1/(1-\beta)}.
\]  

(2.5.32)

The wavefront, whose location is \( x_F = 0 \) at \( t = 0 \), is obtained by integrating

\[
    \frac{dX_F}{dt} = u_F, \text{ etc.}
\]

We have

\[
    x_F = \frac{1}{\lambda(\alpha + 1 - \beta)} \left[ h^{\alpha+1-\beta} - \{h^{1-\beta} - \lambda(1 - \beta)t\}^{\frac{\alpha+1-\beta}{\alpha}} \right].
\]  

(2.5.33)

Now the equation of motion of the shock wave is found from (2.5.32) as

\[
    \frac{dx_s}{dt} = \frac{u^\alpha}{\alpha + 1} = \left[ \frac{h^{1-\beta} - \lambda(1 - \beta)t}^{\alpha/(1-\beta)} \right].
\]  

(2.5.34)

Integrating (2.5.34) and using the condition \( x_s = X \) at \( t = 0 \), we have

\[
    x_s = X + \frac{1}{\lambda(\alpha + 1)(\alpha + 1 - \beta)} \left[ h^{\alpha+1-\beta} - \{h^{1-\beta} - \lambda(1 - \beta)t\}^{\frac{\alpha+1-\beta}{\alpha}} \right].
\]  

(2.5.35)

The time \( t_0 \) at which the front of the rarefaction catches up with the shock is found by equating (2.5.33) and (2.5.35):

\[
    t_0 = \left[ \frac{h^{\alpha+1-\beta} - \frac{\lambda(\alpha+1)(\alpha+1-\beta)}{\alpha} X^{\frac{1-\beta}{\alpha+1-\beta}}}{\lambda(\beta - 1)} - h^{1-\beta} \right].
\]  

(2.5.36)

For \( t_0 \) to exist we must have the right side of (2.5.36) greater than zero; besides, the expression in square brackets must be positive. This requires that

\[
    h > \left[ \frac{\lambda(\alpha + 1)(\alpha + 1 - \beta)X^{\frac{1}{\alpha+1-\beta}}}{\alpha} \right].
\]  

(2.5.37)
From (2.5.35) and (2.5.36) the shock position at \( t = t_0 \) is found to be

\[
x_s = X + \frac{1}{\lambda(\alpha + 1)(\alpha + 1 - \beta)} \cdot \frac{\lambda(\alpha + 1)(\alpha + 1 - \beta)}{\alpha} X
\]

\[
= \frac{\alpha + 1}{\alpha} X.
\]

This is in agreement with the result (2.5.10).

The characteristic solution in an implicit form may easily be found to be

\[
x = \frac{1}{\lambda(\alpha + 1 - \beta)} \left\{ \left[ u^{1-\beta} + \lambda(1 - \beta) t \right]^{\frac{\alpha + 1 - \beta}{\alpha + 1 - \beta}} - u^{\alpha + 1 - \beta} \right\}.
\]

(2.5.38)

(iv) \( \beta = \alpha + 1 \)

As for the derivation of (2.5.33), we have in this case

\[
x_F = \frac{1}{\lambda(1 - \beta)} \ln \left[ \frac{h^{1-\beta}}{h^{1-\beta} - \lambda(1 - \beta) t} \right].
\]

(2.5.39)

The equation of motion of the shock wave is

\[
\frac{dx_s}{dt} = \frac{1}{h^{1-\beta} - \lambda(1 - \beta) t}
\]

which, on integration and use of 1C \( x_s = X \) at \( t = 0 \), gives

\[
x_s = X + \frac{1}{\lambda\beta(1 - \beta)} \ln \left[ \frac{h^{1-\beta}}{h^{1-\beta} - \lambda(1 - \beta) t} \right].
\]

(2.5.40)

The time \( t_0 \) at which the front of the rarefaction wave catches up with the shock is obtained by equating (2.5.39) and (2.5.40):

\[
\lambda\beta(1 - \beta) X = (\beta - 1) \ln \left[ \frac{h^{1-\beta}}{h^{1-\beta} - \lambda(1 - \beta) t_0} \right]
\]

(2.5.41)

or

\[
t_0 = \frac{h^{1-\beta}(e^{\lambda\beta X} - 1)}{\lambda(\beta - 1)}.
\]

(2.5.42)

The value \( t_0 \) in (2.5.42) always exists since \( \beta = \alpha + 1 > 1 \), \( \alpha \) being positive. From (2.5.40) and (2.5.42) the location of the shock at \( t = t_0 \) is

\[
x_s = \frac{(\alpha + 1) X}{\alpha}.
\]
Some smooth solutions of $u_t + u^\alpha u_x + \lambda u^\beta = 0$

We investigate the conditions under which nonnegative, bounded $C^1$ solutions of (2.5.1) exist on a semi-infinite domain: $\epsilon < x \leq \infty$, $\delta < t \leq \infty$, where $\epsilon$ and $\delta$ are positive. These are some classes of initial and boundary conditions for which shock waves do not arise. First we study solutions of the form $u(x, t) = F(x)$ and $u(x, t) = G(t)$, that is, solutions which are functions of one variable only. Consider solutions of the form $u(x, t) = F(x)$. On substitution of this into (2.5.1) we get

$$F^\alpha F' + \lambda F^\beta = 0.$$  \hfill (2.5.43)

Therefore,

$$u(x, t) = F(x) = [(\alpha + 1 - \beta)(C - \lambda x)]^{\alpha - 1 - \beta}$$  \hfill (2.5.44)

where $C$ is constant of integration.

If $\beta > \alpha + 1$ and $C < 0$ (so that the expression within the square brackets in (2.5.44) is positive and the exponent is negative), the solution $u(x, t)$ is bounded. If $\beta = \alpha + 1$, (2.5.43) integrates to give the bounded solution

$$u(x, t) = F(x) = C e^{-\lambda x}$$  \hfill (2.5.45)

where $C$ is the constant of integration.

If $\alpha + 1 > \beta$, there are no bounded solutions of the form $u(x, t) = F(x)$.

Consider now solutions of the form $u(x, t) = G(t)$. Substitution into (2.5.1) gives

$$G' + \lambda G^\beta = 0.$$  \hfill (2.5.46)

If $\beta < 1$, there exist bounded solutions

$$u(x, t) = G(t) \geq \begin{cases} [(1 - \beta)(C - \lambda t)]^{\frac{1}{1-\beta}} & t < \frac{C}{\lambda} \\ 0 & t \geq \frac{C}{\lambda} \end{cases}$$  \hfill (2.5.47)

where the constant of integration $C$ is greater than zero.

These solutions decay to zero in a finite time $C/\lambda$. If $\beta = 1$, Equation (2.5.46) integrates to give

$$u(x, t) = G(t) = C e^{-\lambda t},$$  \hfill (2.5.48)

a solution which decays to zero in infinite time.

If $\beta > 1$, (2.5.46) may be integrated to yield

$$u(x, t) = G(t) = \frac{1}{[(\beta - 1)(\lambda t + C)]^{1/(\beta - 1)}}$$  \hfill (2.5.49)

where $C > 0$ is constant of integration. The solution (2.5.49) is bounded and decays to zero in infinite time.
Special exact solutions of the equation

\[ u_t + u^\alpha u_x + \lambda u^\beta = 0 \]

Consider the one-parameter family of stretching transformations (see Chapter 3)

\[
\begin{align*}
\bar{x} &= e^{\alpha}x, \bar{t} = e^{\beta}t, \bar{u} = e^{\gamma}u \\
\frac{\partial u}{\partial \bar{t}} &= e^{\beta - \gamma} \frac{\partial \bar{u}}{\partial \bar{t}} \frac{\partial \bar{u}}{\partial \bar{x}} = e^{\alpha - \gamma} \frac{\partial \bar{u}}{\partial \bar{x}}
\end{align*}
\]

(2.5.50)

Substituting (2.5.50) into (2.5.1), we get

\[ e^{\beta - \gamma} \frac{\partial \bar{u}}{\partial \bar{t}} + e^{\alpha - \gamma} \frac{\partial \bar{u}}{\partial \bar{x}} \cdot e^{-\gamma} \bar{u} + \lambda e^{-\gamma} \bar{u}^\beta = 0. \]

For invariance of (2.5.1) we must have

\[ b = c(1 - \beta), \quad a = c(1 + \alpha - \beta). \]

For \( \beta \neq 1, 1 + \alpha \) we have solutions of the form

\[ u(x, t) = t^{c/b} H(\eta) = t^{1/(1-\beta)} H(\eta) \]

where

\[ \eta = xt^{-a/b} = xt^{(\beta - a - 1)/(1 - \beta)}. \]

(2.5.51)

Substituting (2.5.51) into (2.5.1), we get

\[ H' = \frac{H - (\beta - 1)\lambda H^\beta}{(\alpha + 1 - \beta)\eta + (\beta - 1)H^\alpha}. \]

(2.5.52)

For \( \alpha + 1 > \beta \) and \( \beta > 1 \), \( H(\eta) \) approaches the constant solution

\[ H^* = [\lambda(\beta - 1)]^{1/(1-\beta)} \]

(2.5.53)

of (2.5.52) as \( \eta \to \infty \).

There are no bounded solutions of (2.5.52) as \( \eta \to \infty \) for \( \alpha + 1 < \beta \), since, in this case, either the denominator in (2.5.53) becomes zero at some finite point or \( H' \) is proportional to \( 1/\eta \) for large \( \eta \) so that \( H \approx O(\ln \eta) \) as \( \eta \to \infty \). There are also no bounded solutions for \( \beta < 1 \) since \( u \) grows with time.

If \( \beta \neq 1, \alpha + 1 \) we also have solutions of the form

\[ u(x, t) = x^{c/a} H(\eta) = t^{(\beta - a - 1)/(1 - \beta)} H(\eta) \]

\[ \eta = xt^{(\beta - a - 1)/(1 - \beta)}. \]

(2.5.54)

Substituting (2.5.54) into (2.5.1) we get, after some simplification,

\[ H' = \frac{(\beta - 1)[(\alpha + 1 - \beta)\lambda H^\beta + H^{\alpha + 1}]}{\eta(\alpha + 1 - \beta)[(\beta - a - 1)\eta H^{\frac{\alpha + 1}{\alpha + 1 - 1}} + (1 - \beta)H^\alpha]}. \]

(2.5.55)
For $\alpha + 1 > \beta$ and $\beta > 1$, $H(\eta) \to 0$ exponentially as $\eta \to \infty$. So bounded solutions exist. There are no bounded solutions to (2.5.55) for $\alpha + 1 < \beta$ or $\beta < 1$ since the denominator in the RHS in these cases vanishes at some finite point $\eta$. Some additional similarity solutions of (2.5.16) exist if $\alpha = 1$.

Bukiet et al. (1996) have studied numerically most of the special cases of (2.5.1) which have exact solutions. Specific values of parameters in the PDEs, IC, and BC (wherever applicable) were chosen. In each case the numerical results agreed closely with the analytic ones. However, the main point of their study was to show the superiority of the proposed numerical scheme over other schemes such as the Lax-Wendroff scheme. The shocks when they formed were accurately located and their progression in time predicted. The shocks were sharp, showing no spurious oscillations. The only drawback of this scheme is that, in its present form, it is applicable only to scalar hyperbolic PDEs.

We conclude this chapter by referring to a result due to Logan (1987) regarding the general first-order, nonlinear PDE

$$F(x, t, u, p, q) = 0 \quad (2.5.56)$$

where $p = u_x$ and $q = u_t$.

For this purpose we need the following definition. Equation (2.5.56) is constant conformally invariant under the one-parameter family of stretching $T_\epsilon$,

$$\bar{x} = \epsilon^a x, \quad \bar{t} = \epsilon^b t, \quad \bar{u} = \epsilon^c u \quad (2.5.57)$$

if, and only if,

$$F(\bar{x}, \bar{t}, \bar{u}, \bar{u}_x, \bar{u}_t) = A(\epsilon)F(x, t, u, u_x, u_t) \quad (2.5.58)$$

for all $\epsilon$ in I, for some function $A$ with $A(1) = 1$. If $A(\epsilon) \equiv 1$, then we say that (2.5.56) is absolutely invariant.

If Equation (2.5.56) is constant conformally invariant under the one-parameter family of stretching transformation (2.5.57), then it can be changed to an ODE of the form

$$H(s, f, f') = 0 \quad (2.5.59)$$

where

$$u = t^{c/b} f(s), \quad s = (x^b/t^a). \quad (2.5.60)$$